Queues with Path-Dependent Arrival Processes

IEOR 8100, PhD Seminar on Queueing Theory, Ward Whitt

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OUTLINE

1. The generalized Polya process (GPP) and path-dependent behavior
2. Stability Properties of the $\Psi - GPP/GI/1$ Queue
3. Heavy-Traffic Limits for the $\sum_{i=1}^{n} P_i/GI/1$ queue

   (tractable approximations for *transient* queue-length distribution)

Source: **"Queues with Path-Dependent Arrival Processes,"** Journal of Applied Probability, 2021, forthcoming, with **Kerry W. Fendick**. See http://www.columbia.edu/∼ww2040/allpapers.html

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Asymptotic Loss of Memory

- The standard notion of steady state for a stochastic Process:

$$X(t) \Rightarrow X(\infty) \quad \text{as} \quad t \to \infty,$$

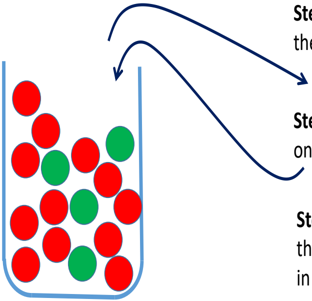
where $X(\infty)$ is independent of $X(t)$ for any fixed t.

- For time-varying model, **asymptotic loss of memory** **(ALOM):** **Large-Time Asymptotics for the $G_t/M_t/s_t + GI_t$ Many-Server Fluid Queue with Abandonment,** Queueing Systems, 67 (2011) 145-182 (with Yunan Liu).

- **Now we consider processes where ALOM does NOT hold**.

Polya urn model

Start with:
r red balls and
g green balls



Step 1. Take one ball out of the urn picked at random.

Step 2. Return that ball plus one more of the same color.

Step 3. Repeat. Let X_n be the proportion of red balls in the urn after n steps.

As $n \to \infty$, $X_n \to$ ***Beta***(r,*g*)

Generalized Polya Process (GPP)

- Definition. A **GPP** $N \equiv \{N(t) : t \geq 0\}$ is a Markov point process with stochastic intensity (defined in terms of the internal histories $\mathcal{H}_t$ by)

$$\lambda(t) \equiv \lambda(t|\mathcal{H}_t) \equiv (\gamma N(t-) + \beta)\kappa(t),$$

where $N(0) = 0$, γ and β are positive constants and $\kappa(t)$ is a positive integrable deterministic real-valued function.

- Definition. A **Polya point process** is the special case in which $\beta = 1$ and

$$\kappa(t) = \frac{1}{\gamma t + 1} \qquad \text{so that} \quad \lambda(t) = \frac{\gamma N(t-) + 1}{\gamma t + 1}.$$

the restarting property, Cha (2014)

- Proposition. If N is a GPP with parameter triple $(\kappa(t), \gamma, \beta)$, then **the conditional future process** $N_u(t) \equiv N(u+t) - N(u)$ given $N(u) = n$ and the history up to time u is itself a GPP with parameter triple $(\kappa(u+t), \gamma, \beta + n\gamma)$, so that

$$\lambda_u(t) \equiv \lambda(u+t | \mathcal{H}_{u+t}, N(u) = n) \equiv (\gamma N_u(t-) + \beta + n\gamma)\kappa(u+t).$$

negative binomial marginal distribution, Cha (2014)

- Proposition. If N is a GPP with parameter triple $(\kappa(t), \gamma, \beta)$, then $N(t)$ has a **negative binomial distribution** with **pmf**

$$P(N(t)=k) = C(\beta,\gamma,k)(1-p(t))^r p(t)^k, \quad k = 0, 1, 2, \ldots$$

$$\text{where} \quad r = \frac{\beta}{\gamma}, \quad p(t) = 1 - e^{-\gamma K(t)}, \quad K(t) = \int_0^t \kappa(s)\, ds,$$

$$C(\beta,\gamma,k) = \frac{\Gamma(r+k)}{\Gamma(r)k!}, \quad E[N(t)] = \frac{rp(t)}{1-p(t)} \quad \text{and} \quad Var[N(t)] = \ldots$$

a stationary point process (Ψ-GPP)

