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Approximating the Admissible Set in Stochastic Dominance

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1. INTRODUCTION AND SUMMARY

In stochastic dominance applications it is often difficult to compute the admissible set. Of course work is being done to develop efficient algorithms, e.g., Bawa *et al.* (1979), but it is often necessary to use approximations, especially when there are many alternatives as with portfolios. A standard approximation procedure is to use a two-parameter admissible set such as the mean-variance admissible set instead of a theoretically appropriate admissible set determined by a set of utility functions (such as the risk-averse admissible set; see Section 4). It is of considerable interest to understand the properties of such an approximation procedure. It is significant that for certain special familes of probability distributions the two-parameter admissible sets coincide with the risk-averse admissible set; see Section 5. However, since a typical set of alternatives at best only can be approximated by such a special family of probability distributions, it is appropriate to ask how well the two-parameter admissible set approximates the risk-averse admissible set when the set of alternatives is close to such a special family. The purpose of this paper is to address this approximation question. This paper is thus similar in spirit to Samuelson (1970).

It should be evident that closely related questions arise in many economics contexts. Hence, we first investigate the approximation question in a more abstract setting and then return to stochastic dominance in Sections 4–8. Before introducing the abstract setting, we indicate how the later sections are organized. In Section 4, we show that the compactness and continuity conditions in Sections 2 and 3 are satisfied in the standard setting for stochastic dominance, i.e., for a large class of probability measures on the real line (with the topology of weak convergence; Billingsley (1968)) and for a large class of all nondecreasing concave utility functions (with the

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topology of uniform convergence over compact sets). In Section 5 we study location-scale families of probability distributions because they are prospective limiting sets of alternatives for which two-parameter rules work exactly. In Section 6 we show that the compactness and continuity conditions in Sections 2 and 3 are satisfied for the two-parameter admissible sets. In Section 7 we show that the continuity results for general sets of alternatives extend to portfolios. Finally, in Section 8 we show how all the pieces can be combined to at least partially answer the original question. In Section 8 we also discuss several extensions, including rates of convergence and first-order stochastic dominance.

Our general framework involves a set $\mathscr{A}$ of alternatives and a set $\mathscr{U}$ of real-valued functions on $\mathscr{A}$. The set $\mathscr{U}$ determines a partial order $\lesssim$ on $\mathscr{A}$, called the $\mathscr{U}$-order: $P \lesssim Q$ for $P, Q \in \mathscr{A}$ if $u(P) \leqslant u(Q)$ for all $u \in \mathscr{U}$. Our notation is motivated by the special case of stochastic dominance, in which the elements of $\mathscr{A}$ are probability measures, the elements of $\mathscr{U}$ are utility functions and $u(P)$ is the expectation of u with respect to P. With stochastic dominance, we thus follow the common practice of simultaneously regarding u as a function on the underlying sample space and as the function on the space of probability measures that is induced by the expectation. Even with stochastic dominance, however, we wish to consider $u(P)$ when u is a function of P not obtained as the expectation of a utility function. For example, $u(P)$ might be the median of P or a measure of dispersion such as the variance.

As usual, let equivalence $P \sim Q$ hold if $P \lesssim Q$ and $Q \lesssim P$. Let strict order $P < Q$ hold if $P \lesssim Q$ and not $P \sim Q$. Hence, $P < Q$ means $u(P) \leqslant u(Q)$ for *all* $u \in \mathscr{U}$ and $u(P) < u(Q)$ for *some* $u \in \mathscr{U}$. Let the subset of dominated elements be

$$\operatorname{dom} \mathscr{A} = \{P \in \mathscr{A} \mid \exists Q \in \mathscr{A} \text{ such that } P < Q\}$$

and let the subset of admissible elements be

$$\operatorname{adm} \mathscr{A} = \mathscr{A} - \operatorname{dom} \mathscr{A}.$$

we write $\operatorname{adm}(\mathscr{A}, \mathscr{U})$ and $\lesssim_{\mathscr{U}}$ when $\mathscr{U}$ is to be emphasized.

We are interested in the way $\operatorname{adm}(\mathscr{A}, \mathscr{U})$ changes as $\mathscr{A}$ and $\mathscr{U}$ change. Before describing the results, we introduce the usual closed-convergence of subsets of a metric space; e.g., pp. 15–21 of Hildenbrand (1974). Let $\{\mathscr{B}_n\}$ be a sequence of nonempty subsets of a metric space (X, d). We denote by $\operatorname{Li}(\mathscr{B}_n)$ (resp. $\operatorname{Ls}(\mathscr{B}_n)$) the subset of all elements x in X for which there exists a sequence $\{x_n\}$ with $x_n \in \mathscr{B}_n$ for all n such that $\{x_n\}$ (resp. a subsequence of $\{x_n\}$) converges to x as $n \to \infty$. Of course, $\operatorname{Li}(\mathscr{B}_n) \subseteq \operatorname{Ls}(\mathscr{B}_n)$. We write $\lim_{n \to \infty} \mathscr{B}_n = \mathscr{B}$ or just $\mathscr{B}_n \to \mathscr{B}$ if $\operatorname{Li}(\mathscr{B}_n) = \mathscr{B} = \operatorname{Ls}(\mathscr{B}_n)$. In general, closed-convergence is not topological, but convergence of closed subsets of a compact metric space is topologizable and can be represented by the Hausdorff metric

