IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3-4, November 18, 2012

SOLUTIONS

Justify your answers; show your work.

1. Forecasting the Weather (12 **points)**

Consider the following probability model of the weather over successive days. First, suppose that on each day we can specify if the weather is rainy or dry. Suppose that the probability that it will be rainy on any given day is a function of the weather on the previous two days. If it was rainy both yesterday and today, then the probability that it will be rainy tomorrow is 0.7. If it was dry yesterday, but rainy today, then the probability that it will be rainy tomorrow is 0.5. If it was rainy yesterday, but dry today, then the probability that it will be rainy tomorrow is 0.4. If it was dry both yesterday and today, then the probability that it will be rainy tomorrow is 0.2. Let X_n be the weather on day n.

(a) (2 points) Calculate the conditional probability that it rains tomorrow but is dry on the next two days, given that it rained both yesterday and today.

$$
\begin{aligned}
&P(X_{n+1} = R, X_{n+2} = D, X_{n+3} = D | X_{n-1} = R, X_n = R) = P(X_{n+1} = R | X_{n-1} = R, X_n = R) \\
&\quad \times P(X_{n+2} = D | X_n = R, X_{n+1} = R) \times P(X_{n+3} = D | X_{n+1} = R, X_{n+2} = D) \\
&\qquad = (0.7)(0.3)(0.6) = 0.126
\end{aligned}
$$

(b) (5 points) Is the stochastic process $\{X_n : n \geq 0\}$ a Markov chain? Why or why not? If not, construct an alternative finite-state stochastic process that is a Markov chain.

No, the stochastic process $\{X_n : n \geq 0\}$ is *not* a Markov chain. It fails to have the Markov property. The probability of a future event, e.g., the weather tomorrow, conditional on present and past states does *not* depend only on the present state.

We now develop a DTMC. We let the state be (X_{n-1}, X_n), combining the states of X_n on days $n-1$ and n. Thus there are four states instead of two. The DTMC then transitions from the state (X_{n-1}, X_n) to the state (X_n, X_{n+1}). Note that part of the new stat is determined by the previous state. The model is the transition matrix

$$
P = \begin{array}{c} 1 \equiv RR \\ 2 \equiv DR \\ 3 \equiv RD \\ 4 \equiv DD \end{array} \begin{pmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix},
$$

where the columns are labeled in the same way, and the same order, as the rows.

(c) (5 points) With your Markov chain model in part (b), calculate the long-run proportion of days that are rainy.

We solve $\pi = \pi P$ with the 4×4 transition matrix in part (b). That gives the limiting probability

$$\lim_{n \to \infty} \{P(((X_{n-1}, X_n) = j)\},$$

where j is one of the four states. To get the desired long-run probability we must add $\pi_{(R,R)} + \pi_{(D,R)} = \pi_{(R,R)} + \pi_{(R,D)}$. (Equality holds because we can look at either the limit of (X_{n-1}, X_n) and sum over the possible values of X_{n-1} to get the marginal distribution of X_n or the limit of (X_n, X_{n+1}) and sum over the possible values of X_{n+1} to get the marginal distribution of X_n.)

When solving $\pi = \pi P$, we get the equations:

$$\begin{aligned} 0.7\pi_1 + 0.5\pi_2 &= \pi_1 \\ 0.4\pi_3 + 0.2\pi_4 &= \pi_2 \\ 0.3\pi_1 + 0.5\pi_2 &= \pi_3 \\ 0.6\pi_3 + 0.8\pi_4 &= \pi_4 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

From equation 1, we get $\pi_1 = (5/3)\pi_2$. Then, using this in equation 3, we get $\pi_3 = 0.3(5/3)\pi_2 + 0.5\pi_2 = \pi_2$. Hence, $\pi_2 = \pi_3$ From equation 4, we get $\pi_4 = 3\pi_3$. Combining the last two equations, we get $\pi_4 = 3\pi_2$.

From the final equation, we then get $\pi_2 = 3/20$. Thus,

$$\pi \equiv (\pi_1, \pi_2, \pi_3, \pi_4) = (5/20, 3/20, 3/20, 9/20).$$

Thus the long run proportion of days that are rainy is

$$\pi_1 + \pi_2 = \pi_1 + \pi_3 = \frac{8}{20} = \frac{2}{5}.$$

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2. Back and Forth to Campus (18 points)

Professor Prhab Hubilliti lives at the bottom of the hill on the corner of 117th Street and 7th Avenue. Going each way - up hill to to teach his class at Columbia or down hill back home - Prhab either runs or walks. Going up the hill, Prhab either walks at 2 miles per hour or runs at 4 miles per hour. Going down the hill, Prhab either walks at 3 miles per hour or runs at 6 miles per hour. In each direction, he always runs the entire way or walks the entire way. Since Prhab often works late into the night, he often gets up late, and has to run up hill to get to his class. On any given day, Prhab runs up hill with probability 3/4 and walks up hill with probability 1/4. On the other hand, Prab is less likely to run going back home. On any given day, he runs down hill with probability 1/3 and walks down hill with probability 2/3. The distance in each direction is 1 mile.

(a) (6 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends going up hill to campus?

