IEOR 6711: Stochastic Models I

Second Midterm Exam, Chapters 3 and 4, November 20, 2007

Justify your answers; show your work.

1. Selling Flour: The Grainery. (30 **points)**

The Grainery is a store that sells flour to bakers. Suppose that bakers come to the Grainery according to a Poisson process with a rate of λ per week. Suppose that each baker asks to buy a random amount of flour, which is distributed according to a random variable X having cumulative distribution function (cdf) G with density g and a mean of m pounds. Let the successive quantities requested by the bakers be independent and identically distributed, independent of the arrival process.

The store uses an (s, S) inventory policy: Whenever the store's inventory level of flour drops below s pounds, the store immediately places an order from a centralized warehouse to bring its inventory level up to S pounds, where $S > s$. Assume that the delivery time from the warehouse to the Grainery is negligible; i.e., assume that these orders are delivered to the Grainery immediately after the last baker whose request takes the inventory level below s leaves. (The baker whose request takes the inventory level below s cannot receive any of the new order. That baker receives only part of his request if his request exceeds the current supply. This is the case of "lost sales;" i.e., there is no commitment to fill the rest of the order later. After that baker leaves, the new inventory is S.)

(a) Give an expression for the long-run rate that the Grainery places orders to replace its stock of flour. Indicate how the numerical value can be obtained.

(b) Let $X(t)$ be the inventory of flour at the Grainery at time t. Explain why the limit of $P(X(t) \geq x)$ as $t \to \infty$ exists and give an expression for that limit.

(c) How does the answer in part (b) simplify when the cdf G is exponential?

(d) Is the cumulative distribution function of the limiting inventory level in part (c) continuous? Explain.

(e) How does the answer (b) simplify when the mean m becomes relatively small compared to $S - s$? (To make this precise, suppose that the mean-m random variable $X \equiv X(m)$ introduced above is constructed from a mean-1 random variable Z by letting $X(m) = mZ$. Then let $m \downarrow 0$.)

2. Random Clockwise Walk Around the Circle. (30 **points)**

Consider a random walk on the circle, where each step is a clockwise random motion. At each step, the angle is at one of the values $k\pi/2$, $0 \leq k \leq 3$. That is, there is a sequence of independent clockwise motions on the circle among the four angles $k\pi/2$, $0 \leq k \leq 3$. Let transitions take place at positive integer times. In each step, the walk moves in a clockwise motion $j\pi/2$ with probability $(j+1)/10$ for $0 \leq j \leq 3$. Henceforth, let state k represent $k\pi/2$, then we have the following transition matrix for transitions among the four states 0, 1, 2 and 3:

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 & 0.1 \end{pmatrix} .$$

(a) Show that there exists a distance (metric) d on the space of probability vectors of length 4 and a constant c with $0 < c < 1$ such that, for any two probability vectors $u \equiv (u_1, u_2, u_3, u_4)$ and $v \equiv (v_1, v_2, v_3, v_4)$,

$$d(uP, vP) \leq cd(u, v) \ .$$

(b) Find the smallest such constant c in part (a), such that the inequality is valid for all u and v, and prove that it is smallest.

(c) Use part (a) to show that there exists a unique stationary probability vector for P, i.e., a probability vector $\pi \equiv (\pi_1, \pi_2, \pi_3, \pi_4)$ satisfying

$$\pi = \pi P.$$

(d) Use part (a) to show that

$$d(uP^n, \pi) \leq Kc^n, \quad n \geq 1,$$

where c and K are constants, independent of u, and identify the best possible constant K (the smallest constant that is valid for all u).

(e) Find the stationary probability vector π.

3. Customers at an ATM. (40 **points)**

Suppose that customers arrive at a single automatic teller machine (ATM) according to a Poisson process with rate λ per minute. Customers use the ATM one at a time. There is unlimited waiting space. Assume that all potential customers join the queue and wait their turn. (There is no customer abandonment.)

Let the successive service times at the ATM be mutually independent, and independent of the arrival process, with a cumulative distribution function G having density g and mean $1/\mu$. Let $Q(t)$ be the number of customers at the ATM at time t, including the one in service, if any.

(a) Prove that $Q(t) \to \infty$ as $t \to \infty$ with probability 1 if $\rho \equiv \lambda/\mu > 1$.

(b) Is the stochastic process $\{Q(t) : t \geq 0\}$ a Markov process? Explain.

(c) Identify random times T_n, $n \geq 1$, such that the stochastic process $\{X_n : n \geq 1\}$ is an irreducible infinite-state discrete-time Markov chain (DTMC), when $X_n = Q(T_n)$ for $n \geq 1$.

(d) Find conditions under which the state X_n at time n of the DTMC $\{X_n : n \geq 1\}$ in part (c) converges in distribution to a proper steady-state limit as $n \to \infty$, and determine that limiting steady-state distribution.

(e) In the setting of part (d), what is the steady-state probability that the system is empty (at these embedded random times)?

(f) How does the steady-state distribution determined in part (d) simplify when the service-time cdf is

$$G(x) = 1 - e^{-\mu x}, \quad x \geq 0 \ ?$$

(g) Find

$$\lim_{n \to \infty} P(X_n = j)$$

in the setting of part (f)

(h) What is the heavy-traffic approximation for the steady-state distribution found in part (d)?

(i) State and prove a limit theorem justifying the approximation in part (h).