$$h(\mathscr{B}_1, \mathscr{B}_2) = \inf\{\varepsilon > 0: \mathscr{B}_1 \subseteq \mathscr{B}_2^\varepsilon \text{ and } \mathscr{B}_2 \subseteq \mathscr{B}_1^\varepsilon\},$$

where

$$\mathscr{B}_1^\varepsilon = \{x \in X: \exists y \in \mathscr{B}_1, d(x, y) \leqslant \varepsilon\}.$$

To get a feel for closed-convergence and the possible applications, suppose $\mathscr{B}_n$, $n \geqslant 1$, and $\mathscr{B}$ are subsets of the real line with $\mathscr{B} = [0, 1]$ and $\mathscr{B}_n = \{k/n \mid k = 1,..., n\}$; then $\mathscr{B}_n \to \mathscr{B}$.

In this paper we establish sufficient conditions for continuity, i.e., for $\text{adm}(\mathscr{A}_n, \mathscr{U}_n) \to \text{adm}(\mathscr{A}, \mathscr{U})$ and $\text{dom}(\mathscr{A}_n, \mathscr{U}_n) \to \text{dom}(\mathscr{A}, \mathscr{U})$ when $\mathscr{A}_n \to \mathscr{A}$ and $\mathscr{U}_n \to \mathscr{U}$. Part of the difficulty is illustrated by the fact that strict inequality is not preserved by convergence: if $\{a_n\}$ and $\{b_n\}$ are sequence of real numbers such that $a_n \to a$, $b_n \to b$, and $a_n < b_n$ for each n, then $a \leqslant b$ but not necessarily $a < b$. For this reason, we either restrict attention to finite sets of alternatives (Section 2) or use a different (new) ordering $\lesssim_{c,d}$ (Section 3), defined by $P \lesssim_{c,d} Q$ if $u(P) \leqslant u(Q) - c$ for all $u \in U$ and $u(P) \leqslant u(Q) - d$ for some $u \in U$. Of course, the case of interest is $c < d$. For $d > 0$, $\lesssim_{c,d}$ is irreflexive and, for $c < 0 < d$, $\lesssim_{c,d}$ is not transitive. Of course, $P < Q$ if and only if $P \lesssim_{0,\varepsilon} Q$ for some $\varepsilon < 0$. We introduce (c, d)-dominance as a technical device to obtain positive results, but we also believe it is a concept of independent interest. It is related to approximate equilibria and ε-cores; e.g., Hildenbrand *et al.* (1973).

For brevity, many of the proofs are omitted. For additional details, see the unabridged version, Goroff and Whitt (1977), henceforth referred to as GW. For additional discussion, also see Whitt (1978).

2. Finite Sets of Alternatives

Our basic assumptions for the next two sections are:

A1. $\mathscr{A}_n$, $n \geqslant 1$, and $\mathscr{A}$ are subsets of a metric space X.

A2. $\mathscr{U}_n$, $n \geqslant 1$, and $\mathscr{U}$ are subsets of a metric space Y.

A3. Joint continuity of $u(P)$: if $u_n \to u$ and $P_n \to P$, where $u_n \in \mathscr{U}_n$ and $P_n \in \mathscr{A}_n$ for each n, then $u_n(P_n) \to u(P)$.

In this section we also assume:

A4. The subsets A_n, $n \geqslant 1$, and A are finite, each having k elements.

A5. The elements of $\mathscr{A}$ are $\mathscr{U}$-distinct: $P \not\sim Q$ for all $P, Q \in \mathscr{A}$ with $P \neq Q$.

For part of our results, we also need to assume

A6. Ordered elements of $\mathscr{A}$ do not touch: $u(P) < u(Q)$ for all $u \in \mathscr{U}$ if $P \lesssim Q$ for all $P, Q \in \mathscr{A}$ with $P \neq Q$.

Note that A5 and A6 apply to $(\mathscr{A}, \mathscr{U})$ but not necessarily $(\mathscr{A}_n, \mathscr{U}_n)$, $n \geqslant 1$. Also note that A6 implies A5.

THEOREM 2.1. *Suppose* A1–A5 *hold*, $\mathscr{A}_n \to \mathscr{A}$ *and* $\mathscr{U}_n \to \mathscr{U}$. *Then*

(a) $\mathrm{Ls}(\mathrm{dom}(\mathscr{A}_n, \mathscr{U}_n)) \subseteq \mathrm{dom}(\mathscr{A}, \mathscr{U})$ *and* $\mathrm{adm}(\mathscr{A}, \mathscr{U}) \subseteq \mathrm{Li}(\mathrm{adm}(\mathscr{A}_n, \mathscr{U}_n))$.