The main idea is to exploit renewal theory. Successive round trips can be considered the i.i.d. renewals. In more detail, we can regard this as an alternating renewal process, based on the sequence $\{(U_n, D_n); n \geq 1\}$ of i.i.d. random vectors of nonnegative random variables, where U_n is the time to go uphill and D_n is the time to go uphill on the n^{th} trip. In this case, U_n and D_n are independent as well. The cycles are $X_n = U_n + D_n$. Let each cycle start with an uphill trip and end with a downhill trip.

However, the analysis can be quite simple for this problem. The idea is to apply the simple **renewal reward theory** from §3.6 of Ross. Indeed, this problem is a minor variation of the truck driver problem in the lecture notes of October 4.

Let U equal the time to go up and let D equal the time to go down. Use the renewal reward formula

$$\begin{aligned}
\text{Long run average reward} &= \frac{\text{average reward per cycle}}{\text{average length of cycle}} \\
&= \frac{E[U]}{E[U]+E[D]} = \frac{(5/16)}{(5/16)+(5/18)} = \frac{18}{34} = \frac{9}{17} \approx 0.529
\end{aligned}$$

To elaborate, **actual calculation requires care**. we need to use the formula $D = RT$, i.e., "distance equals rate multiplied by time." Hence, $T = D/R$. Since here $D = 1$, the time is simply the reciprocal of the speed. hence

$$E[U] = \left(\frac{3}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{5}{16} \quad \text{hour}$$

Similarly, $E[D] = 5/18$.

(b) (6 points) What is the long-run proportion of Prhab's total travel time going to and from campus that he spends walking up hill to campus?

The general approach is the same. The mean cycle length is the same, but now there is a reward only if he is walking up hill.

$$\begin{aligned}
\text{Long run average reward} &= \frac{\text{average reward per cycle}}{\text{average length of cycle}} \\
&= \frac{E[U;\text{walking}]}{E[T_U]+E[T_D]} = \frac{(1/8)}{(5/16)+(5/18)} = \frac{36}{170} = \frac{18}{85} \approx 0.21
\end{aligned}$$

(c) (6 points) Suppose that Prhab's sister in India happens to call him (at a time that can be taken to be at random, independent of his travel schedule, which Prhab has been following for a long time) while he is going up hill to campus. If he talks to her throughout the rest of his trip uphill, ending the call at his usual destination, and if his mode of travel is unaltered by the phone call, then what is the expected length of the phone call?

There are two parts to this problem: (i) determining what the answer should be and (ii) proving that the answer is correct. The second part is much harder than the first, because the time between renewals is lattice in this example.

But determining the answer is not hard. We can create a renewal process out of the successive trips uphill, ignoring the trips downhill and the times spent on campus. An interval between renewals is a time to go uphill. If the time to go up hill U has cdf F, then the remaining time during the phone call has cdf F_e, the stationary forward excess. (Recall the lecture notes of October 11 for more on the excess.) The cdf is F_e because the system is viewed as being in steady state, i.e., in equilibrium, since the call is assumed to be "at a time that can be taken to be at random, independent of his travel schedule. See Theorem 3.5.2.

Since U has a two-point distribution, this F and F_e are very similar to the distribution of X_n in problem 1. Let X and X_e be random variables with cdf's F and F_e, respectively. A fast answer follows from the formula

$$E[X_e^k] = \frac{E[X^{k+1}]}{(k+1)E[X]}, \quad k \geq 1,$$

which we only need to consider for $k = 1$. First, $E[T_U] = 5/16$ from part (a). Second,

$$E[T_U^2] = \left(\frac{3}{4} \times \frac{1}{16}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{7}{64}.$$

Hence we have

$$E[X_e] = \frac{E[X^2]}{(2)E[X]} = \frac{(7/64)}{2(5/16)} = \frac{7}{40} \quad \text{hours} \quad = 10.5 \quad \text{minutes.}$$

But it is not so hard to first calculate the pdf $f_e(x) = F^c(x)/E[X_1]$, and then calculate its mean.

more careful analysis.

To justify that the length of the call is distributed according to F_e, we can reason more carefully. It is directly defined as a conditional distribution. Let $X(t) = 1$ if Prhab is walking up hill during his travel time, and let $X(t) = 0$ if he is walking down hill. Let $L(t)$ be the remaining length of the call at time t. Let $U(t)$ be the remaining length of the uphill trip during the cycle in progress at time t. (If Prhab is going down hill at time t, then $U(t) = 0$.)

$$P(L(t) > x) = P(X(t) = 1, U(t) > x | X(t) = 1) = \frac{P(X(t) = 1, U(t) > x)}{P(X(t) = 1)}.$$

But we want to consider the system in equilibrium. The stochastic process $\{X(t) : t \geq 0\}$ is a regenerative process, as in §3.7 and the lecture notes of November 13. Let $\{X_e(t) : t \geq 0\}$ be the equilibrium or stationary version. Introduce e subscripts everywhere. Then we get

$$P(L_e(t) > x) = P(X_e(t) = 1, U_e(t) > x | X_e(t) = 1) = \frac{P(X_e(t) = 1, U_e(t) > x)}{P(X_e(t) = 1)}.$$

However,

$$P(X_e(t) = 1) = \frac{E[U]}{E[U] + E[D]} \tag{1}$$

and

$$P(X_e(t) = 1, U_e(t) > x) = \frac{E[U]F_e^c(x)}{E[U] + E[D]}. \tag{2}$$

That is, the stationary version X_e requires starting in state 1 with probability $p \equiv E[U]/(E[U] + E[D])$ and starting in state 0 otherwise. In addition, it requires that, given that we start in state 1, the length of time we remain there is distributed as U_e. Similarly, given that we start in state 0, the length of time we remain there is distributed as D_e. Since X is not a Markov process, we must initialize more than just the possible values of $X(0)$.

