IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3-4, November 17, 2013

Justify your answers; show your work.

1. Random Movement on a Chessboard (25 **points**)

The king (a chess piece) is placed on one corner square of an empty $8 \times 8 = 64$-square chessboard. The king then makes a sequence of random moves from square to square, making each of its legal moves with equal probability on each move, independent of how it reached its current square. In each move, the king is allowed to move one square in any direction, including diagonally, as long as it stays on the board. Thus, the king has three possible moves from its initial corner square, but 8 possible moves from each interior square, away from any side of the board.

(a) Does the probability that the king is in its initial square after n moves converge to a limit as $n \to \infty$? If so, what is that limit?

(b) What is the expected number of moves until the king first returns to its initial square?

(c) Justify your answers in parts (a) and (b). (Style points for careful complete answers, including supporting details and proofs.)

(d) Give an expression (carefully identifying all components) for the probability that the king visits the opposite corner square (the corner square that is on a different row and in a different column) before it visits the other corner square on the same row as its initial square?

(e) Justify your answer in part (d).

2. Finite-State Markov Chains (25 **points**)

Consider an m-state Markov chain for $m < \infty$ with transition probabilities $P_{i,j}$ that are strictly positive for all i and j. Consider the following three statements:

(i) There are positive numbers x_i such that $\sum_i^m x_i P_{i,j} = x_j$ for all j.

(ii) There are positive numbers x_i such that $x_i P_{i,j} = x_j P_{j,i}$ for all i and j.

(iii) For all triples of states (i, j, k), $P_{i,j} P_{j,k} P_{k,i} = P_{i,k} P_{k,j} P_{j,i}$.

Indicate whether or not each of the following claims is valid. Then support your answer with a proof, quoting established theorems where appropriate. Finally, prove all quoted theorems used to answer (c) and (d).

(a) Statement (i) implies statement (ii).

(b) Statement (ii) implies statement (i).

(c) Statement (ii) implies statement (iii).

(d) Statement (iii) implies statement (ii).

(e) Statement (i) is always valid.

(f) Statement (ii) is always valid.

(g) If Statement (i) holds for vectors $x \equiv (x_1, \ldots, x_m)$ and $y \equiv (y_1, \ldots, y_m)$, then necessarily $y = cx$ for some constant $c > 0$.

3. Automobile Replacement (25 points)

Mr. Brown has a policy that he buys a new car as soon as his old one breaks down or reaches the age of 6 years, whichever occurs first. Suppose that the successive lifetimes (time until they breakdown) of the cars he buys can be regarded as independent and identically distributed random variables, each uniformly distributed on the interval $[0, 10]$ years. Suppose that each new car costs $\$20,000$. Suppose that Mr. Brown incurs an additional random cost each time the car breaks down. Suppose that this additional breakdown cost is exponentially distributed with mean $\$4,000$. Suppose that he can trade his car in after it is 6 years old if it does not break down, and only if it does not break down, and receive a random dollar value uniformly distributed in the interval $[1000, 3000]$.

(a) What is the long-run average cost per year of Mr. Brown's car-buying strategy?

(b) What is the long-run average age of the car currently is use?

(c) Suppose that the car buying policy started at time 0 with a purchase of a new car. Give an explicit expression for the distribution of the remaining time he will use the current car at time t via its Laplace transform.

(d) Prove that the remaining time he will use the car currently in use at time t converges in distribution as $t \to \infty$ and identify the limiting distribution.

4. I.I.D. Uniform Random Variables (25 points)

Let U_n, $n \geq 1$, be independent and identically distributed (i.i.d.) random variables, each uniformly distributed on the interval $[0, 2]$. Let

$$\begin{aligned} g(n,x) &\equiv P(U_1 + \cdots + U_n \leq x) \quad \text{for} \quad x \geq 0, \quad n \geq 1 \quad \text{and} \\ g(x) &\equiv \sum_{n=1}^{\infty} g(n,x) \quad \text{for} \quad x \geq 0. \end{aligned}$$

Indicate whether or not each of the following statements is valid. Then support your answer with a proof, quoting established theorems where appropriate. Finally, *either* prove all quoted theorems that you used to answer (ii) *or* prove all quoted theorems that you used to answer (iii).

(i) $g(x) < \infty$ for all x, $0 < x < \infty$.

(ii) $g(x)/x \to 1$ as $x \to \infty$.

(iii) $g(x+1) - g(x) \to 1$ as $x \to \infty$.

(iv) $2g(x) = x \wedge 2 + \int_0^{x \wedge 2} g(x-y)dy$ for $x \geq 0$, where $a \wedge b \equiv \min\{a, b\}$.

(v) $\frac{g(x)-x}{\sqrt{x}} \Rightarrow N(0, \sigma^2)$ as $x \to \infty$, for some $\sigma^2 > 0$ where $N(a,b)$ denotes a Gaussian random variable with mean a and variance b and $\Rightarrow$ denotes convergence in distribution.

(vi) $g(x) - x \Rightarrow Y$ as $x \to \infty$, for some positive random variable Y with $E[Y^2] < \infty$.