(b) *If Y is compact and* A6 *holds, then* $\mathrm{dom}(\mathscr{A}_n, \mathscr{U}_n) \to \mathrm{dom}(\mathscr{A}, \mathscr{U})$ *and* $\mathrm{adm}(\mathscr{A}_n, \mathscr{U}_n) \to \mathrm{adm}(\mathscr{A}, \mathscr{U})$.

The proof is based on the following elementary properties of closed convergence:

LEMMA 2.1. *If* $\mathscr{A}_n \to \mathscr{A}$ *and* $\mathscr{B}_n \subseteq \mathscr{A}_n$ *for each n, then* $\mathscr{A} - \mathrm{Li}(\mathscr{B}_n) \subseteq \mathrm{Ls}(\mathscr{A}_n - \mathscr{B}_n)$.

LEMMA 2.2. *If* $\mathscr{A}_n \to \mathscr{A}$ *and* $\mathscr{B}_n \subseteq \mathscr{A}_n$ *for all n. then the following are equivalent*:

(i) $\mathscr{B}_n \to \mathscr{B}$ *and* $\mathscr{A}_n - \mathscr{B}_n \to \mathscr{A} - \mathscr{B}$

and

(ii) $\mathrm{Ls}(\mathscr{B}_n) \subseteq \mathscr{B}$ *and* $\mathrm{Ls}(\mathscr{A}_n - \mathscr{B}_n) \subseteq \mathscr{A} - \mathscr{B}$.

Proof of Theorem 2.1. Note that A1, A4, and $\mathscr{A}_n \to \mathscr{A}$ imply that the elements of $\mathscr{A}_n$ and $\mathscr{A}$ may be relabled so that

$$\mathscr{A}_n = \{P_{n1}, \ldots, P_{nk}\}, \quad \mathscr{A} = \{P_1, \ldots, P_k\} \quad \text{and} \quad P_{nj} \to P_j, \quad j = 1, \ldots, k.$$

(a) By Lemma 2.1, the second conclusion follows from the first. To establish the first, suppose that $\{P_{n'}\}$ is a subsequence of a sequence $\{P_n\}$ with $P_n \in \mathscr{A}_n$ for all n, $P_{n'} \in \mathrm{dom}(\mathscr{A}_{n'}, \mathscr{U}_{n'})$ for all n' and $P_{n'} \to P$ as $n' \to \infty$. Since $P_{n'} \to P$, $P = P_j$ for some j and there exists n'_0 such that $P_{n'} = P_{n'j}$ for all $n' \geqslant n'_0$. Henceforth, assume $n' \geqslant n'_0$. Since $P_{n'j} \in \mathrm{dom}(\mathscr{A}_{n'}, \mathscr{U}_{n'})$, there exists $Q_{n'} \in \mathscr{A}_{n'}$ such that $P_{n'j} < Q_{n'}$ for all n'. By A4, there exists an i, $i \neq j$, such that $Q_{n'} = P_{n'i}$ for infinitely many n'. Hence, there is a subsequence $\{Q_{n''}\}$ of the sequence $\{Q_{n'}\}$ such that $Q_{n''} = P_{n''i}$ and $P_{n''j} < P_{n''i}$ for all n''. Hence, $u_{n''}(P_{n''j}) \leqslant u_{n''}(P_{n''i})$ for all $u_{n''} \in \mathscr{U}_{n''}$ and $u^0_{n''}(P_{n''j}) < u^0_{n''}(P_{n''i})$ for some $u^0_{n''} \in \mathscr{U}_{n''}$. Since $\mathscr{U}_n \to \mathscr{U}$, for any $u \in \mathscr{U}$, there is a sequence $\{u_n\}$ with $u_n \in \mathscr{U}_n$ and $u_n \to u$. For such a u and $\{u_n\}$, $u_{n''}(P_{n''j}) \to u(P_j)$ and $u_{n''}(P_{n''i}) \to$

$u(P_i)$ by A3. Hence, $u(P_j) \leqslant u(P_i)$ for all $u \in \mathscr{U}$. The strict inequality $u^0_{n''}(P_{n''j}) < u^0_{n''}(P_{n''i})$ may be lost in the limit, but A5 implies that $u(P) < u(Q)$ for some $u \in \mathscr{U}$, so that $P \in \text{dom}(\mathscr{A}, \mathscr{U})$, i.e., $\text{Ls}\,\text{dom}(\mathscr{A}_n, \mathscr{U}_n) \subseteq \text{dom}(\mathscr{A}, \mathscr{U})$.

