IEOR 6711: Stochastic Models I

SOLUTIONS to the Second Midterm Exam, November 17, 2013

1. Random Movement on a Chessboard (25 points)

The king (a chess piece) is placed on one corner square of an empty $8 \times 8 = 64$-square chessboard. The king then makes a sequence of random moves from square to square, making each of its legal moves with equal probability on each move, independent of how it reached its current square. In each move, the king is allowed to move one square in any direction, including diagonally, as long as it stays on the board. Thus, the king has three possible moves from its initial corner square, but 8 possible moves from each interior square, away from any side of the board.

(a) Does the probability that the king is in its initial square after n moves converge to a limit as $n \to \infty$? If so, what is that limit?

Note in this case the chain is aperiodic. Hence the answer is yes. We use the fact that this is a special case of a random walk on the vertices of a graph with weights on its arcs. The nodes are the squares. There is an arc between two nodes if the king can move from one square to the other. All weights are 1. The probability of moving from one square to another then is the weight on the arc divided by the sum of the weights out of the originating square. In this case, the steady-state probability is

$$\pi_j = \frac{\sum_k w_{i,k}}{\sum_j \sum_k w_{j,k}} = \frac{3}{(4 \times 3) + (24 \times 5) + 36 \times 8)} = \frac{3}{420} = \frac{1}{140}.$$

as shown in Proposition 4.71.

(b) What is the expected number of moves until the king first returns to its initial square?

Let $T_{1,1}$ be the first time the king to returns to the initial square. Then

$$E[T_{1,1}] = \frac{1}{\pi_1} = 140.$$

(c) Justify your answer in parts (a) and (b). (Style points for careful complete answers, including supporting details and proofs.)

See lecture notes on reversibility for November 7 and Section 4.7 of the textbook. Use renewal theory to justify (b), recalling that successive visits to the initial square are renewals.

(d) Give an expression (carefully identifying all components) for the probability that the king visits the opposite corner square (the corner square that is on a different row and in a different column) before it visits the other corner square on the same row as its initial square?

Here we have an issue in absorbing Markov chain theory, from Section 2 of the lecture notes of October 24. We put the 64×64 transition matrix in canonical form. in block matrix form, we have

$$P = \begin{pmatrix} I & 0 \\ R & Q \end{pmatrix} ,$$

where I is an identity matrix (1's on the diagonal and 0's elsewhere) and 0 (zero) is a matrix of zeros. In this case, I would be 2×2, R is 62×2 and Q is 62×62). The two possible destination corner squares are the two absorbing states. Let the opposite corner be labeled square 1 and the other square on the same row as the initial square be labeled square 2. The matrix Q gives the transition probabilities among the transient states, which includes the initial corner square. Let the initial corner square be labeled as square 64. Then what we want to calculate is

$$B_{64,1},$$

the entry on the 64^{th} row and in the first column of the 62×2 matrix B. (I label the rows of R, Q and B with the integers $3, 4, \ldots, 64$. I label the columns of Q the same way. I label the columns of R and B by 1 and 2. I label the columns of Q the same ways as the rows. The matrix B is computed by

$$B = NR$$

where

$$N = (I - Q)^{-1}.$$

(e) Justify your answer in part (d).

See Section 2 of the lecture notes of October 24.

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2. Finite-State Markov Chains (25 points)

Consider an m-state Markov chain for $m < \infty$ with transition probabilities $P_{i,j}$ that are strictly positive for all i and j. Consider the following three statements:

(i) There are positive numbers x_i such that $\sum_i^m x_i P_{i,j} = x_j$ for all j.

(ii) There are positive numbers x_i such that $x_i P_{i,j} = x_j P_{j,i}$ for all i and j.

(iii) For all triples of states (i, j, k), $P_{i,j} P_{j,k} P_{k,i} = P_{i,k} P_{k,j} P_{j,i}$.

Indicate whether or not each of the following claims is valid. Then support your answer with a proof, quoting established theorems where appropriate. Finally, prove all quoted theorems used to answer (c) and (d).

