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ON ARRIVALS THAT SEE TIME AVERAGES: A MARTINGALE APPROACH

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Abstract

This paper is a sequel to our previous paper investigating when *arrivals see time averages* (ASTA) in a stochastic model; i.e., when the steady-state distribution of an embedded sequence, obtained by observing a continuous-time stochastic process just prior to the points (arrivals) of an associated point process, coincides with the steady-state distribution of the observed process. The relation between the two distributions was also characterized when ASTA does not hold. These results were obtained using the conditional intensity of the point process given the present state of the observed process (assumed to be well defined) and basic properties of Riemann–Stieltjes integrals. Here similar results are obtained using the stochastic intensity associated with the martingale theory of point processes, as in Brémaud (1981). In the martingale framework, the ASTA result is almost an immediate consequence of the definition of a stochastic intensity. In a stationary framework, the results characterize the Palm distribution, but stationarity is not assumed here. Watanabe's (1964) martingale characterization of a Poisson process is also applied to establish a general version of anti-PASTA: if the points of the point process are appropriately generated by the observed process and the observed process is Markov with left-continuous sample paths, then ASTA implies that the point process must be Poisson.

EMBEDDED PROCESSES; PALM MEASURE; PASTA; QUEUES; CUSTOMER AVERAGES AND TIME AVERAGES; THE ARRIVAL THEOREM; POINT PROCESSES; STOCHASTIC INTENSITY

1. Introduction

The purpose of this paper is to extend and complement Melamed and Whitt (1990), hereafter referred to as MW, where it is shown that (not necessarily Poisson) *arrivals see time averages* (ASTA) in a stochastic model under a *lack of bias assumption* (LBA); see Definitions 1 and 3 below. This LBA condition generalizes the *lack of anticipation assumption* (LAA) in Wolff (1982) and a related condition in König and Schmidt (1980) (see Theorem 1.6.6 of Franken et al. (1981)), and is in fact necessary and sufficient for ASTA given that a conditional intensity or stochastic intensity exists for the point process. In MW we used basic properties of Riemann–Stieltjes integrals; here we use

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basic properties of the martingale theory of point processes, as in Brémaud (1981) and Varaiya and Walrand (1981). As noted in MW, closely related work has been done by Brémaud (1989), König et al. (1989), Serfozo (1989a,b) and Stidham and El Taha (1989). (Our work was done independently.)

Our model consists of two stochastic processes $X \equiv \{X(t): t \geqq 0\}$ and $N \equiv \{N(t): t \geqq 0\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$. The process X, which is intended to represent the state or a partial description of the state of some system such as a queue or a network of queues, takes values in a complete separable metric space E endowed with its Borel σ-field. The process N is a simple stochastic point process on $[0, \infty)$, i.e., it has non-decreasing sample paths with values in the non-negative integers and jumps of size 1. We assume that the sample paths of X and N have left and right limits at all t (all $t > 0$ for left limits). We also assume that $N(0) = 0$, $N(t) \to \infty$ as $t \to \infty$ w.p.1 and that there exists t_0 such that $0 < E[N(t)] < \infty$ for *all* $t > t_0$. Of most importance, we assume that *the sample paths of X are left-continuous, while the sample paths of N are right-continuous*. Let

$$T_n = \inf\{t > 0 : N(t) \geqq n\}, \qquad n \geqq 1. \tag{1.1}$$

Here is the basic problem: assuming that

$$X(t) \Rightarrow X(\infty) \quad \text{as } t \to \infty \quad \text{and} \quad X(T_n) \Rightarrow \tilde{X}(\infty) \quad \text{as } n \to \infty, \tag{1.2}$$

where $\Rightarrow$ denotes convergence in distribution or weak convergence, as in Billingsley (1968), we want to know when

$$X(\infty) \stackrel{d}{=} \tilde{X}(\infty), \tag{1.3}$$

where $\stackrel{d}{=}$ means equality in distribution. ASTA is said to hold when (1.3) holds. Of course, ASTA is something of a misnomer, because N need not be an arrival process and averages have not been considered yet. However, we consider averages below in a generalization of (1.2) and (1.3). We use the suggestive ASTA terminology because of the many applications and long history in queueing theory; see MW for further discussion.