(b) By Lemma 2.2, it suffices to show that $\text{Ls}(\text{dom}(\mathscr{A}_n, \mathscr{U}_n)) \subseteq \text{dom}(\mathscr{A}, \mathscr{U})$ and $\text{Ls}(\text{adm}(\mathscr{A}_n, \mathscr{U}_n)) \subseteq \text{adm}(\mathscr{A}, \mathscr{U})$. The first inclusion has been established in part (a). Now suppose that $\{P_{n'}\}$ is a subsequence of a sequence $\{P_n\}$ with $P_n \in \mathscr{A}_n$ for all n, $P_{n'} \in \text{adm}(\mathscr{A}_{n'}, \mathscr{U}_{n'})$ for all n' and $P_{n'} \to P$. Again, we can assume $P = P_j$ and $P_{n'} = P_{n'j}$ for all n'. Let P_i be an arbitrary element of $\mathscr{A}$ with $P_i \neq P_j$. Then $P_{ni} \to P_i$. Since $P_{n'j} \in \text{adm}(\mathscr{A}_{n'}, \mathscr{U}_{n'})$, there exists a $u_{n'}$ such that $u_{n'}(P_{n'j}) \geqslant u_{n'}(P_{n'i})$ for all n'. Since Y is compact, the sequence $\{u_{n'}\}$ has a convergent subsequence $\{u_{n''}\}$ with limit u. Since $\mathscr{U}_n \to \mathscr{U}$, $u \in \mathscr{U}$. By A3, $u(P_j) \geqslant u(P_i)$. By A5, either $u(P_j) > u(P_i)$ or there exists some other $u_0 \in \mathscr{U}$ such that $u_0(P_j) > u_0(P_i)$. If no such u_0 existed, then $u(P_j) \leqslant u(P_i)$ for all u, which implies by A6 that $u(P_j) < u(P_i)$ for all u, which is a contradiction. Hence, $P \equiv P_j \in \text{adm}(\mathscr{A}, \mathscr{U})$, so that $\text{Ls}(\text{adm}(\mathscr{A}_n, \mathscr{U}_n)) \subseteq \text{adm}(\mathscr{A}, \mathscr{U})$, which completes the proof. ∎

We now state an elementary convergence result in which only the sets $\mathscr{U}_n$ change. Write $\mathscr{B}_n \uparrow \mathscr{B}$ if $\mathscr{B}_n \subseteq \mathscr{B}$ for all n and $\bigcup_{n=1}^{\infty} \mathscr{B}_n = \mathscr{B}$. Write $\mathscr{B}_n \downarrow \mathscr{B}$ if $\mathscr{B}_n^c \uparrow \mathscr{B}^c$, where $\mathscr{B}^c = X - \mathscr{B}$.

THEOREM 2.2. *If* A1–A4 *hold,* $\mathscr{A}_n = \mathscr{A}$ *for all n and* $\mathscr{U}_n \uparrow \mathscr{U}$, *then there exists* n_0 *such that* $\text{adm}(\mathscr{A}, \mathscr{U}_n) = \text{adm}(\mathscr{A}, \mathscr{U})$ *for* $n \geqslant n_0$.

Proof [GW, Theorem 5.2(b)].

Remark. One cannot say that $\text{adm}(\mathscr{A}, \mathscr{U}_n) \to \text{adm}(\mathscr{A}, \mathscr{U})$ if A4 is not assumed in Theorem 2.2; Example 5.2 of GW.

We conclude this section with some elementary set inclusion relations that complement the limit theorems. They obviously do not use the assumptions (except A5 in part (b)).

THEOREM 2.3. (a) *If* $\mathscr{A}_1 \subseteq \mathscr{A}_2$, *then* $\text{adm}(\mathscr{A}_1, \mathscr{U}) \supseteq \text{adm}(\mathscr{A}_2, \mathscr{U}) \cap \mathscr{A}_1$.

(b) *If* $\mathscr{U}_1 \subseteq \mathscr{U}_2$ *and* A5 *holds for* $\mathscr{U}_1$, *then* $\text{adm}(\mathscr{A}, \mathscr{U}_1) \subseteq \text{adm}(\mathscr{A}, \mathscr{U}_2)$.

Remark. The inclusion in (b) can fail if assumption A5 for $\mathscr{U}_1$ does not hold; p. 21 of GW.

3. (c, d)-DOMINANCE

We now consider more general sets of alternatives using the ordering $\lesssim_{c,d}$ with $c < d$. Recall that $P \lesssim_{c,d} Q$ if $u(P) \leqslant u(Q) - c$ for *all* $u \in \mathscr{U}$ and $u(P) \leqslant u(Q) - d$ for *some* $u \in \mathscr{U}$.

Let

$$\mathrm{dom}_{c,d}(\mathscr{A},\mathscr{U})=\{P\in\mathscr{A}\mid \exists Q\in\mathscr{A}\text{ with }P\lesssim_{c,d}Q\}$$

and

$$\mathrm{adm}_{c,d}(\mathscr{A},\mathscr{U})=\mathscr{A}-\mathrm{dom}_{c,d}(\mathscr{A},\mathscr{U}).$$

It is significant that in this setting we no longer need assumptions A5 and A6, but we still assume A1–A3.

For the proofs of the following results, see Section 3 of GW.

THEOREM 3.1. *If X is compact,* $\mathrm{Ls}(\mathscr{A}_n)\subseteq\mathscr{A}$ *and* $\mathscr{U}_n\to\mathscr{U}$, *then* $\mathrm{Ls}(\mathrm{dom}_{c,d}(\mathscr{A}_n,\mathscr{U}_n))\subseteq\mathrm{dom}_{c,d}(\mathscr{A},\mathscr{U})$.