- **Theorem**. If N is a GPP with parameter triple $(\kappa(t), \gamma, \beta)$ and if

$$\begin{aligned} \kappa(t) &= \frac{1}{\gamma t + 1}, \quad \text{so that} \\ \lambda(t) &= \frac{\gamma N(t-) + \beta}{\gamma t + 1} \quad (\text{need not have } \beta = 1), \end{aligned}$$

then N is a **stationary point process** (has stationary increments) plus

$$E[N(t)] = \beta t \quad \text{and} \quad Cov(N(s), N(t)) = \beta s(1+\gamma t) \quad \text{for} \quad 0 \le s \le t < \infty,$$

so that $\quad Var(N(t)) = \beta t(1 + \gamma t), \quad t \ge 0.$ **(of order t^2 as t grows)**

non-ergodic LLN

Theorem

. If N is a (β, γ) Ψ-GPP (for which $\kappa(t) = 1/(\gamma t + 1)$), then

$$t^{-1}N(t) \to L(\gamma, \beta) \quad as \quad t \to \infty \quad w.p.1$$

where $L \equiv L(\gamma, \beta)$ has a gamma distribution with shape β/γ and rate $1/\gamma$, and thus mean $E[L] = \beta$ and variance $Var[L] = \beta\gamma$.

If a point process N satisfies a non-ergodic LLN, then we say that the point process N exhibits path-dependent behavior.

asymptotically Poisson with a random rate

- **Corollary.** If N is a (β, γ) Ψ-GPP with stochastic intensity $\lambda(t)$, then

$$\lambda(t) \to L(\gamma, \beta) \quad \text{as} \quad t \to \infty \quad \text{w.p.1}$$

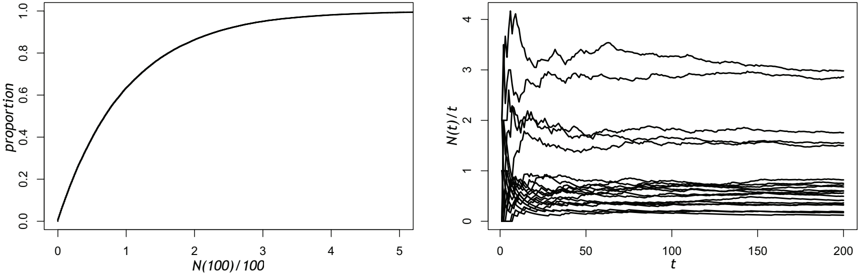
  where $L \equiv L(\gamma, \beta)$ has a gamma distribution with shape β/γ and rate $1/\gamma$, and thus mean $E[L] = \beta$ and variance $Var[L] = \beta\gamma$. Hence, asymptotically as $t \to \infty$, the point process behaves as a **Poisson process with random rate L.**

simulation support of the non-ergodic LLN

The empirical distribution of $N(100)/100$ based on $50,000$ iid samples (left)

and 25 paths of $N(t)/t$ over $[0, 200]$ (right)

for a Ψ-GPP with $(\beta, \gamma) = (1, 1)$.



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Source: Remark 2 and Section 6 of **"Queues with Path-Dependent Arrival Processes,"** Journal of Applied Probability, 2021, forthcoming (with Kerry W. Fendick). See http://www.columbia.edu/∼ww2040/allpapers.html

explosion in $\Psi - GPP/GI/1$ queue

- **Corollary.** If $Q(t)$ is the queue length process starting empty in the $\Psi - GPP/GI/1$ queue with service times having mean 1 and N is a (β, γ) Ψ-GPP with stochastic intensity $\lambda(t)$, then

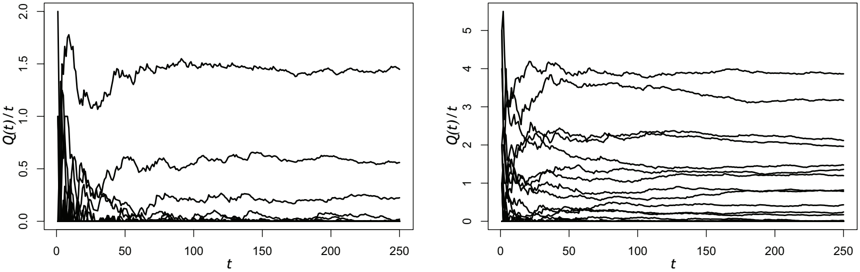
$$t^{-1}Q(t) \to max\{L(\gamma, \beta) - 1, 0\} \quad \text{as} \quad t \to \infty \quad \text{w.p.1, so that}$$

$$P(Q(t) \to \infty \quad \text{as} \quad t \to \infty) = P(L(\gamma, \beta) > 1), \text{where}$$

$$0 < P(L(\gamma, \beta) > 1) < 1.$$

simulation illustration of instability

Figure: Display of twenty-five individual sample paths of $Q(t)/t$ for $0 \leq t \leq 250$, starting empty, for a $P/D/1$ queue with parameter pairs $(\beta, \gamma) = (0.5, 1)$ (left) and $(\beta, \gamma) = (1.5, 1)$ (right).



Note weak link between traffic intensity and performance!