As in MW, we focus on the *expectations of averages*, in particular,

$$\bar{V}(t) \equiv \frac{E\left[\int_0^t f[X(s)]r(s)ds\right]}{\int_0^t r(s)ds} \quad \text{and} \quad \bar{W}(t) \equiv \frac{E\left[\sum_{k=1}^{N(t)} f[X(T_k)]\right]}{E[N(t)]}, \qquad t \geqq 0, \tag{1.4}$$

where f is a bounded measurable real-valued function on E and $r(t) = E[\nu(t)]$, where ν is the stochastic intensity of N, which will be defined in Section 2. (The notation is chosen to suggest that $\bar{V}(t)$ might be an average related to the *Virtual* waiting time in a queue, while $\bar{W}(t)$ might be an average related to the actual *Waiting* time per customer.) Usually, $r(t)$ in (1.4) will be independent of t so that $\bar{V}(t)$ will reduce to a simple time average, but we include weighted time averages to permit treating non-stationary processes (e.g., periodic processes), as in Section 3 of Wolff (1982). Our main result relates $\bar{V}(t)$ and $\bar{W}(t)$ given that N has a stochastic intensity ν with respect to (X, N). Let

$\mathrm{Cov}(Y_1, Y_2)$ be the covariance between Y_1 and Y_2, i.e., $\mathrm{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$.

Theorem 1. *Suppose that* N *has a stochastic intensity* v *with respect to* (X, N) *with* $E[v(t)] = r(t) < \infty$ *and*

$$E\left[\int_0^t v(s)ds\right] = \int_0^t r(s)ds < \infty, \qquad t \geqq 0.$$

If f *is continuous, then*

$$\bar{W}(t) = \frac{\int_0^t E[f(X(s))v(s)]ds}{\int_0^t r(s)ds} = \bar{V}(t) + \frac{\int_0^t \mathrm{Cov}[f(X(s)), v(s)]ds}{\int_0^t r(s)ds}.$$

Corollary 1. *If, in addition,* $\bar{V}(t) \to \bar{V}(\infty)$, $\bar{W}(t) \to \bar{W}(\infty)$ *and* $t^{-1}\int_0^t r(s)ds \to r(\infty)$ *as* $t \to \infty$, *then*

$$r(\infty)[\bar{W}(\infty) - \bar{V}(\infty)] = \lim_{t\to\infty} t^{-1} \int_0^t \mathrm{Cov}[f(X(s)), v(s)]ds.$$

Definition 1. *The lack of bias assumption (LBA) holds for* $f(X)$ *and* N *if*

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{Cov}[f(X(s)), v(s)]ds = 0.$$

Corollary 2. *Suppose that* $r(\infty) > 0$ *under the assumptions of Corollary* 1. *Then* $\bar{W}(\infty) = \bar{V}(\infty)$ *if and only if LBA holds for* $f(X)$ *and* N.

Of course, in a stationary framework $\mathrm{Cov}[X(t), v(t)]$ will be independent of t, in which case Theorem 1 and its corollaries simplify.

Corollary 3. *If, in addition to the assumptions of Theorem* 1, *the distribution of* $[f(X(t)), v(t)]$ *is independent of* t, *then*

$$\bar{W}(t) = \frac{E[f(X(t))v(t)]}{E[v(t)]} = \bar{V}(t) + \frac{\mathrm{Cov}[f(X(t)), v(t)]}{E[v(t)]},$$

which is independent of t.

To obtain the desired (1.3) from Corollary 2, we *assume* that $r(t)$ is independent of t and

(1.5) $\bar{V}(t) \to \bar{V}(\infty) = E[f(X(\infty))]$ and $\bar{W}(t) \to \bar{W}(\infty) = E[f(\tilde{X}(\infty))]$ as $t \to \infty$

for a large class of functions f, in particular, for all bounded continuous real-valued f, because expectations of these functions determine distributions; see p. 9 of Billingsley (1968). Corollary 2 establishes what we consider to be the most interesting part of (1.3).