It is easy to see that

$$\mathrm{Ls}(\mathrm{adm}_{c,d}(\mathscr{A}_n,\mathscr{U}_n))\subseteq\mathrm{adm}_{c,d}(\mathscr{A},\mathscr{U})$$

does *not* hold in general under the assumptions of Theorem 3.1 when $\mathscr{A}_n\to\mathscr{A}$. However, positive results can be obtained by modifying the parameters c and d. First, it is easy to verify

LEMMA 3.1. $\mathrm{adm}_{c,d}(\mathscr{A},\mathscr{U})\subseteq\mathrm{adm}_{c+\epsilon_1,d+\epsilon_2}(\mathscr{A},\mathscr{U})$ *for all* $\varepsilon_1,\varepsilon_2>0$.

Lemma 3.1 leads us to define

$$\mathrm{adm}_{c+,d+}(\mathscr{A},\mathscr{U})=\bigcap_{\epsilon>0}\mathrm{adm}_{c+\epsilon,d+\epsilon}(\mathscr{A},\mathscr{U}).$$

THEOREM 3.2. *If* $\mathscr{A}_n\to\mathscr{A}$ *and* $\mathscr{U}_n\to\mathscr{U}$, *then* $\mathrm{Ls}(\mathrm{adm}_{c,d}(\mathscr{A}_n,\mathscr{U}_n))\subseteq\mathrm{adm}_{c+,d+}(\mathscr{A},\mathscr{U})$.

We now combine Theorems 3.1 and 3.2 to obtain

THEOREM 3.3. *If X is compact,* $\mathscr{A}_n\to\mathscr{A}$ *and* $\mathscr{U}_n\to\mathscr{U}$, *then for any* ε_1, $\varepsilon_2>0$

$$\mathrm{adm}_{c,d}(\mathscr{A},\mathscr{U})\subseteq\mathrm{Li}(\mathrm{adm}_{c,d}(\mathscr{A}_n,\mathscr{U}_n))\subseteq\mathrm{Ls}(\mathrm{adm}_{c,d}(\mathscr{A}_n,\mathscr{U}_n))$$
$$\subseteq\mathrm{adm}_{c+,d+}(\mathscr{A},\mathscr{U})\subseteq\mathrm{adm}_{c+\epsilon_1,d+\epsilon_2}(\mathscr{A},\mathscr{U}).$$

We conclude this section with an analog of Theorem 2.2.

THEOREM 3.4. *If* A1–A3 *hold, X is compact,* $\mathscr{A}_n=\mathscr{A}$ *for all n and* $\mathscr{U}_n\uparrow\mathscr{U}$, *then* $\mathrm{adm}(\mathscr{A},\mathscr{U}_n)\uparrow\mathrm{adm}(\mathscr{A},\mathscr{U})$.

Proof [GW, Theorem 5.2(c)].

4. Risk Averse Decision Makers

We now focus on the special case of stochastic dominance for risk averse decision markers.

Let Y be the set of continuous real-valued functions on the real line R with convergence $u_n \to u$ being uniform convergence on every compact set. This space is known to be metrizable; Chapter XII of Dugundji (1966). Let $\mathscr{U}_2$ be the subset of Y consisting of all concave nondecreasing functions. Since utility functions are typically unique only up to a positive linear transformation, we normalize the elements of $\mathscr{U}_2$ by requiring that $u(0)=0$ and $u(1)=1$ for all $u \in \mathscr{U}_2$. We thus rule out total indifference or even indifference in the interval $[0, 1]$. We consider a subset of $\mathscr{U}_2$ given in terms of a (fixed) function $g \in \mathscr{U}_2$ with $g^+(0) \geqslant 1$. (Of course the right derivative $u^+(x)$ exists for each $u \in \mathscr{U}_2$.) Define

C1. $\mathscr{U}^g = \{u \in \mathscr{U}_2 \mid u^+(x) \leqslant g^+(x) \text{ for all } x \leqslant 0\}$.

A possible choice of g is $g(x) = Mx$, which can be used whenever the choice of relevant x values is bounded below.

Let X be the set of probability measures on the real line with convergence being weak convergence, i.e., $P_n \to P$ if $\int f\, dP_n \to \int f\, dP$ for all bounded continuous real-valued functions f; see Billingsley (1968). This space is known to be metrizable. In order to have $u(P)$ defined for all $u \in \mathscr{U}^g$, consider the subset $\mathscr{P}^g$ of X given in terms of the fixed function $g \in \mathscr{U}_2$ by

C2. $\mathscr{P}^g = \{P \in X \mid \int |h(x)|^{1+\delta}\, dP(x) \leqslant M_2\}$,

where δ and M_2 are arbitrary positive constants and

$$\begin{aligned} h(x) &= g(x), && x \leqslant 0, \\ &= \min\{1, xg^+(0)\}, && 0 \leqslant x \leqslant 1, \\ &= x, && x \geqslant 1. \end{aligned}$$

Obviously, $\mathscr{U}^g$ increases and $\mathscr{P}^g$ decreases as $g(x)$ decreases (as $g^+(x)$ increases). If $g(x) = M_1 x$, then C2 just means that the $(1+\delta)$th moments are uniformly bounded. If the space of possible outcomes is contained in a bounded interval, then we can have $\mathscr{P}^g = X$.