We can justify equations (1) and (2) by applying the renewal reward theorem. The stationary probabilities must coincide with the average reward per cycle divided by the average length of a cycle, for an appropriately defined reward function. Let the reward at time t be 1 is $X(t) = 1$ and $U(t) > x$. Then

$$
\begin{aligned}
P(X_e(t) = 1, U_e(t) > x) &= \frac{\text{ave reward per cycle}}{\text{ave cycle length}} \\
&= \frac{E[(U_1 - x)^+]}{E[X_1]} == \frac{\int_x^\infty P(U_1 > y)\, dy}{E[X_1]} \\
&= \left(\frac{E[U_1]}{E[X_1]}\right) F_e^c(x) = \frac{E[U_1] F_e^c(x)}{E[U_1] + E[D_1]}
\end{aligned}
$$

as given in (2).

The reasoning could be more straightforward for nonlattice distributions. Then we could apply Theorem VI.1.2 on p. 170 of Asmussen (2003). Or we could regard this as a simple semi-Markov process and apply Theorem 4.8.4 in Ross, which is very similar. Ti apply Asmussen, we let the regenerative process be the vector process $\{(X(t), U(t), D(t)) : t \geq 0\}$, where $X(t)$ and $U(t)$ are defined as above, while $D(t)$ is the length of the remaining downhill trip in the cycle in progress at time t. Note that this process also is a regenerative process, but it also is a Markov process. We now can apply Theorem VI.1.2 on p. 170 of Asmussen (2003). It gives the stationary version of the expectation of any function of the regenerative process. Let $f(X(t), U(t), D(t)) = 1$ if $X(t) = 1$ and $U(t) > x$. Then the theorem states that

$$
\begin{aligned}
P(X_e(t) = 1, U_e(t) > x) &= E_e[f(X(t), U(t), D(t))] \\
&= \frac{1}{E[X_1]} E[\int_0^{X_1} f(X(t), U(t), D(t))\, dt] = \frac{E[(U_1 - x)^+]}{E[X_1]} \\
&= \frac{\int_x^\infty P(U_1 > y)\, dy}{E[X_1]} = \left(\frac{E[U_1]}{E[X_1]}\right) F_e^c(x) = \frac{E[U_1] F_e^c(x)}{E[U_1] + E[D_1]}
\end{aligned}
$$

The righthand side of the second line expresses the expected value of an integral over the first cycle, stating at the beginning of a renewal cycle. This justifies the more difficult step above. There is no issue with

$$
P(X_e(t) = 1) = \frac{E[U_1]}{E[U_1] + E[D_1]},
$$

which can be obtained in the same way.

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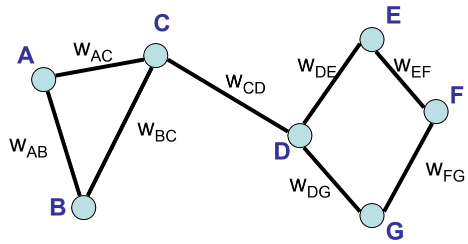
3. Random Walk on a Graph (30 points)

Consider the graph shown in the figure above. There are 7 nodes, labelled with capital letters and 8 arcs connecting some of the nodes. On each arc is a numerical weight specified by the letter w with two subscripts, one for each node the arc connects. Consider a random walk on this graph, where we move randomly from node to node, always going to a neighbor, via a connecting arc. Let each move be to one of the current node's neighbors, with a probability proportional to the weight on the connecting arc. Thus the probability of going from node C to node A in one step is $w_{AC}/(w_{AC} + w_{BC} + w_{CD})$, while the probability of moving from node C to node B in one step is $w_{BC}/(w_{AC} + w_{BC} + w_{CD})$. Let X_n be the node occupied by the random walk on the n^{th} step.

The scoring is 5 points for each of the 6 parts.

Random Walk on a Graph



(a) Prove or disprove: The stochastic process $\{X_n : n \geq 0\}$ is a time reversible irreducible discrete-time Markov chain.

Yes, the stochastic process $\{X_n : n \geq 0\}$ is a time reversible irreducible discrete-time Markov chain. The process is a Markov chain, because the transition probabilities depend only on the present state and not the past. Since the graph is connected, the DTMC is irreducible. The chain can get from any state to any other state in some finite number of moves.

To establish reversibility, it suffices to show that the detailed or local balance conditions hold, i.e.,

$$\pi_i P_{i,j} = \pi_j P_{j,i} \quad \text{for all} \quad i \quad \text{and} \quad j.$$

This is done in Proposition 4.7.1 in the book.

