Algorithms for Extremal Queues in $GI/GI/1$

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Motivation

- Extremal queues in $GI/GI/1$: $F_0/G_u/1$ (Chen and Whitt (2018)) (a special two-point inter-arrival and service times queue)

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- **GOAL: effective algoritms to estimate $E[W(F_0/G_u/1)]$?**

Basic Settings

- Steady-state Waiting Time ($W_0 = 0$):

$$W \stackrel{\mathrm{d}}{=} [W + V - U]^+.$$

$$w : \mathcal{P}_{a,2}(M_a) \times \mathcal{P}_{s,2}(M_s) \to \mathbb{R}, w(F,G) \equiv E[W(F,G)]$$

where $0 < \rho < 1$ and

- Mean Waiting Time:

$$E[W] = \sum_{k=1}^{\infty} \frac{E[S_k^+]}{k} < \infty.$$

$S_k \equiv X_1 + \cdots + X_k$ and $X_k \equiv V_k - U_k$, $k \geq 1$.

Background

$(1, c_a^2 + 1)$ for interarrival F and $(\rho, \rho^2(c_s^2 + 1))$ for service G.

- Kingman (1962) bound:

$$E[W(F,G)] \leq \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1-\rho)}.$$

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$$E[W(M,G)] = \frac{\rho^2(1+c_s^2)}{2(1-\rho)}.$$

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- Ott (1987) bound:

$$E[W(D,A_3)] = \frac{\rho((1+c_s^2)\rho - 1)^+}{2(1-\rho)}.$$

Extremal Distributions

Two-point cdfs under $(1, 1 + c_a^2)$ and $(\rho, \rho^2(1 + c_s^2))$:

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- F: $c_a^2/(c_a^2 + (b_a - 1)^2)$ at b_a, $(b_a - 1)^2/(c_a^2 + (b_a - 1)^2)$ at $1 - c_a^2/(b_a - 1)$ where $1 + c_a^2 \le b_a \le M_a$.

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- G: $c_s^2/(c_s^2 + (b_s - 1)^2)$ at ρb_s, $(b_s - 1)^2/(c_s^2 + (b_s - 1)^2)$ at $\rho(1 - c_s^2/(b_s - 1))$ where $1 + c_s^2 \le b_s \le M_s$.

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- G: $c_s^2/(c_s^2 + (b_s - 1)^2)$ at ρb_s, $(b_s - 1)^2/(c_s^2 + (b_s - 1)^2)$ at $\rho(1 - c_s^2/(b_s - 1))$ where $1 + c_s^2 \le b_s \le M_s$.
- $F = F_0$ for $b_a = 1 + c_a^2$ and $F = F_u$ when $b_a = M_a$.

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- G: $c_s^2/(c_s^2 + (b_s - 1)^2)$ at ρb_s, $(b_s - 1)^2/(c_s^2 + (b_s - 1)^2)$ at $\rho(1 - c_s^2/(b_s - 1))$ where $1 + c_s^2 \leq b_s \leq M_s$.
- $F = F_0$ for $b_a = 1 + c_a^2$ and $F = F_u$ when $b_a = M_a$.
- $G = G_0$ for $b_s = 1 + c_s^2$ and $G = G_u$ when $b_s = M_s$.

Reduction

- Three-point reduction:

$$\sup_{F \in \mathcal{P}_{a,2}(M_a), G \in \mathcal{P}_{s,2}(M_s)} E[W(F,G)] = E[W(F^*,G^*)]$$

where $F^* \in \mathcal{P}_{a,2,3}(M_a)$, $G^* \in \mathcal{P}_{s,2,3}(M_s)$.

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Research Question: define $E[W(F_0, G_{u^*})] \equiv \lim_{M_s \to \infty} E[W(F_0, G_u)]$, how to evaluate $E[W(F_0, G_{u^*})]$?

Reduction for Interarrival Time

Theorem

For any service-time V with cdf G having mean ρ and scv c_s^2, the steady-state waiting time is distributed as (denoted by $\stackrel{\mathrm{d}}{=}$)

$$W(F_0/G/1) \stackrel{\mathrm{d}}{=} W(D(1/p)/RS(V,p)/1) + \sum_{k=1}^{N(p)-1} V_k$$

for $N(p)$ and $RS(V,p)$. Hence, the mean is

$$\begin{aligned} E[W(F_0/G/1)] &= E[W(D(1/p)/RS(V,p)/1)] + (E[N(p)] - 1)E[V] \\ &= E[W(D(1/p)/RS(V,p)/1)] + \rho(1-p)/p \\ &= E[W(D(1/p)/RS(V,p)/1)] + \rho c_a^2. \end{aligned}$$