Similarly, if we are interested in limiting averages, as in Wolff (1982), then we consider

$$(1.6)\quad V(t) \equiv \frac{\int_0^t f[X(s)]r(s)ds}{\int_0^t r(s)ds} \quad \text{and} \quad W(t) \equiv [N(t)]^{-1} \sum_{k=1}^{N(t)} f[X(T_k)] \qquad t \geqq 0,$$

with the hope of showing that $V(t) \to V(\infty)$ w.p.1 as $t \to \infty$ if and only if $W(t) \to W(\infty)$ w.p.1 as $t \to \infty$, in which case $V(\infty) = W(\infty)$. (As in (1.4), $r(t)$ is included in $V(t)$ to treat non-stationary processes; see Section 3 of Wolff (1982).) We establish conditions for $V(\infty) = W(\infty)$, which we regard as the most interesting part, by establishing Theorem 1 and its corollaries. To obtain $V(\infty) = W(\infty)$ from Corollary 2, we *assume* that

$$(1.7)\qquad V(t) \to V(\infty) = \bar{V}(\infty) \quad \text{and} \quad W(t) \to W(\infty) = \bar{W}(\infty) \quad \text{as } t \to \infty.$$

Note that $\bar{V}(t) = EV(t)$, but in general we do *not* have $\bar{W}(t) = EW(t)$. It suffices to have both the numerators and the denominators in $W(t)$ and $\bar{W}(t)$ converge after dividing by t.

As noted in MW, there are many sufficient conditions for (1.5) and (1.7), such as stationarity plus ergodicity. In a stationary framework, as in König and Schmidt (1980) and Franken et al. (1981), $X(\infty)$ and $\tilde{X}(\infty)$ have the one-dimensional marginal distributions associated with the underlying measure P and the associated Palm measure P^0, respectively. In this context, Corollary 3 above describes the marginal distribution of $\tilde{X}(\infty)$ even when (1.3) does not hold; i.e., Corollary 3 establishes the covariance formula in (8) of MW and Papangelou's formula in Brémaud (1989), i.e.,

$$(1.8)\qquad E[f(\tilde{X}(\infty))] = \frac{E[f(X(t))\nu(t)]}{E[\nu(t)]} = E[f(X(t))] + \frac{\mathrm{Cov}[f(X(t)), \nu(t)]}{E[\nu(t)]}.$$

As noted in MW, (1.8) holds for all measurable real-valued f for which the expectations are well defined if it holds for all bounded continuous real-valued f, by standard limiting arguments; see pp. 7–9 of Billingsley (1968).

A significant feature of our results here and in MW is that we may have ASTA without the point process N being Poisson. (Applications to queueing networks are discussed in MW.) However, the Poisson property is quite closely linked to ASTA, so that many examples of ASTA involve Poisson processes. In fact, in certain situations, there is anti-PASTA; i.e., ASTA implies that the arrival process must be Poisson; see Miyazawa (1982), König et al. (1983), Section 2.10 of Walrand (1988) and Green and Melamed (1990). In Section 3, we obtain a general anti-PASTA result with a revealing proof. A key condition for our anti-PASTA result is that X be Markov with left-continuous sample paths. (This condition is not satisfied by the examples of non-Poisson ASTA in MW.)

2. Stochastic intensities

We work with the martingale theory of stochastic point processes, as in Brémaud (1981), to which we refer for background. Let $\{\mathscr{F}_t : t \geqq 0\}$ be the *internal history* of (X, N), i.e., $\mathscr{F}_t$ is the sub-σ-field of $\mathscr{F}$ generated by $\{[X(s), N(s)] : 0 \leqq s \leqq t\}$. Since the sample paths of X are left-continuous, $f(X)$ is adapted to $\mathscr{F}_t$ ($f(X(t))$ is $\mathscr{F}_t$-measurable) for all t and all measurable f. If, in addition, $U(t) \equiv f(X(t))$ has left-continuous sample paths with probability 1, then U is $\mathscr{F}_t$*-predictable*; see p. 9 of Brémaud (1981). For what follows, it is vital that U be predictable, so that we always assume that U has left-continuous sample paths. If X is a pure jump process, then U has this desired property for any measurable f. However, to treat more general cases, we assume that f is also continuous. Since expectations of bounded continuous real-valued functions determine distributions (p. 9 of Billingsley (1968)), we obtain the desired (1.3) from only considering these special functions.

The key to our results is the definition of a stochastic intensity; see p. 27 of Brémaud (1981). (For definitions of the terms progressive and predictable, see pp. 288 and 8 of Brémaud (1981).)