(b) Starting from node A, what is the expected number of steps required to return to node A?

Since the DTMC is irreducible and has a finite state space, there necessarily is a uniques solutuion to the equation $\pi = \pi P$. Let T_A be the number of steps to return to state A for the first time. We have

$$E[T_A] = \frac{1}{\pi_A},$$

where

$$\pi_A = \frac{\sum_j w_{A,j}}{\sum_i \sum_j w_{i,j}}$$

where i and j are understood to run over the nodes, and the value is 0 where no arc exists, as given in Proposition 4.7.1

(c) Prove that your answer in part (b) is correct. (You may quote theorems without proof as part of your proof.)

From parts (a) and (b), we have determined π_A. Since the DTMC is aperiodic, we have

$$P^n_{i,j} \to \pi_j \quad \text{as} \quad n \to \infty.$$

for any i and j, which in turn implies that

$$\frac{1}{n}\left(\sum_{k=1}^{n} P^k_{i,j}\right) \to \pi_j \quad \text{as} \quad n \to \infty.$$

(This limit of averages also holds in periodic chains.) Let $N(n)$ be the number of visits to state A in the first n steps. Since successive visits to state A are renewals, the process $\{N(t) : t \geq 0\}$ is a renewal process. By the elementary renewal theorem

$$\frac{E[N(t)]}{t} \to \frac{1}{E[T_A]} \quad \text{as} \quad t \to \infty.$$

However, letting j denote state A, we have

$$E[N(n)] = \sum_{k=1}^{n} P^n_{i,j}$$

Hence, we have $\pi_j = \pi_A = 1/E[T_A]$, which implies the result in part (b).

(d) Give an expression for the expected number of visits to node G, starting in node A, before going to either node B or node F.

We use the theory of absorbing Markov chains. To do so, we make states B and F absorbing states and let the rest of the states be transient states. If we reorder the states so that the transient states appear first, then we can write the transition matrix in block matrix form as

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix},$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 2×2, R is 5×2 and Q is 5×5). The matrix Q describes the probabilities of motion among the transient states, while the matrix R gives the probabilities of absorption in one step (going from one of the transient states to one of the absorbing states in a single step).

The answer is then $N_{A,G}$, the entry of the matrix N from the row corresponding to transient state A and the column corresponding to transient state G.

(e) Prove that your answer in part (d) is correct. (You may quote theorems without proof as part of your proof.)

As in the lecture notes of October 23, first suppose that we want to calculate the expected number of times the chain spends in transient state j starting in transient state i. Let $T_{i,j}$ be the total number times and let $N_{i,j} \equiv E[T_{i,j}]$ be the expected number of times. It is convenient to write

$$T_{i,j} = T_{i,j}^{(0)} + T_{i,j}^{(1)} + T_{i,j}^{(2)} + T_{i,j}^{(3)} + T_{i,j}^{(4)} + T_{i,j}^{(5)} + \cdots$$

where $T_{i,j}^{(k)}$ is the number of times at the k^{th} transition. Clearly, $T_{i,j}^{(k)}$ is a random variable that is either 1 (if the chain is in transient state j on the k^{th} transition) or 0 (otherwise). By definition, we say that $T_{i,j}^{(0)} = 1$ if $i = j$, but $= 0$ otherwise. Since these random variables assume only the values 0 and 1, we have

$$\begin{aligned} N_{i,j} \equiv E[T_{i,j}] &= E[T_{i,j}^{(0)} + T_{i,j}^{(1)} + T_{i,j}^{(2)} + T_{i,j}^{(3)} + T_{i,j}^{(4)} + T_{i,j}^{(5)} + \cdots] \\ &= E[T_{i,j}^{(0)}] + E[T_{i,j}^{(1)}] + E[T_{i,j}^{(2)}] + E[T_{i,j}^{(3)}] + E[T_{i,j}^{(4)}] + E[T_{i,j}^{(5)}] + \cdots \\ &= P(T_{i,j}^{(0)} = 1) + P(T_{i,j}^{(1)} = 1) + P(T_{i,j}^{(2)} = 1) + P(T_{i,j}^{(3)} = 1) + P(T_{i,j}^{(4)} = 1) + \cdots \\ &= Q_{i,j}^{0} + Q_{i,j}^{1} + Q_{i,j}^{2} + Q_{i,j}^{3} + Q_{i,j}^{4} + Q_{i,j}^{5} + \cdots \end{aligned}$$

To summarize,

$$N_{i,j} \equiv Q_{i,j}^{(0)} + Q_{i,j}^{(1)} + Q_{i,j}^{(2)} + Q_{i,j}^{(3)} + \cdots \ .$$

In matrix form, we have

$$\begin{aligned} N &= Q^{(0)} + Q^{(1)} + Q^{(2)} + Q^{(3)} + \cdots \\ &= I + Q + Q^2 + Q^3 + \cdots \end{aligned}$$

where the identity matrix I here has the same dimension m as Q. (Since Q is the submatrix corresponding to the transient states, $Q^n \to 0$ as $n \to \infty$, where here 0 is understood to be a matrix of zeros.)