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Additional Source: **Kerry W. Fendick**,**"Brownian motion minus independent increments: representation and queueing application,"** *Probability in the Engineering and informational Sciences* (PEIS), published online 2020. (Draws on 1994 paper by **B. Hajek**)

preservation under superposition, Cha (2014)

- Proposition. If N_1 and N_2 are independent GPP's with parameter triples $(\kappa(t), \gamma, \beta_1)$ and $(\kappa(t), \gamma, \beta_2)$, respectively, then **the superposition process** $N_1(t) + N_2(t)$ is itself a GPP with parameter triple $(\kappa(t), \gamma, \beta_1 + \beta_2)$, so that
- for each n, **the sum (superposition) of n i.i.d. GPP's** with parameter triple $(\kappa(t), \gamma, \beta)$ is itself a GPP with parameter triple $(\kappa(t), \gamma, n\beta)$.

superposition FCLT for Ψ-GPP (a Ψ-GMP limit)

- **Theorem**. If $\{N_i(t) : i \geq 1\}$ is a sequence of i.i.d. (γ, β) Ψ-GPPs, then

  $$A_n \Rightarrow A \quad \text{as} \quad n \to \infty \quad \text{(FCLT in function space } \mathcal{D}\text{), where,}$$

  $$A_n(t) \equiv n^{-1/2}\left(\sum_{i=1}^{n} N_i(t) - n\beta t\right), \quad t \geq 0,$$

  and A is a **Ψ-GMP**, stationary Gaussian Markov process (see **Fendick (2021)**) with the covariance fct. of $N_i(t)$, $cov(A(s), A(t)) = \beta s(1 + \gamma t)$.

  Also, A satisfies the sde $dA(t) = \mu(t)A(t)dt + \sigma dB(t)$ for $B(t)$ standard BM, $\sigma \equiv \sqrt{\beta}$ and $\mu(t) \equiv (t + (1/\gamma))^{-1}$.

  Proof. Apply Hahn's theorem for sums of processes, as in Thm 7.2.1 of WW book.

convergence to $\Psi - GMP$ with drift

- **Theorem**. If $\{N_i(t) : i \geq 1\}$ is a sequence of i.i.d. (γ, β) Ψ-GPPs and if $\sqrt{n}(\mu_n - 1) \to \mu$ as $n \to \infty$ (adding drift), then $A_n^d(t) \Rightarrow A(t) + \beta\mu t \quad \text{as} \quad n \to \infty$ (FCLT in function space $\mathcal{D}$), where,

$$A_n^d(t) \equiv n^{-1/2}(\sum_{i=1}^{n} N_i(\mu_n t) - n\beta t), \quad t \geq 0,$$

and A is a Ψ-GMP, stationary Gaussian Markov process (see **Fendick (2021)**) with the covariance fct. of $N_i(t)$, $cov(A(s), A(t)) = \beta s(1 + \gamma t)$.

heavy-traffic FCLT for $\sum_{i=1}^{n}(\Psi - GPP_i)/GI/1$ queue

- **Theorem**. Let $C(t)$ be the renewal counting process of i.i.d. service times with mean $1/\beta$ and scv c_s^2. Let
$$S_n(t) \equiv n^{-1/2}(C(nt) - \beta nt),$$
$$X_n(t) \equiv A_n^d(t) - S_n(t) \quad \text{and}$$
$$Q_n(t) \equiv n^{-1/2}Q^n(t), \quad t \geq 0.$$

  If $Q_n(0) \Rightarrow Q(0)$ in addition to previous assumptions, then
$$(A_n^d, S_n, X_n, Q_n) \Rightarrow (A + \beta\mu e, S, X, Q) \quad \text{in} \quad \mathcal{D}^4 \quad \text{as} \quad n \to \infty,$$

Remaining Proof standard HT theory from Chapter 9 of 2002 WW book.

marginal distributions

- **Corollary**. Under the assumptions above,

$$(X_n(s), Q_n(s), Q_n(s+t)) \Rightarrow (X(s), Q(s), Q(s+t)) \quad \text{in} \quad \mathbb{R}^3 \quad \text{as} \quad n \to \infty,$$

where the joint limiting distribution has joint pdf

$$f(x_s, q_s, q_{s+t}) = f(x_s) f(q_s | x_s) f(q_{s+t} | x_s, q_s),$$

with all given **explicitly** in terms of the normal cdf, the exponential function and the model parameters, so that $\mathbb{P}(Q(s+t) \leq q_{s+t} | X(s) = x_s, Q(s) = q_s)$ and $\mathbb{P}(Q(t) \leq q_t)$ are also given **explicitly** in the same way. (**See paper for details.**)

Thank you!