Reduction for Service Time

Theorem
(*the Daley decomposition*) *Consider the $F/G_u/1$ model with arbitrary interarrival-time cdf F and two-point service-time cdf G_u. Then*

$$\begin{aligned}\lim_{M_s\to\infty} E[W(F/G_u/1)] &= E[W(F/D(\rho)/1)] + \lim_{M_s\to\infty} E[W(D(1)/G_u/1)]. \\ &= E[W(F/D(\rho)/1)] + \frac{\rho^2 c_s^2}{2(1-\rho)}.\end{aligned}$$

Overall Reduction

Theorem
(*overall decomposition of the upper bound*) *For the $GI/GI/1$ model with extremal interarrival-time cdf F_0 and extremal service-time cdf G_{u^*},*

$$\begin{aligned} E[W(F_0, G_{u^*})] &\equiv \lim_{M_s \to \infty} E[W(F_0/G_u/1)] \\ &= E[W(D(1/p)/RS(D(\rho), p)/1)] + \rho c_a^2 + \frac{\rho^2 c_s^2}{2(1-\rho)} \end{aligned}$$

for $p \equiv 1/(c_a^2 + 1)$.

Numerical Algorithm

A new service time:

$$RS(V, p) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N(p)} V_k, \tag{1}$$

where $N(p)$ is a geometric random variable on the positive integers, having mean $E[N(p)] = 1/p$ with $p = 1/(1 + c_a^2)$ and $\{V_k : k \geq 1\}$ is i.i.d. random variables distributed as a service time V.

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- Apply Inter-arrival Time Reduction:

$$E[W(F_0, G)] = E[W(D(1/p), RS(V,p))] + \rho c_a^2.$$

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- Apply Inter-arrival Time Reduction:

$$E[W(F_0, G)] = E[W(D(1/p), RS(V,p))] + \rho c_a^2.$$

- Apply *Daley Decomposition*:

$$E[W(F_0, G_{u^*})] \quad = \quad E[W(D(1/p), RS(D(\rho), p))] + const.$$

Numerical Algorithm (Negative Binomial)

Lemma
(*NB representation*) *For the* $D(1/p)/RS(D(\rho), p)/1$ *model,*

$$S_k \stackrel{\mathrm{d}}{=} \rho(NB(n, 1-p) + n) - (n/p),$$

so that

$$E[W] = \sum_{n=1}^{\infty} \frac{E[(\rho(NB(n, 1-p) + n) - (n/p))^+]}{n}.$$

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pmf: $P(NB(n, 1-p) = k) \equiv \left(\frac{(n+k-1)!}{k!(n-1)!}\right) p^n (1-p)^k.$

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recursion: $\mathbb{P}(NB = k) = \mathbb{P}(NB = k-1)(n + k - 1)/k)(1 - p)$.

Numerical Algorithm (Negative Binomial)

Algorithm 1 Basic Negative Binomial Recursion (k in outer loop)

```
for k ∈ [K] do
    S(k) ← 0, nbpdf ← p(1 − p)^k
    for n ∈ [n] do
        S(k) ← S(k) + nbpdf max((n + k)ρ − n/p, 0)/n
        nbpdf ← nbpdf((n+k)/n)p
    E[W] ← E[W] + S(k)
Output E[W]
```

Numerical Algorithm (Negative Binomial)

Refinement: Staring at mean and going up and down.

Algorithm 2 NB Recursion (Up and Down from the Mean)

for $n \in [1, N]$ **do**
  $nbpdf(1, n(1-p)/p) \leftarrow 1$
  for $k \in [m(n) - \alpha\sqrt{N}, m(n)]$ **do**
    $nbpdf(1, k-1) \leftarrow nbpdf(1, k)/(n+k-1)(k)/(1-p)$
  for $k \in [m(n), m(n) + \alpha\sqrt{N} - 1]$ **do**
    $nbpdf(1, k+1) \leftarrow nbpdf(1, k)(n+k)/(k+1)(1-p)$
  Normalize *nbpdf* to obtain $\mathbb{P}(NB(n, 1-p) = k)$
  $S(n) \leftarrow \sum_k \mathbb{P}(NB(n, 1-p) = k) \max((n+k)\rho - n/p, 0)$
  $E[W] \leftarrow E[W] + S(n)/n$
Output $E[W]$

Numerical Algorithm (Negative Binomial)