Definition 2. *The stochastic point process N has a stochastic intensity $\nu \equiv \{\nu(t): t \geqq 0\}$ (also called a $(P, \mathscr{F}_t)$-intensity) if ν is a non-negative $\mathscr{F}_t$-progressive process such that for all t*

$$\int_0^t \nu(s)ds < \infty \qquad \text{w.p.1} \tag{2.1}$$

and for all non-negative $\mathscr{F}_t$-predictable processes $C \equiv \{C(t): t \geqq 0\}$

$$E\left[\int_0^\infty C(s)dN(s)\right] = E\left[\int_0^\infty C(s)\nu(s)ds\right]. \tag{2.2}$$

If ν is also predictable, which can always be achieved, then ν is essentially unique; see pp. 30–31 of Brémaud (1981). Let $N(t, t+u) = N(t+u) - N(t)$ for $t, u > 0$. Paralleling (16) of MW, we often also have

$$\nu(t) = \lim_{u \downarrow 0} u^{-1}E[N(t, t+u) \mid \mathscr{F}_t] \qquad \text{w.p.1}, \tag{2.3}$$

but (2.3) does not hold automatically; see p. 28 of Brémaud (1981).

Proof of Theorem 1. Apply Definition 2 to characterize the expectations in the expressions for $\bar{V}(t)$ and $\bar{W}(t)$ in (1.4); e.g., by appropriate choice of C, we see that

$$\bar{W}(t) = \frac{E\left[\int_0^t U(s)dN(s)\right]}{E\left[\int_0^t dN(s)\right]} = \frac{\int_0^t E[U(s)\nu(s)]ds}{\int_0^t E[\nu(s)]ds}. \tag{2.4}$$

Under the stationarity condition ((15) of MW)

$$[X(t), N(t, t+u)] \stackrel{d}{=} [X(t+h), N(t+h, t+h+u)] \tag{2.5}$$

for all positive t, u and h, we have $E[\nu(t)]$ and $\mathrm{Cov}[U(t), \nu(t)]$ are independent of t and we obtain Corollary 3 to Theorem 1. However, the stochastic intensity $\nu(t)$ here is not the same as the conditional intensity $\mu(t)$ in (16) of MW, even assuming (2.3) and (2.5) above, because $\mu(t)$ is the intensity conditional on only $X(t)$, whereas $\nu(t)$ is the intensity conditional on $\mathscr{F}_t$, i.e., the entire past of X and N. Since $\mu(t)$ is a coarsening of $\nu(t)$ (X is adapted to $\mathscr{F}_t$), we can now *define* $\mu(t)$ by

(2.6) $$\mu(t)=E[\nu(t)\mid X(t)]$$

and conclude that $E[\mu(t)]=E[\nu(t)]$ and

(2.7) $$E[U(t)\nu(t)]=E[U(t)E[\nu(t)\mid X(t)]]=E[U(t)\mu(t)].$$

Hence, under the stationarity condition (2.5), Theorem 1 is essentially equivalent to Theorem 3 of MW; i.e., Theorem 3 of MW holds if $\mu(t)$ is defined, not by (16) and (17) in MW, but by (2.6) here, assuming the existence of the stochastic intensity ν. Even with this modification, evidently neither theorem completely contains the other; Theorem 3 of MW requires the existence of the conditional intensity $\mu(t)$ as defined by (16) and (17) there, whereas Theorem 1 here requires the existence of the stochastic intensity $\nu(t)$ defined by Definition 1. Theorem 1 here evidently tends to be more general, but Theorem 3 of MW is more elementary.

Paralleling Corollary 1 to Theorem 3 of MW, we now indicate how to establish ASTA, i.e., (1.3) in Section 1.

Definition 3. *LBA holds for X and N if LBA holds for U and N* (*Definition* 1) *for all bounded continuous real-valued f.*

Corollary 4. *Suppose that, in addition to the conditions of Corollary* 2, (1.2) *and* (1.5) *hold for all bounded continuous real-valued f. Then* $X(\infty)\overset{d}{=}\tilde{X}(\infty)$ *if and only if LBA holds for X and N.*

Proof. Theorem 1 plus the conditions establish that $E(f[X(\infty)])=E(f[\tilde{X}(\infty)])$ for all bounded continuous real-valued f. The desired (1.3) holds because expectations of these functions determine distributions; see p. 9 of Billingsley (1968). If LBA does not hold, then the distributions obviously must be different.

3. ASTA as a characterization of a point process

In general, ASTA need not imply much about N, because N could be independent of X. However, under appropriate conditions, we have anti-PASTA: ASTA implies that N is Poisson. The following result (essentially the same as Theorem 4 of MW) is a first step.