Multiplying by $(I - Q)$ on both sides, we get a simple formula, because there is cancellation on the righthand side. In particular, we get

$$(I - Q) * N = I \ ,$$

so that, multiplying on the left by the inverse $(I - Q)^{-1}$, which can be shown to exist, yields

$$N = (I - Q)^{-1}$$

We can be a little more careful and write

$$N_n \equiv I + Q + Q^2 + Q^3 + \cdots + Q^n \ .$$

Then the cancelation yields

$$(I - Q)N_n = I - Q^{n+1} \ .$$

We use the fact that Q is the part of P corresponding to the transient states, so that Q^n converges to a matrix of zeros as $n \to \infty$. (We invoke the proposition that $Q^n \to 0$ as $n \to \infty$.) Hence, $(I - Q)N_n \to I$ as $n \to \infty$. That in turn implies, by a theorem from linear algebra, that

$$(I - Q)N = I \ ,$$

which in turn implies that both N and $I - Q$ are nonsingular, and thus invertible, yielding $N = (I - Q)^{-1}$, as stated above.

(f) Give an expression for the probability of going to B before going to node F, starting in node A.

We use $B_{A,B}$, where B is the matrix $B = NR$ for the matrices R and N above, i.e., $B_{A,B}$ is the entry from the row corresponding to transient state A and the column corresponding to absorbing state B in the 5×2 matrix B.

4. A Renewal Process (52 points)

Let $\{N(t) : t \geq 0\}$ be a renewal process with times between renewals X_n having probability distribution

$$P(X_n = 5) \equiv \frac{1}{3} \equiv 1 - P(X_n = 2).$$

Let $S_n \equiv X_1 + \cdots + X_n$, $n \geq 1$, with $S_0 \equiv 0$ (but there is no renewal at time 0). Let $m(t) \equiv E[N(t)]$ and $Y(t) \equiv S_{N(t)+1} - t$, $t \geq 0$.

The scoring on Problem 4 is 4 points for each correct answer. Thus the maximum score is $13 \times 4 = 52$. Thus the maximum score on the entire test is 112.

(a) What is $m(4)$?

Since

$$m(t) = \sum_{n=1}^{\infty} P(S_n \leq t),$$

$P(S_1 \leq 4) = P(X_1 \leq 4) = P(X_1 = 2) = 2/3$, $P(S_2 \leq 4) = P(X_1 + X_2 \leq 4) = P(X_1 = 2, X_2 = 2) = (2/3)^2$ and $P(S_n \leq 4) = 0$ for all $n \geq 3$,

$$m(4) = \frac{2}{3} + \frac{4}{9} = \frac{10}{9}.$$

(b) Prove or disprove:

$$\lim_{t \to \infty} m(t)/t = \frac{1}{3} \quad \text{as} \quad t \to \infty.$$

(You may quote a theorem without proof as part of your proof.)

From the definition above, $E[X_n] = 3$. By the elementary renewal theorem, Theorem 3.3.4 of Ross, the limit holds, as stated.

(c) Prove or disprove:

$$\lim_{t \to \infty} m(t + a) - m(t) = \frac{a}{3} \quad \text{as} \quad t \to \infty \quad \text{for all} \quad a > 0.$$

(You may quote a theorem without proof as part of your proof.)

The obvious candidate theorem is Blackwell's renewal theorem, but it does not apply in the form above because the distribution is lattice with period 1. Thus, the correct answer here is simply "No." You could observe that a positive statement would hold if we let $t \to \infty$ through an appropriate subsequence. That is, you could prove the lattice version of Blackwell's theorem, but that is not being asked for here. The lattice problem is looked at in more detail in part (e) below.

What is asked for, though, is a proof that the answer is no. So, we do not stop, but give a counterexample. Here is one:In particular,

$$m(t+a) - m(t) = 0 \quad \text{for} \quad a = 0.1 \quad \text{and} \quad t = k + 0.1 \ge 1,$$

where k is an integer, because there are no points in intervals of the form $[k + 0.1, k + 0.2]$ for an integer k. On the other hand, by the lattice version of Blackwell's theorem, Theorem 3.4.1 (ii),

$$m(t+a) - m(t) \to \frac{1}{3} > 0 \quad \text{for} \quad a = 0.2 \quad \text{and} \quad t = k - 0.1 \ge 1,$$

because the left side becomes the expected number of renewals at $kd = d$ where $d = 1$ is the period of the lattice distribution. Thus, there are subsequences of time points $\{t_k : k \ge 1\}$ with $t_k \to \infty$ with different limits. We have exhibited one subsequence with limit 0 and another with limit 1/3. As a consequence, we have *proved* that the limit does not hold. (Note that Blackwell's theorem does not actually state that directly.)

(d) Prove or disprove:

$$P(X_{N(t)+1} > x) \ge P(X_1 > x) \quad \text{for all} \quad t > 0 \quad \text{and} \quad x \ge 0.$$

Here and below, let F be the cdf of X_n. The assertion is correct for *any* cdf F, not just the special one we have. The random variable $X_{N(t)+1}$ is the lifetime at t. This problem was in Section 2.1 of the lecture notes of October 11.