Table 1: Comparison of Two Approaches Generating Negative Binomial Probabilities

k	$n_1 = 10$	$n_2 = 10$	k	$n_1 = 100$	$n_2 = 100$	k	$n_1 = 1000$	$n_2 = 1000$
40	0.0279638	0.0279638	400	0.0089128	0.0089128	4000	0	0.0028207
41	0.0272818	0.0272818	401	0.0088906	0.0088906	4001	0	0.0028200
42	0.0265023	0.0265023	402	0.0088641	0.0088641	4002	0	0.0028192
43	0.0256394	0.0256394	403	0.0088333	0.0088333	4003	0	0.0028182
44	0.0247071	0.0247071	404	0.0087983	0.0087983	4004	0	0.0028170
45	0.0237188	0.0237188	405	0.0087592	0.0087592	4005	0	0.0028158
46	0.0226875	0.0226875	406	0.0087160	0.0087160	4006	0	0.0028144
47	0.0216256	0.0216256	407	0.0086689	0.0086689	4007	0	0.0028128
48	0.0205443	0.0205443	408	0.0086179	0.0086179	4008	0	0.0028111
49	0.0194542	0.0194542	409	0.0085631	0.0085631	4009	0	0.0028093
50	0.0183647	0.0183647	410	0.0085047	0.0085047	4010	0	0.0028074

Limit When $M_s \to \infty$

Theorem
(*the idle-time representation*) *In the $GI/GI/1$ queue with cdf's F and G having parameter 4-tuple $(1, c_a^2, \rho, c_s^2)$,*

$$E[W] \equiv E[W(F,G)] = \psi(1, c_a^2, \rho, c_s^2) - \phi(I),$$

where

$$\psi(1, c_a^2, \rho, c_s^2) \equiv \frac{E[(U-V)^2]}{2E[U-V]} = \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1-\rho)} + \frac{1-\rho}{2}$$

and

$$\phi(I) \equiv \phi(F,G) = \frac{E[I^2]}{2E[I]} = E[I_e].$$

Limit When $M_s \to \infty$

Corollary

(*reduction to idle time*) *For the $GI/GI/1$ model with extremal interarrival-time cdf F_0 and extremal service-time cdf G_u,*

$$\begin{aligned} E[W(F_0, G_{u^*})] &\equiv \lim_{M_s \to \infty} E[W(F_0/G_u/1)] \\ &= \frac{c_a^2 + \rho^2 c_s^2}{2(1-\rho)} + \frac{1-\rho}{2} - \phi(I; 1, c_a^2, \rho, c_s^2), \end{aligned}$$

where I is the idle time in an $F_0/G_{u^*}/1$ queue or, equivalently, in a $F_0/D/1$ queue for an appropriate D.

Numerical Algorithm (Negative Binomial)

Table 2: Performance of Algorithm with Different Truncation Levels

	Algorithm 2					Minh and Sorli Algorithm	
$\rho \backslash N$	2E+03	4E+03	8E+03	1.6E+04	2E+04	$T = 1E + 07$	95%CI
0.1	0.422229	0.422229	0.422229	0.422229	0.422229	0.422	7.79E-05
0.2	0.903885	0.903885	0.903885	0.903885	0.903885	0.904	1.30E-04
0.3	1.499234	1.499234	1.499234	1.499234	1.499234	1.499	1.71E-04
0.4	2.304105	2.304105	2.304105	2.304105	2.304105	2.304	1.90E-04
0.5	3.470132	3.470132	3.470132	3.470132	3.470132	3.470	2.25E-04
0.6	5.294825	5.294825	5.294825	5.294825	5.294825	5.294	2.43E-04
0.7	8.441305	8.441305	8.441305	8.441305	8.441305	8.442	3.05E-04
0.8	14.916937	14.916937	14.916937	14.916937	14.916937	14.917	3.22E-04
0.9	34.721476	34.721484	34.721484	34.721484	34.721484	34.722	5.17E-04
0.95	74.552341	74.619631	74.620917	74.620937	74.620937	74.621	7.11E-04

Performance in the Heavy-traffic

Table 3: Performance of Algorithm 2 in Heavy Traffic

Case	Algorithm 2				Minh and Sorli	
$\rho \backslash N$	1×10^4	2×10^4	3×10^4	4×10^4	$T = 1 \times 10^7$	
0.98	194.0544167173	194.5385548017	194.5559125683	194.5567071265	194.556	9.29E-04
	5×10^4	1×10^5	2×10^5	3×10^5		
0.98	194.5567179973	194.5567742874	194.5567742874	194.5567742874	194.556	9.29E-04
$\rho \backslash N$	1×10^4	3×10^4	5×10^4	1×10^5	$T = 1 \times 10^7$	
0.99	372.0880005430	372.0880005430	391.8858614678	394.5238008176	394.532	1.45E-03
	2×10^5	3×10^5	4×10^5	5×10^5		
0.99	394.5331823499	394.5331886695	394.5331886695	394.5331886695	394.532	1.45E-03