Theorem 2. *Suppose that N has a stochastic intensity* ν *as in Theorem* 1 *and* $\mu(t)=E[\nu(t)\mid X(t)]$. *Then* $\mathrm{Cov}[f(X(t)),\mu(t)]=0$ *for all bounded continuous real-valued f if and only if* $\mu(t)=E[\mu(t)]$ *w.p.1. for* $t\geqq 0$.

Proof. Suppose that

(3.1) $$E[f[X(t)]\mu(t)]=E[f[X(t)]]E[\mu(t)]$$

for all bounded continuous real-valued f. Then, by taking limits (pp. 7–9 of Billingsley (1968)), (3.1) is valid for all bounded measurable f, so that $E[\mu(t)]$ is a version of the conditional expectation $E[\mu(t)\mid X(t)]$. Since $E[\mu(t)\mid X(t)]=\mu(t)$, we have $\mu(t)=E[\mu(t)]$ w.p.1. The other direction is trivial.

Corollary. *If the stationarity condition* (2.5) *holds in addition to the assumptions of Theorem* 2, *then the following are equivalent*:

(i) $\mathrm{Cov}[f(X(t)), \mu(t)] = 0$, $t \geqq 0$, *for all bounded continuous real-valued* f;

(ii) *LBA holds for* X *and* N;

(iii) *ASTA holds in the form of* (1.3), *i.e.*, $X(\infty) \stackrel{d}{=} \tilde{X}(\infty)$;

(iv) $\mu(t) = E[\mu(0)]$ *w.p.*1, $t \geqq 0$;

(v) $\mu(t)$ *is independent of* $X(t)$, $t \geqq 0$.

As indicated above, if N is independent of X, then Theorem 2 says nothing; but if the evolution of N beyond t (i.e., the process $\{N(t, t+u) : u \geqq 0\}$ depends on the past $\{N(s) : s < t\}$ of N only through the present state $X(t)$ of X, then we can conclude that N is Poisson by applying Watanabe's (1964) martingale characterization of a Poisson process; see p. 25 of Brémaud (1981). Let $\{\mathscr{F}_t^X : t \geqq 0\}$ and $\{\mathscr{F}_t^N : t \geqq 0\}$ be the separate internal histories of X and N, respectively. Of course, these are sub-σ-fields of the internal history of (X, N), which we have denoted by $\{\mathscr{F}_t : t \geqq 0\}$. Since the sample paths of N are right-continuous, while the sample paths of X are left-continuous, we cannot expect to have $\mathscr{F}_t^N \subseteq \mathscr{F}_t^X$, because a jump at t registers in $\mathscr{F}_t^N$ but not in $\mathscr{F}_t^X$. However, we can expect to have $\mathscr{F}_{t-}^N \subseteq \mathscr{F}_t^X$ for all t, where $\mathscr{F}_{t-}^N$ is the σ-field generated by $\{N(s) : s < t\}$. For example, we have $\mathscr{F}_{t-}^N \subseteq \mathscr{F}_t^X$ for all t whenever the occurrence of a jump in N at t can be identified from the two values $X(t-)$ and $X(t+)$ for all t.

Before stating our anti-PASTA result (Theorem 5 below), we state a preliminary result which applies to non-stationary processes and clearly shows the role ASTA plays in anti-PASTA.

Proposition 3. *Suppose that* N *has a stochastic intensity* $\nu(t)$ *with respect to* $\mathscr{F}_t \equiv \mathscr{F}_t^{X,N}$ *as in Theorem* 1, $\mu(t) = E[\nu(t) \mid X(t)]$ *as in* (2.6) *and the following four conditions hold*:

(i) $P(N(t) - N(t-) \geqq 1) = 0$, $t > 0$,

(ii) $\mathscr{F}_{t-}^N \subseteq \mathscr{F}_t^X$, $t > 0$,

(iii) $E[\nu(t) \mid \mathscr{F}_t^X] = E[\nu(t) \mid X(t)] \equiv \mu(t)$ *w.p.*1, $t > 0$,

(iv) $\mu(t) = E[\mu(t)]$ *w.p.*1, $t > 0$.

Then $\nu(t) = E[\nu(t)]$ *w.p.*1 *for* $t > 0$ *and* N *is a* $(P, \mathscr{F}_t^N)$*-Poisson process with* (*possibly time-dependent*) *intensity* $E[\mu(t)] = E[\nu(t)]$, $t \geqq 0$.

