IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3-4, November 18, 2012

There are four questions, each with multiple parts.

Justify your answers; show your work.

1. Forecasting the Weather (12 points)

Consider the following probability model of the weather over successive days. First, suppose that on each day we can specify if the weather is rainy or dry. Suppose that the probability that it will be rainy on any given day is a function of the weather on the previous two days. If it was rainy both yesterday and today, then the probability that it will be rainy tomorrow is 0.7. If it was dry yesterday, but rainy today, then the probability that it will be rainy tomorrow is 0.5. If it was rainy yesterday, but dry today, then the probability that it will be rainy tomorrow is 0.4. If it was dry both yesterday and today, then the probability that it will be rainy tomorrow is 0.2. Let X_n be the weather on day n.

(a) Calculate the conditional probability that it rains tomorrow but is dry on the next two days, given that it rained both yesterday and today.

(b) Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Why or why not? If not, construct an alternative finite-state stochastic process that is a Markov chain.

(c) With your Markov chain model in part (b), calculate the long-run proportion of days that are rainy.

2. Back and Forth to Campus (18 points)

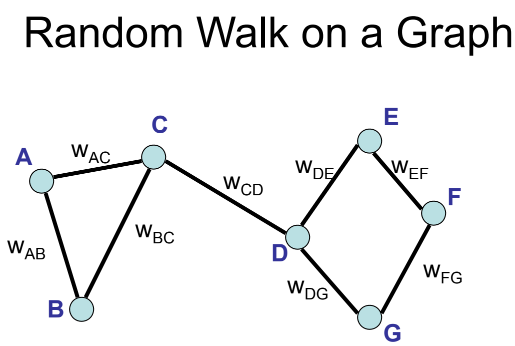
Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117th Street and 7th Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 4 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability 3/4 and walks up hill with probability 1/4. On the other hand, Prab is less likely to run going back home. On any given day, he runs down hill with probability 1/3 and walks down hill with probability 2/3. The distance in each direction is 1 mile.

(a) (6 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

(b) (6 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends walking up hill to campus?

(c) (6 points) Suppose that Prhab's sister in India happens to call him (at a time that can be taken to be at random, independent of his travel schedule, which Prhab has been following for a long time) while he is going up hill to campus. If he talks to her throughout the rest of his trip uphill, ending the call at his usual destination, and if his mode of travel is unaltered by the phone call, then what is the expected length of the phone call?

3. Random Walk on a Graph (30 points)



Consider the graph shown in the figure above. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight specified by the letter w with two subscripts, one for each node the arc connects. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move on any step be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $w_{AC}/(w_{AC} + w_{BC} + w_{CD})$, while the probability of moving from node C to node B in one step is $w_{BC}/(w_{AC} + w_{BC} + w_{CD})$. Let X_n be the node occupied by the random walk on the n^{th} step.

(a) Prove or disprove: The stochastic process $\{X_n : n \geq 0\}$ is a time reversible irreducible discrete-time Markov chain.

(b) Starting from node A, what is the expected number of steps required to return to node A?

(c) Prove that your answer in part (b) is correct. (You may quote theorems without proof as part of your proof.)

(d) Give an expression for the expected number of visits to node G, starting in node A, before going to either node B or node F.

(e) Prove that your answer in part (d) is correct. (You may quote theorems without proof as part of your proof.)

(f) Give an expression for the probability of going to B before going to node F, starting in node A.

4. A Renewal Process (52 points)

Let $\{N(t) : t \geq 0\}$ be a renewal process with times between renewals X_n having probability distribution

$$P(X_n = 5) \equiv \frac{1}{3} \equiv 1 - P(X_n = 2).$$

Let $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$, with $S_0 \equiv 0$ (but there is no renewal at time 0). Let $m(t) \equiv E[N(t)]$ and $Y(t) \equiv S_{N(t)+1} - t$, $t \geq 0$.

(a) What is $m(4)$?

(b) Prove or disprove:

$$\lim_{t \to \infty} m(t)/t = \frac{1}{3} \quad \text{as} \quad t \to \infty.$$

(You may quote a theorem without proof as part of your proof.)

(c) Prove or disprove:

$$\lim_{t \to \infty} m(t+a) - m(t) = \frac{a}{3} \quad \text{as} \quad t \to \infty \quad \text{for all} \quad a > 0.$$

(You may quote a theorem without proof as part of your proof.)

(d) Prove or disprove:

$$P(X_{N(t)+1} > x) \geq P(X_1 > x) \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \geq 0.$$

(e) Let $u_n \equiv P(N(n+0.5) - N(n-0.5) = 1)$, $n \geq 1$, and $u_0 \equiv 1$ (without there being a renewal at 0). Give explicit expressions for u_j, $1 \leq j \leq 9$.

(f) Does the limit of u_n as $n \to \infty$ exist? Why or why not? If the limit exists, what is its value?

(g) Derive the generating function $\hat{u}(z) \equiv \sum_{n=0}^{\infty} u_n z^n$ of the sequence $\{u_n : n \geq 0\}$ in (e).

(h) How could you use the generating function in part (g) to compute u_n for any n?

(i) Give an expression for $P(Y(t) > 4)$ in terms of u_n, where $Y(t)$ is the excess, defined at the outset.

(j) Set up a renewal equation for $P(Y(t) > 4)$, solve it, and relate your answer to part (i).

(k) Let $N_G(t)$ be a new counting process obtained by letting X_1 be distributed according to the cdf G instead of the given distribution, while the other random variables X_n for $n \geq 2$ remain unchanged. Let $m_G(t) \equiv E[N_G(t)]$. Exhibit all cdf's G such that $m_G(t) = t/3$, $t \geq 0$. As usual, justify your answer.

(l) Prove or disprove: There exists a cdf G (with $G(t) \to 1$ as $t \to \infty$) such that the stochastic process $\{N_G(t); t \geq 0\}$ has stationary increments.

(m) Prove or disprove: There exists a cdf G (with $G(t) \to 1$ as $t \to \infty$) such that the stochastic process $\{N_G(t); t \geq 0\}$ has stationary and independent increments.

The scoring on Problem 4 is 4 **points for each correct answer. Thus the maximum score is** $13 \times 4 = 52$**. Thus the maximum score on the entire test is** 112**.**