(a) Statement (i) implies statement (ii).

Of course, (i) is just the stationarity equation $\pi = \pi P$ (assuming that the numbers x_i are normalized to sum to one, while (ii) is the definition of reversibility. False. Give a counterexample. Here is one:

$$P = \begin{pmatrix} 0.05 & 0.90 & 0.05 \\ 0.05 & 0.05 & 0.90 \\ 0.90 & 0.05 & 0.05 \end{pmatrix},$$

Since the transition matrix is doubly stochastic, $\pi = (1/3, 1/3, 1/3)$, as can be checked, but (ii) does not hold, as can be checked.

(b) Statement (ii) implies statement (i).

This is correct. Just sum on either i or j to show it.

(c) Statement (ii) implies statement (iii).

This is correct, by the Kolmogorov cycle theorem, Theorem 4.72. The proof is given on the top of p. 209. But it does only cycles of length 3.

Given reversibility, we can write

$$P_{i,j} = \frac{P_{j,i}\pi_j}{\pi_i}$$

Hence given any cycle, such as $1 \to 2 \to 3 \to 4 \to 5 \to 1$, we can write

$$\begin{aligned} P_{1,2}P_{2,3}P_{3,4}P_{4,5}P_{5,1} &= \left(\frac{P_{2,1}\pi_2}{\pi_1}\right)\left(\frac{P_{3,2}\pi_3}{\pi_2}\right)\cdots\left(\frac{P_{1,5}\pi_1}{\pi_5}\right) \\ &= P_{2,1}P_{3,2}P_{4,3}P_{5,4}P_{1,5} = P_{1,5}P_{5,4}P_{4,3}P_{3,2}P_{2,1} \end{aligned}$$

because the π_j terms cancel in the first line on the right. More generally, the proof can be done by induction.

(d) Statement (iii) implies statement (ii).

This is also correct. This is a combination of the Kolmogorov cycle theorem, Theorem 4.72, and an additional argument showing that it suffices to have the cycle relation for cycles of length 3 if the elements of the matrix P are all positive, see Exercise 4.45.

Here is a direct argument for (iii) implies (ii): Let

$$\pi_j = \frac{CP_{1,j}}{P_{j,1}},$$

where 1 is a fixed state and C is chosen so that $\sum_j \pi_j = 1$. Since all the probabilities are positive, we can do this. Then observe that

$$\pi_j P_{j,k} = \frac{CP_{i,j}P_{j,k}}{P_{j,i}} = \frac{CP_{i,k}P_{k,j}}{P_{k,i}} = \pi_k P_{k,j}$$

which is (ii).

(e) Statement (i) is always valid.

This claim is correct. A proof could be by Theorem 4.3.3, for which we must have (ii) there. A proof could also be by the contraction fixed point theorem in the lecture of October 29.

(f) Statement (ii) is always valid.

That claim is not correct. The counterexample in part (a) applies.

(g) If Statement (i) holds for vectors $x \equiv (x_1, \ldots, x_m)$ and $y \equiv (y_1, \ldots, y_m)$, then necessarily $y = cx$ for some constant $c > 0$.

The claim is correct. The proofs of existence in part (e) imply uniqueness among probability vectors. But scalar multiples also satisfy the same relation.

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3. Automobile Replacement (25 points)

Mr. Brown has a policy that he buys a new car as soon as his old one breaks down or reaches the age of 6 years, whichever occurs first. Suppose that the successive lifetimes (time until they breakdown) of the cars he buys can be regarded as independent and identically distributed random variables, each uniformly distributed on the interval $[0, 10]$ years. Suppose that each new car costs \$20,000. Suppose that Mr. Brown incurs an additional random cost each time the car breaks down. Suppose that this additional breakdown cost is exponentially distributed with mean \$4,000. Suppose that he can trade his car in after it is 6 years old if it does not break down, and only if it does not break down, and receive a random dollar value uniformly distributed in the interval $[1000, 3000]$.