Proof. Since ν is a stochastic intensity of N with respect to $\mathscr{F}_t$, ν is adapted to $\mathscr{F}_t$. Hence, $\nu(t) = E[\nu(t) \mid \mathscr{F}_t]$ w.p.1, $t > 0$. By conditions (i) and (ii), $E[\nu(t) \mid \mathscr{F}_t] = E[\nu(t) \mid \mathscr{F}_t^X]$ w.p.1, $t \geqq 0$. Combining conditions (i)–(iii), we see that $\nu(t) = E[\nu(t) \mid X(t)] \equiv \mu(t)$ w.p.1, $t > 0$. Given this, condition (iv) implies that $\nu(t) = E[\mu(t)] = E[\nu(t)]$ w.p.1, $t > 0$. Since ν is assumed to be a stochastic intensity with respect to $\mathscr{F}_t$ for which $E[\int_0^t \nu(s)ds] < \infty$ (see Theorem 1), $N(t) - \int_0^t \nu(s)ds$ is an $\mathscr{F}_t$-martingale, by the integration theorem, p. 27 of Brémaud (1981). Since $\nu(t) = E[\nu(t)]$ w.p.1, $t > 0$, N is a $(P, \mathscr{F}_t)$-Poisson process with intensity $E[\nu(t)]$ by Watanabe's (1964) theorem, p. 25 of Brémaud (1981). Since $\mathscr{F}_t^N \subseteq \mathscr{F}_t$, $E[\nu(t) \mid \mathscr{F}_t^N] = E[\nu(t)]$ is a stochastic intensity of N with respect to $\mathscr{F}_t^N$; see p. 32 of Brémaud (1981). Hence, N is also a $(P, \mathscr{F}_t^N)$-Poisson process with intensity $E[\nu(t)]$.

Remark 3.1. Condition (i) is a minor regularity condition, which we usually expect to hold; see Proposition 4 below. The standard way to obtain conditions (ii) and (iii) is to have

$$N(t+)-N(t-)=g[X(t-),X(t+)], \qquad t \geqq 0, \tag{3.2}$$

for some $\{0, 1\}$-valued measurable function g (an indicator function) with X Markov. By Theorem 2, condition (iv) is nearly equivalent to LBA for X and N and thus ASTA; it is equivalent under the stationarity condition (2.5).

Proposition 4. *Under the stationarity condition* (2.5), $P(N(t)-N(t-)\geqq 1)=0$ *for all* t.

Proof. Note that

$$\begin{aligned} P(N(t)-N(t-)>\varepsilon) &= \lim_{n\to\infty} P(N(t-n^{-1},t+n^{-1})>\varepsilon) \\ &\leqq \lim_{n\to\infty} P(N(0,2/n)>\varepsilon) \quad \text{by (2.5)} \\ &\leqq \lim_{n\to\infty} \varepsilon^{-1}E[N(0,2/n)] \quad \text{by Markov's inequality} \\ &= \lim_{n\to\infty} (\varepsilon n)^{-1}E[N(0,2)]=0 \quad \text{by (2.5).} \end{aligned}$$

Finally, we combine Propositions 3 and 4 and Remark 3.1 to obtain our basic anti-PASTA result.

Theorem 5 (anti-PASTA). *Suppose that N has a stochastic intensity* ν *with respect to* $\mathscr{F}_t$ *as in Theorem* 1. *Also suppose that* (2.5), (2.6) *and* (3.2) *hold. If, in addition, X is Markov and ASTA holds, then N is a* $(P, \mathscr{F}_t^N)$*-Poisson process with constant intensity* $E[\nu(0)]$.

Proof. By Proposition 4, condition (i) of Proposition 3 is satisfied. Since X is Markov, has left-continuous sample paths and (3.2) holds, conditions (ii) and (iii) of Proposition 3 hold. By the Corollary to Theorem 2, ASTA implies that condition (iv) of Proposition 3 holds with $E[\nu(t)]=E[\nu(0)]$ for all t.

Remark 3.2. For the case in which X is a continuous-time Markov chain and (3.2) holds, there are many closely related results; see Melamed (1982), Section 2.10 of Walrand (1988), Green and Melamed (1990) and Serfozo (1989a); an additional result is Wolff (1990).

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