Theorem 4.1. *$\mathscr{U}^g$ and $\mathscr{P}^g$ in* C1 *and* C2 *are compact metric spaces such that $u_n(P_n) \to u(P)$ when $u_n \to u$ and $P_n \to P$.*

Proof. We first use the Arzela–Ascoli Theorem to show that $\mathscr{U}^g$ is compact; p. 267 of Dugundji (1966). Note that $u \in \mathscr{U}^g$ oscillates at most

$g^+(x \wedge 0)\varepsilon$ in the interval $[x, x+\varepsilon]$; so the elements of $\mathscr{U}^g$ are equicontinuous. Also,

$$\{u(x) \mid u \in \mathscr{U}^g\} = [1, x], \qquad x \geqslant 1,$$
$$= [x, \min\{1, xg^+(0)\}, \qquad 0 \leqslant x \leqslant 1,$$
$$= [g(x), x], \qquad x \leqslant 0,$$

and the above sets are compact. We finally argue that $\mathscr{U}^g$ is closed in Y. It is easy to verify that $\mathscr{U}_2$ is closed in Y. It remains to show that if $u_n \in \mathscr{U}^g$ and $u_n \to u$ then $u^+(x) \leqslant g^+(x)$ for all $x \leqslant 0$. The latter property is equivalent to requiring that $u(x_2) - u(x_1) \leqslant g(x_2) - g(x_1)$ for all $x_1 < x_2 < 0$, a property which is clearly preserved under pointwise limits.

We now use Prohorov's Theorem to show that $\mathscr{P}^g$ is compact; p. 35 of Billingsley. By Prohorov's Theorem, we will have shown that $\mathscr{P}^g$ has compact closure if we show that $\mathscr{P}^g$ is uniformly tight, but this follows from a minor variant of Chebyshev's inequality because, for any $P \in \mathscr{P}^g$ and any interval of the form $[-n, n]$,

$$P([-n, n]^c) \leqslant \frac{\int_{[-n,n]^c} |h(x)|^{1+\delta}\, dP(x)}{(\min\{|h(n)|, |h(-n)|\})^{1+\delta}} \leqslant \frac{M_2}{g(-n)^{1+\delta}}.$$

To show that $\mathscr{P}^g$ is itself closed, it is convenient to work with random variables X_n, $n \geqslant 1$, and X with probability laws P_n, $n \geqslant 1$, and P; let $X_n \to X$ mean convergence in law (weak convergence); p. 22 of Billingsley (1968). If $X_n \to X$, then $|h(X_n)|^{1+\delta} \to |h(X)|^{1+\delta}$ by the continuous mapping theorem; Theorem 5.1 of Billingsley (1968). Moreover, by Fatou (Theorem 5.3 of Billingsley (1968)),

$$E\,|h(X)|^{1+\delta} \leqslant \liminf_{n\to\infty} E\,|h(X_n)|^{1+\delta} \leqslant M_2.$$

Hence the law of X is in $\mathscr{P}^g$, i.e., $\mathscr{P}^g$ is closed.

We now verify the joint continuity. First, however, note that $u(P)$ is defined for every $u \in \mathscr{U}^g$ and $P \in \mathscr{P}^g$ because $|u(x)| \leqslant |h(x)|$ for all x and $|h(P)| \leqslant M_2^{(1+\delta)^{-1}}$, by Jensen's inequality. Finally, suppose that $u_n \to u$ in $\mathscr{U}^g$ and $X_n \to X$ ($P_n \to P$) in $\mathscr{P}^g$. By Theorem 5.5 of Billingsley (1968), $u_n(X_n) \to u(X)$ (convergence in law). Finally, $Eu_n(X_n) \equiv u_n(P_n) \to u(P) \equiv Eu(X)$ because $\{u_n(X_n)\}$ is uniformly integrable; p. 32 of Billingsley (1968). To see this, note that $|u_n(X_n)| \leqslant |h(X_n)|$ and $\{h(X_n)\}$ is uniformly integrable because $\sup_n E\,|h(X_n)|^{1+\delta} \leqslant M_2$. ∎

We now consider the relationship between $\mathscr{U}^g$-order and $\mathscr{U}_2$-order. Let $\mathscr{P}_2$ be a closed subset of $\mathscr{P}^g$ such that $u(P)$ is defined for all $P \in \mathscr{P}_2$ and $u \in \mathscr{U}_2$. It is easy to see that $\mathscr{U}_2$-order need not coincide with $\mathscr{U}_c$-order in $\mathscr{P}_2$;

Example 5.1 of GW. However, we can apply Theorems 2.2, 2.3, and 3.4 to obtain:

THEOREM 4.2. (a) $\operatorname{adm}(\mathscr{A},\mathscr{U}_c) \subseteq \operatorname{adm}(\mathscr{A},\mathscr{U}_2)$ *for any* $\mathscr{A} \subseteq \mathscr{P}_2$.