A simple proof is obtained by conditioning upon the last renewal prior to time t. First,

$$P(X_{N(t)+1} > x | S_{N(t)} = t - s) = 1 \quad \text{if} \quad s > x.$$

Second, for $s \le x$, we also condition on $N(t) + 1$ and then uncondition, getting

$$\begin{aligned} P(X_{N(t)+1} > x | S_{N(t)} = t - s, N(t) + 1 = n) &= P(X_n > x | X_n > s) \\ &= \frac{1 - F(x)}{1 - F(s)} \ge 1 - F(x). \end{aligned}$$

Unconditioning on n, we get (still for $s \le x$),

$$P(X_{N(t)+1} > x | S_{N(t)} = t - s) \ge 1 - F(x).$$

Combining the above two results, we have

$$P(X_{N(t)+1} > x | S_{N(t)} = t - s) \ge 1 - F(x)$$

for all x and s. But now unconditioning on $S_{N(t)}$, we get the desired result, as stated above.

In general, there is some issue about the meaning of the conditional probability $P(X_{N(t)+1} > x|S_{N(t)} = t - s)$ because the conditioning event may have probability 0. Note that there is no issue in this problem because $S_{N(t)}$ can take only finitely many values, since each X_n can take only two values. Hence only some values $t - s$ need be considered. More generally, the conditional probability can easily be defined assuming that X_n has a pdf and conditioning upon $N(t) = n$. If we condition upon $N(t) = n$, then $S_{N(t)} = S_n$ has a density, defined by convolution, given the pdf of X_1. So the conditional probability is well defined in the usual elementary way. More generally, we have a regular conditional probability, which is covered in measure-theoretic probability.

(e) Let $u_n \equiv P(N(n + 0.5) - N(n - 0.5) = 1)$, $n \geq 1$, and $u_0 \equiv 1$ (without there being a renewal at 0). Give explicit expressions for u_j, $1 \leq j \leq 9$.

Note that we are now looking at a discrete analog of renewal theory, with the sequence $\{u_n; n \geq 0\}$. From the distribution of X_n, we easily derive the following explicit expressions:

$$u_0 \equiv 1 \quad \text{(by definition)}, \quad u_1 = 0, \quad u_2 = 2/3, \quad u_3 = 0, \quad u_4 = (2/3)^2 = 4/9,$$
$$u_5 = 1/3, \quad u_6 = (2/3)^3 = 8/27, \quad u_7 = (2/3) \times (1/3) + (1/3) \times (2/3) = 4/9,$$
$$u_8 = (2/3)^4 = 16/81, \quad u_9 = 3 \times (2/3)^2 \times (1/3) = 4/9.$$

(f) Does the limit of u_n as $n \to \infty$ exist? Why or why not? If the limit exists, what is its value?

Yes, the limit exists, by virtue of the lattice version of Blackwell's theorem, Theorem 3.4.1 (ii) on p. 110. Here the period is $d = 1$ because that is the greatest common divisor of 2 and 5, the possible values of X_n. Hence,

$$u_n \to \frac{1}{E[X_1]} = \frac{1}{3} \quad \text{as} \quad n \to \infty.$$

(g) Derive the generating function $\hat{u}(z) \equiv \sum_{n=0}^{\infty} u_n z^n$ of the sequence $\{u_n : n \geq 0\}$ in (e).

We derive the generating function just as in the derivation of the Pollaczek-Khintchine transform for the $M/G/1$ queue; see the lecture notes for November 1 and the textbook. We can do that by developing a recursion for u_n for $n \geq 5$. In particular, we have

$$u_n = \frac{2}{3}u_{n-2} + \frac{1}{3}u_{n-5}, \quad n \geq 5.$$

We have already calculated the explicit expression for the initial terms up to u_9 in part (e) above. [As an aside, note that we can compute u_n for any n numerically directly from this recursion, given the initial values from part (e).]

But we want the generating function. We now multiply both sides by z^n and then add over all n such that $n \geq 5$. Then, if we multiply by z^n and add over $n \geq 5$ in the recursion, then we get on the left

$$\sum_{n=5}^{\infty} u_n z^n = \hat{u}(z) - \sum_{n=0}^{4} u_n z^n = \hat{u}(z) - u_0 z^0 - u_2 z^2 - u_4 z^4 = \hat{u}(z) - 1 - \frac{2z^2}{3} - \frac{4z^4}{9}.$$

Reasoning similarly, the righthand side becomes

$$\begin{aligned}\sum_{n=5}^{\infty} \left(\frac{2}{3}u_{n-2} + \frac{1}{3}u_{n-5}\right) z^n &= \frac{2z^2}{3}[\hat{u}(z) - \sum_{n=0}^{2} u_n z^n] + \frac{z^5}{3}[\hat{u}(z)] \\ &= \frac{2z^2}{3}[\hat{u}(z) - u_0 z^0 - u_2 z^2] + \frac{z^5}{3}[\hat{u}(z)].\end{aligned}$$

Hence, we have

$$\hat{u}(z) - u_0 z^0 - u_2 z^2 - u_4 z^4 = \frac{2z^2}{3}[\hat{u}(z) - u_0 z^0 - u_2 z^2] + \frac{z^5}{3}[\hat{u}(z)]$$

or, inserting the known u_k and recalling that $z^0 = 1$, we have

$$\hat{u}(z) - 1 - \frac{2}{3}z^2 - \frac{4}{9}z^4 = \frac{2z^2}{3}[\hat{u}(z) - 1 - \frac{2}{3}z^2] + \frac{z^5}{3}[\hat{u}(z)].$$

Now, solving for $\hat{u}(z)$, we get

$$\hat{u}(z) = \frac{1}{1 - \frac{2z^2}{3} - \frac{z^5}{3}}.$$

(h) How could you use the generating function $\hat{u}(z)$ in part (g) to compute u_n for any n?