Upper Bound Approximation

Theorem

In the setting of previous theorems,

$$W(D(1/p), RS(D(\rho), p)) \leq_{icx} W(D(1/p), M(\rho/p)), \tag{2}$$

so that well known results for the $D/M/1$ queue yield

$$E[W(F_0, G_{u^*})] \leq \frac{[2(1-\rho)\rho/(1-\delta)]c_a^2 + \rho^2 c_s^2}{2(1-\rho)}, \tag{3}$$

where $\delta \in (0,1)$ solves the equation

$$\delta = \exp(-(1-\delta))/\rho). \tag{4}$$

Upper Bound Inequalities

Overall Upper Bound Inequalities:

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$$E[W(F,G)] \quad \leq \quad E[W(F_0, G_{u^*})](\text{Tight UB})$$

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Overall Upper Bound Inequalities:

$$\begin{aligned} E[W(F,G)] &\leq E[W(F_0, G_{u^*})] \text{(Tight UB)} \\ &\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)} \text{(UB Approx)} \end{aligned}$$

Upper Bound Inequalities

Overall Upper Bound Inequalities:

$$
\begin{aligned}
E[W(F,G)] &\leq E[W(F_0, G_{u^*})] (\text{Tight UB}) \\
&\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)} (\text{UB Approx}) \\
&< \frac{\rho^2([(2-\rho)c_a^2/\rho] + c_s^2)}{2(1-\rho)} (\text{Daley(1977)})
\end{aligned}
$$

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Overall Upper Bound Inequalities:

$$
\begin{aligned}
E[W(F,G)] &\leq E[W(F_0, G_{u^*})]\text{(Tight UB)} \\
&\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}\text{(UB Approx)} \\
&< \frac{\rho^2([(2-\rho)c_a^2/\rho] + c_s^2)}{2(1-\rho)}\text{(Daley(1977))} \\
&< \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1-\rho)}\text{(Kingman(1962))}
\end{aligned}
$$

Upper Bound Inequalities

Overall Upper Bound Inequalities:

$$\begin{aligned}
E[W(F,G)] &\leq E[W(F_0, G_{u^*})](\text{Tight UB}) \\
&\leq \frac{2(1-\rho)\rho/(1-\delta)c_a^2 + \rho^2 c_s^2}{2(1-\rho)}(\text{UB Approx}) \\
&< \frac{\rho^2([(2-\rho)c_a^2/\rho] + c_s^2)}{2(1-\rho)}(\text{Daley(1977)}) \\
&< \frac{\rho^2([c_a^2/\rho^2] + c_s^2)}{2(1-\rho)}(\text{Kingman(1962)})
\end{aligned}$$

where $\delta \in (0,1)$ and $\delta = \exp(-(1-\delta)/\rho)$.

Summary for Upper Bound

Table 4: A comparison of the bounds and approximations for the steady-state mean $E[W]$ as a function of ρ for the case $c_a^2 = c_s^2 = 4.0$ and $c_s^2 = 4.0$.

ρ	Tight LB	HTA	Tight UB	UB Approx	δ	MRE	Daley	Kingman
0.10	0.00	0.044	0.422	0.422	0.000	0.003%	0.44	2.24
0.20	0.00	0.200	0.904	0.906	0.007	0.19%	1.00	2.60
0.30	0.00	0.514	1.499	1.51	0.041	0.60%	1.71	3.11
0.40	0.00	1.07	2.304	2.33	0.107	0.94%	2.67	3.87
0.50	0.25	2.00	3.470	3.51	0.203	1.15%	4.00	5.00
0.60	1.00	3.60	5.295	5.35	0.324	1.07%	6.00	6.80
0.70	2.42	6.53	8.441	8.52	0.467	0.93%	9.33	9.93
0.80	5.50	12.80	14.92	15.02	0.629	0.67%	16.00	16.40
0.90	15.25	32.40	34.72	34.84	0.807	0.35%	36.00	36.20
0.95	35.13	72.20	74.62	74.76	0.902	0.18%	76.00	76.10
0.98	95.05	192.1	194.6	194.7	0.960	0.07%	196.0	196.0
0.99	195.0	392.0	394.5	394.7	0.980	0.04%	396.0	396.0

Thank You!