(b) *If* $\mathscr{A}$ *is a finite subset of* $\mathscr{P}_2$ *and* $\mathscr{U}_n \uparrow \mathscr{U}_2$, *then there exists* n_0 *such that* $\operatorname{adm}(\mathscr{A},\mathscr{U}_n) = \operatorname{adm}(\mathscr{A},\mathscr{U}_2)$ *for* $n \geqslant n_0$.

(c) *If* $\mathscr{U}_n \uparrow \mathscr{U}_2$, *then* $\operatorname{adm}_{c,d}(\mathscr{A},\mathscr{U}_n) \uparrow \operatorname{adm}_{c,d}(\mathscr{A},\mathscr{U}_2)$ *for* $d > 0$.

Proof. (a) By Theorem 2.3(b), it suffices to verify A5. If $u(P_1) = u(P_2)$ for all $u \in \mathscr{U}^g$, then $\operatorname{mean}(P_1) = \operatorname{mean}(P_2)$. Using utility functions of the form $a + bx + cxI_{(-\infty,t]}(x)$, where $I_B(x)$ is the indicator function of the set B, we see that $\int_{-\infty}^t x\,dP_1(x) = \int_{-\infty}^t x\,dP_2(x)$ for all t, which implies that $P_1 = P_2$.

Remarks. (1) Similar results hold for higher-order stochastic dominance, i.e., using $\mathscr{U}_3$ or $\mathscr{U}_4$ instead of $\mathscr{U}_2$; Bawa (1975, p. 100) or Fishburn (1976). For example, to treat $\mathscr{U}_3$, consider the subset of functions u in $\mathscr{U}^g$ such that $u(x+h) - 2u(x) + u(x-h)$ is nondecreasing in x for all x and h. Since this is a closed subset of $\mathscr{U}^g$, it is a compact metric space too.

(2) Similar results also hold if the underlying space of possible outcomes is k-dimensional Euclidean space R^k with the ordering $x \equiv (x_1,\dots,x_k) \leqslant (y_1,\dots,y_k) \equiv y$ if $x_j \leqslant y_j$, $1 \leqslant j \leqslant k$, or even a non-Euclidean space; see pp. 23–24 of GW.

(3) There is a problem with the ordering $\lesssim_{c,d}$ applied to $\mathscr{U}_2$. Suppose the space of possible outcomes is $[0,1]$ and $P(\{0\}) = Q(\{0\}) = 0$ for all P, $Q \in \mathscr{A}$. Then, for any $c > 0$, there exists a $u \in \mathscr{U}_2$ such that $u(P) > u(Q) - c$. Hence, $\operatorname{adm}_{c,d}(\mathscr{A},\mathscr{U}_2) = \mathscr{A}$ for all $0 < c \leqslant d$. To see this, consider $u_M(x) = \min\{Mx, 1\}$. For any $P \in \mathscr{A}$, $u_M(P) \to 1$ as $M \to \infty$. There are two possible resolutions to this problem. The first is to work with $c < 0 < d$. The second is to further restrict the set of test functions $\mathscr{U}_2$, for example, by bounding $u^+(x)$ below as well as above for each x. Both methods seem reasonable and both yield useful conclusions.

5. LOCATION-SCALE FAMILIES

One interesting possible set $\mathscr{A}$ of limiting alteratives in the setting of Section 4 is a location-scale family of probability measures; see Hanoch and Levy (1969) and Section 5 of Bawa (1975). To specify a location-scale family, let Ψ be a c.d.f. such that $0 < \Psi(x) < 1$ for some x and let Δ be a subset of $R \times (0,\infty)$. Then the set $\mathscr{A}$ is defined in terms of c.d.f.'s as

$$\mathscr{A} = \{F \mid F(x) = \Psi([x-l]/s) \text{ for all } x;\ (l,s) \in \Delta\}.$$

In other words, $\mathscr{A}$ is the set of probability laws of $sZ+l$ for $(l, s) \in \Delta$, where Z is a fixed real-valued random variable with distribution Ψ, i.e., $P(Z \leqslant x) = \Psi(x)$.

It is intuitively clear that a location-scale family $\mathscr{A}$ is essentially two-dimensional, so that $\mathscr{U}_2$-order in $\mathscr{A}$ should be easy to check. To make this precise, endow X with the weak convergence topology as in Section 4. The following result is easy to verify; it is also a consequence of the convergence of types theorem; p. 253 of Feller (1971).

THEOREM 5.1. *The location-scale family $\mathscr{A}$ in X is homeomorphic to Δ in R^2.*

COROLLARY. *$\mathscr{A}$ is a compact metric space if and only if Δ is compact.*

We next show how the U_2-order in a location-scale family is determined by an order of Δ. We remark that we do not assume that $\int x^2 \, d\Psi < \infty$. The first result follows from Bawa (1975) where a smaller set U_4 is considered. However it is shown there that for location-scale families U_2-order and the U_4-order coincide. Also, notice that Bawa does not require that $u(0) = 0$ and $u(1) = 1$.