We now can calculate u_n for any $n \geq 0$ by numerically inverting the generating function. Let $\mathcal{G}^{-1}$ be the operator mapping the generating function into its inverse. Then we have

$$u_n = \mathcal{G}^{-1}(\hat{u}(z))$$

for $\hat{u}(z)$ given explicitly above. We thus have an explicit formula for u_n:

$$u_n = \mathcal{G}^{-1}\left(\frac{1}{1 - \frac{2z^2}{3} - \frac{z^5}{3}}\right).$$

We can compute the inverse numerically by applying a numerical inversion algorithm for generating functions An algorithm based on the Fourier-series method is similar to the algorithm for Laplace transforms. See J. Abate and WW, "Numerical Inversion of Probability Generating Functions," *Operations Research Letters*, vol. 12, No. 4, 1992, pp. 245-251. But, as noted above, we could numerically solve the recursion in part (g) too.

Let $\hat{u}^{(k)}(z_0)$ be the k^{th} derivative of $\hat{u}(z)$ with respect to the variable z, evaluated at $z = z_0$. Note that $\hat{u}(0) = u_0$ and

$$\hat{u}^{(k)}(0) = k!u_k, \quad k \geq 1.$$

Hence any numerical algorithm for computing derivatives could be used to calculate u_n.

(i) Give an expression for $P(Y(t) > 4)$ in terms of u_n, where $Y(t)$ is the excess, defined at the outset.

It is easily seen that we can have $Y(t) > 4$ if and only if there is a point at $\lfloor t \rfloor$, or if $t < 1$, where $\lfloor t \rfloor$ is the "floor" function, giving the greatest integer less than or equal to t, *and* the next interval is of length 5. Thus,

$$P(Y(t) > 4) = \frac{u_{\lfloor t \rfloor}}{3},$$

where u_n is defined as in part (e) above.

(j) Set up a renewal equation for $P(Y(t) > 4)$, solve it, and relate your answer to part (i).

The excess has special structure because the X_n has a two-point distribution. Note that $P(Y(t) > x) = 0$ for all $x \geq 5$.

Let F be the cdf of X_n. As in §3 of the lecture notes for October 11, we derive the renewal equation for $P(Y(t) > x)$. Let F be the cdf of X_n. Then

$$\begin{aligned} P_4(t) \equiv P(Y(t) > 4) &= P(Y(t) > 4, X_1 > t) + P(Y(t) > 4, X_1 \leq t) \\ &= P(X_1 > t + 4) + \int_0^t P_4(t - s)\, dF(s) \end{aligned}$$

But note that $P(X_1 > t + 4) = 0$ unless $t + 4 < 5$, i.e., unless $0 \leq t < 1$.

So far, we have an equation with the desired $P_4(t)$ on both sides, so this is not the desired end result. We now solve the renewal equation, getting

$$\begin{aligned} P_4(t) &\equiv P(Y(t) > 4) = P(X_1 > t + 4) + \int_0^t P(X_1 > t + 4 - s)\, dm(s) \\ &= F^c(t + 4) + \int_0^t F^c(t + 4 - s)\, dm(s), \end{aligned}$$

where $F^c(x) \equiv 1 - F(x)$,

$$m(t) = \sum_{n=1}^{\infty} P(S_n \leq t) \quad \text{and} \quad P(S_n \leq t) = \int_0^t P(S_{n-1} \leq t - s)\, dF(s), \quad n \geq 2,$$

with $P(S_1 \leq x) = P(X_1 \leq x) = F(x)$. Thus, we have given an expression for $m(t)$ in terms of F and then an expression for $P_4(t) \equiv P(Y(t) > 4)$ in terms of F^c and $m(t)$ (and thus F). This is valid for any F, and thus for our specific F.

But now we look at what we have more closely. Observe that, for our specific F, $F^c(t+4) = 0$ if $t \geq 1$ and $F^c(t + 4) = 1/3$ for $0 \leq t < 1$. Moreover, since X_n is integer-valued, we have

$$m(t) = \sum_{k=1}^{\lfloor t \rfloor} u_k$$

for the u_k in part (e). Hence, from the solution of the renewal equation, we get an alternative derivation of the formula in part (i):

$$P(Y(t) > 4) = \frac{u_{\lfloor t \rfloor}}{3},$$

where u_n is defined as in part (e) above.