(a) What is the long-run average cost per year of Mr. Brown's car-buying strategy?

We apply renewal reward theory:

$$\text{long run average cost} \quad = \frac{\text{average cost per cycle}}{\text{average length of cycle}}$$

In particular we apply the SLLN for renewal reward processes, Theorem 3.6.1 (i). Here it is important to justify the simple method.

The times that Mr. Brown has each car constitute the cycles. The instants that Mr. Brown gets a new car are the renewal epochs. Let T be the time Mr. Brown has each car. Calculating the mean $E[T]$ is somewhat complicated because the distribution has a (uniform) density over the interval $[0, 6]$ plus a discrete mass at the point 6. We can write

$$E[T] = \int_0^6 \frac{x dx}{10} + 6P(L > 6) = \frac{36}{20} + 6(0.4) = 1.8 + 2.4 = 4.2 \,\text{years}$$

Next, we need the expected cost per car. For that, we only need the mean turn-in value, which is of course \$2000. Let $\bar{C}$ be the long-run average cost. (The cost can be treated just like the reward. Directly, we can regard the cost as negative reward.) Then

$$\bar{C} = \frac{20,000 + (0.6)4,000 - (0.4)2,000}{E[T]} = \frac{20,000 + 2,400 - 800}{4.2} = \frac{21,600}{4.2} \approx 5143$$

The average cost per year is about \$5143.

(b) What is the long-run average age of the car currently is use?

We can again use renewal reward theory, just as above. The same theory applies. This is worked out in detail in Example 3.6(B). We will need the second moment of T. We can write

$$E[T^2] = \int_0^6 \frac{x^2 dx}{10} + 6^2 P(L > 6) = \frac{216}{30} + 36(0.4) = 7.2 + 14.4 = 21.6$$

Let $A(t)$ be the age of the car currently in use at time t. In the long-run, the age of the car currently in use at time t, $A(t)$, is distributed according to T_e a random variable with the stationary-excess cdf F_e associated with the cdf F of T. The mean of this equilibrium age has the formula

$$\lim_{t\to\infty} \frac{\int_0^t A(t)\, dt}{t} = E[T_e] = \frac{E[T^2]}{2E[T]} = \frac{21.6}{2(4.2)} = \frac{21.6}{8.4} = 2.57 \quad \text{years.}$$

(c) Suppose that the car buying policy started at time 0 with a purchase of a new car. Give an explicit expression for the distribution of the remaining time he will use the current car at time t via its Laplace transform.

The question asks for the distribution of the residual life $Y(t)$. Its cdf $P(Y(t) \le x)$ satisfies a renewal equation; see §3.1 of the lecture notes on October 15, where $h(t) = F(t+x) - F(t)$. By the lecture notes of October 10, we have the general renewal equation

$$g(t) = h(t) + \int_0^t g(t-y)\, dF(y)$$

with solution

$$g(t) = h(t) + \int_0^t h(t-y)\, dm(y)$$

and Laplace transform

$$\hat{g}(s) = \frac{\hat{h}(s)}{1 - \hat{f}(s)},$$

so it suffices to exhibit the two Laplace transforms that occur in this case

$$\hat{f}(s) = \frac{1 - e^{-6s}}{10s} + \frac{4e^{-6s}}{10}$$

and

$$\hat{h}(s) = \int_0^\infty e^{-st}(F(t+x) - F(t)\,dt = \frac{(e^{sx}-1)\hat{f}(s)}{s}$$

for $\hat{f}(s)$ above.

(d) Prove that the remaining time he will use the car currently in use at time t converges in distribution as $t \to \infty$ and identify the limiting distribution.