Remark. You could observe that

$$P(Y(t) > 4) = P(Y(\lfloor t \rfloor) > 4) \quad \text{for all} \quad t > 0,$$

because renewals can only occur at integer times. You could then observe that $p_j \equiv P(Y(j) > 4)$ satisfies the same recursion as u_j in part (g) above. By looking ath the initial values, you then see that

$$p_j = u_j/3, \quad j \geq 0.$$

(k) Let $N_G(t)$ be a new counting process obtained by letting X_1 be distributed according to the cdf G while the other random variables X_n for $n \geq 2$ remain unchanged. Let $m_G(t) \equiv E[N_G(t)]$. Exhibit all cdf's G such that $m_G(t) = t/3$, $t \geq 0$.

This part and the final two below concern the equilibrium renewal process, as treated in §3.5 of Ross, see especially Theorem 3.5.2; also see the lecture notes of October 18. There is one and only one such cdf G, namely $G = F_e$, where

$$\begin{aligned} F_e(x) &\equiv \frac{1}{E[X]} \int_0^x P(X_1 > s)\, ds \\ &= (x/3)1_{[0,2]}(x) + [(2/3) + (1/9)(x-2)]1_{[2,5]}(x) + 1_{[5,\infty)}(x), \end{aligned}$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. That is, F_e has density $f_e \equiv F^c(x)/E[X_1]$, where $f_e(t) = 1/3$, $0 \leq t \leq 2$, and $f_e(t) = 1/9$, $2 \leq t \leq 5$.

To see that there is only one such cdf G (in general), note that the Laplace transform $\hat{m}_G(s) \equiv \int_0^\infty e^{-st} m_G(t)\, dt$ can be represented as

$$\hat{m}_G(s) = \frac{\hat{g}(s)}{s(1 - \hat{f}(s))},$$

which equals $1/E[X]s^2$, the Laplace transform of $t/E[X]$, if and only if $\hat{g}(s) = \hat{f}_e(s) = (1 - \hat{f}(s))/sE[X]$.

(l) Prove or disprove: There exists a cdf G (with $G(t) \to 1$ as $t \to \infty$) such that the stochastic process $\{N_G(t); t \geq 0\}$ has stationary increments.

If the stochastic process $\{N_G(t) : t \geq 0\}$ has stationary increments, then $m_G(t) = ct$. Thus, necessarily it must be the stationary renewal process with $G = F_e$, by part (k), because we have $m_G(t)/t \to 1/E[X_1]$ by the generalization of the elementary renewal theory, so the constant c must be $c = 1/E[X_1]$. Hence, there is at most one cdf G that will work: F_e.

It remains to show that the delayed renewal process with initial cdf $G = F_e$ has stationary increments. As in Ross, we observe that the increment $N_G(t+u) - N_G(t)$ may be interpreted as the number of renewals in an interval of length u starting with the time until the first renewal of $Y_G(t)$. Thus we will show that, when $G = F_e$, that distribution is independent of the time t. Hence, the stochastic process $\{N_e(t) : t \geq 0\}$ has stationary increments.

Thus, we show that the distribution of $Y_G(t)$ is independent of t. From part (e), we have

$$P_x(t) \equiv P(Y(t) > x) = F^c(t+x) + \int_0^t F^c(t+x-s)\, dm(s).$$

for the ordinary renewal process. If we instead consider the delayed renewal process, where X_1 has cdf G, then we obtain instead

$$P_x(t) \equiv P(Y_G(t) > x) = G^c(t+x) + \int_0^t F^c(t+x-s)\, dm(s).$$

If we then let $G = F_e$, we have $m(t) = t/E[X_1]$ and

$$\begin{aligned} P_x(t) &\equiv P(Y_{F_e}(t) > x) = F_e^c(t+x) + \frac{1}{E[X_1]} \int_0^t F^c(t+x-s)\, ds \\ &= F_e^c(t+x) + F_e^c(x) - F_e^c(t+x) = F_e^c(x), \end{aligned}$$

as shown on p. 132 of Ross. In particular, note that $P(Y_{F_e}(t) > x)$ is independent of t. Hence, the stochastic process $\{N_e(t) : t \geq 0\}$ has stationary increments.

(m) Prove or disprove: There exists a cdf G (with $G(t) \to 1$ as $t \to \infty$) such that the stochastic process $\{N_G(t); t \geq 0\}$ has stationary and independent increments.

No such cdf G exists. To prove it, we could first note by parts (k) and (l), that there is only one cdf G, namely, F_e, such that the process $\{N_G(t); t \geq 0\}$ has stationary increments. We consider that cdf $G = F_e$ that yields stationary increments. Then, for any $t \geq 0$, $1 > P(N_G(t+0.3) - N_G(t) > 0) > 0$, independent of t. Then observe that

$$P(N_G(t+0.6) - N_G(t+0.3) > 0 | N_G(t+0.3) - N_G(t) > 0) = 0$$

while

$$P(N_G(t+0.6) - N_G(t+0.3) = 0) = P(N_G(t+0.3) - N_G(t) = 0) > 0.$$

Hence the increments over the intervals $(t, t+0.3]$ and $(t+0.3, t+0.6]$ cannot be independent.

An **alternative proof** could be based on the theorem that the only counting process with both unit jumps and stationary and independent increments is the Poisson process, and the present process is *not* a Poisson process. That theorem is the definition given in the Wikipedia entry for a Poisson process. But we did not cover that theorem. It is not one of the definitions in Chapter 2 of Ross. A counterexample is the best way to disprove.