Apply the key renewal theorem to get convergence in distribution $Y(t) \Rightarrow T_e$, where

$$P(T_e \le x) = \frac{1}{ET}\int_0^x P(T > y)\,dy$$

It is important to show that the conditions are satisfied. First the mean time between renewals is finite, $E[T] < \infty$. Second the distribution of T is non-lattice, because it has the component that is uniformly distributed on $[0,6]$; the atom at 6 does not cause any problems. Finally, it is important to note that the function h here is directly Riemann integrable (d.R.i.). That last step is somewhat complicated, because h here is not decreasing. If $x = 2$, then there is a jump up in $h(t)$ at $t = 4$. However, we can use condition (iv) in Proposition V.4.1 on p. 154 of Asmussen (2003). We can observe that h is bounded and continuous a.e. with respect to Lebesgue measure. (For $0 < x < 6$, it has only two discontinuity points, when $t + x = 6$ and when $t = 6$.) And we see that $h(t) \le 1 - F(t)$, where $1 - F(t)$ is d.R.i. because it is nondecreasing and integrable. Its integral is $E[T]$ by the tail integral formula..

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4. I.I.D. Uniform Random Variables (25 points)

Let U_n, $n \ge 1$, be independent and identically distributed (i.i.d.) random variables, each uniformly distributed on the interval $[0,2]$. Let

$$\begin{aligned} g(n,x) &\equiv P(U_1 + \cdots + U_n \le x) \quad \text{for} \quad x \ge 0, \quad n \ge 1 \quad \text{and} \\ g(x) &\equiv \sum_{n=1}^{\infty} g(n,x) \quad \text{for} \quad x \ge 0. \end{aligned}$$

Indicate whether or not each of the following statements is valid. Then support your answer with a proof, quoting established theorems where appropriate. Finally, *either* prove all quoted theorems that you used to answer (ii) *or* prove all quoted theorems that you used to answer (iii).

The critical observation is that $g(x) = m(x)$, the renewal function, which follows from Proposition 3.2.1.

(i) $g(x) < \infty$ for all x, $0 < x < \infty$.

This is Proposition 3.2.2.

(ii) $g(x)/x \to 1 \quad \text{as} \quad x \to \infty$.

This is the elementary renewal theorem, Theorem 3.3.4. I am expecting that you will give a detailed proof of this theorem as the final part.

(iii) $g(x+1) - g(x) \to 1 \quad \text{as} \quad x \to \infty$.

This is Blackwell's theorem, Theorem 3.4.1.

(iv) $2g(x) = x \wedge 2 + \int_0^{x \wedge 2} g(x-y)dy$ for $x \ge 0$, where $a \wedge b \equiv \min\{a, b\}$.

This is the renewal equation for $g(x) \equiv m(x)$, written in general as

$$m(t) = 0 + \int_0^t [(1 + m(t-y)]\, dF(y) = F(t) + \int_0^t m(t-y)] f(y)\, dy$$

(See the third display on page 2 of the lecture notes for October 10.)

(v) $\frac{g(x)-x}{\sqrt{x}} \Rightarrow N(0, \sigma^2) \quad \text{as} \quad x \to \infty$, for some $\sigma^2 > 0$ where $N(a, b)$ denotes a Gaussian random variable with mean a and variance b and $\Rightarrow$ denotes convergence in distribution.

This statement is invalid. A deterministic function of x cannot converge in distribution to a nondegenerate stochastic limit (with positive variance) as $x \to \infty$. The actual limit is 0. That follows as an easy consequence of the next part, by dividing the left side by $\sqrt{x}$.

(vi) $g(x) - x \Rightarrow Y \quad \text{as} \quad x \to \infty$, for some positive random variable Y with $E[Y^2] < \infty$.

This statement IS correct, because we have convergence everywhere, which implies convergence with probability 1, which in turn implies convergence in distribution. The limiting random variable is deterministic. Thus $Y = y$ and $E[Y^2] = y^2$ for some constant y. In particular, we have

$$g(x) - x \to \frac{E[U[0,2]^2]}{2} - 1 = \frac{4/3}{2} - 1 = -\frac{1}{3} \quad \text{as} \quad x \to \infty$$

as a consequence of Corollary 3.4.7 on p. 121. But the full argument uses the key renewal theorem; see §3.4.2 for the details.