THEOREM 5.2 (Bawa (1975)). *If $\mathscr{A}$ is a location-scale family for which $0 < \Psi(x) < 1$ for all x and the means exist, then $F_1 \geqslant F_2$ in $(\mathscr{A}, \mathscr{U}_2)$ if and only if mean $(F_1) \geqslant$ mean (F_2) and $s_1 \leqslant s_2$.*

Remarks. (1) It is easy to obtain corresponding characterizations when the condition $0 < \Psi(x) < 1$ for all x is relaxed; Bawa (1975). Then in some cases the condition for order requires $s_1 \geqslant s_2$ instead of $s_1 \leqslant s_2$.

(2) Note that if $F(x) = \Psi[(x-l)/s]$ and the variance of Ψ exists, then variance $(F_i) = s_i^2$ variance (Ψ). So, when $\int x^2 \, d\Psi$ exists, the order in Theorem 5.2 is the mean-variance rule.

The next result gives a different way to detemine the order in $\mathscr{A}$ when $\mu = \int x \, d\Psi = 0$. We use the quantiles of the distributions. Let $F^{-1}(p) = \inf\{t \mid F(t) > p\}$, $0 \leqslant p \leqslant 1$.

THEOREM 5.3. *Let $\mathscr{A}$ be a location-scale family for which Ψ is a* c.d.f. *with $\mu = 0$ and $0 < \Psi(x) < 1$ for all x. Then $F_1 \geqslant F_2$ in $(\mathscr{A}, \mathscr{U}_2)$ if and only if $F_1^{-1}(p_0) \geqslant F_2^{-1}(p_0)$, where $\Psi^{-1}(p_0) = 0$ and $F_1^{-1}(p_2) - F_1^{-1}(p_1) \leqslant F_2^{-1}(p_2) - F_2^{-1}(p_1)$ for $p_1 < p_2$.*

Proof. The equivalence is immediate from

$$F_i^{-1}(p_0) = s_i \Psi^{-1}(p_0) + l_i = l_i$$

and

$$F_i^{-1}(p_2) - F_i^{-1}(p_1) = s_i[\Psi^{-1}(p_2) - \Psi^{-1}(p_1)].$$

Remark. $\mathscr{U}^g$-order and $\mathscr{U}_2$-order need not agree in a location-scale family; p. 28 of GW.

Additional interesting two-parameter orderings are the mean-lower-partial-moment orderings advocated by Bawa (1975, 1978) and others. For general c.d.f.'s F_1 and F_2 with finite αth moments, we say $F_1 \geqslant_{m(t_0,\alpha)} F_2$ if mean $(F_1) \geqslant$ mean (F_2) and

$$\mathrm{LPM}_\alpha(t_0, F_1) = \int_{-\infty}^{t_0} (t_0 - y)^\alpha \, dF_1(y)$$

$$\leqslant \int_{-\infty}^{t_0} (t_0 - y)^\alpha \, dF_2(y) \equiv \mathrm{LPM}_\alpha(t_0, F_2).$$

We shall focus on the case $\alpha = 1$, but what follows can be extended to other α. Write $m(t_0)$ for $m(t_0, 1)$. Note that $m(t_0)$-ordering for all t_0 corresponds to $\mathscr{U}_2$-order; see Theorem 2 of Bawa (1975) or Theorem 2.3 of Brumelle and Vickson (1975). Considering only one t_0 together with the mean (which can be interpreted as $t_0 = \infty$) is a natural approximation. In general there is no simple characterization of $m(t_0)$-order.

THEOREM 5.4. *Let $\mathscr{A}$ be a location-scale family for which $\int x \, d\Psi(x) = 0$. Then $F_1 \geqslant_{m(t_0)} F_2$ in $\mathscr{A}$ if and only if*

$$l_1 \geqslant l_2 \quad \textit{and} \quad s_1^{-1} \int_{-\infty}^{s_1 t_0 + l_1} \Psi(x) \, dx \leqslant s_2^{-1} \int_{-\infty}^{s_2 t_0 + l_2} \Psi(x) \, dx.$$

Proof. By a change of variables,

$$\mathrm{LPM}(t_0, F_i) \equiv \int_{-\infty}^{t_0} F_i(y) \, dy = s_i^{-1} \int_{-\infty}^{s_i t_0 + l} \Psi(x) \, dx.$$

Remark. Closely related to the mean-lower-partial-moment ordering is the generalized safety-first rule in which we minimize $\mathrm{LPM}(t_0, F)$ over the subset of F in $\mathscr{A}$ with a given mean and then take a union over all means; Bawa (1978). In particular, let $\mathrm{SF}(t, \mu)$ be the subset of $\mathscr{A}$ attaining the minimum of $\mathrm{LPM}(t, F)$ given that mean $(F) = \mu$. Let $\mathrm{SF}(\mathscr{A}, t) = \bigcup_\mu \mathrm{SF}(t, \mu)$. Obviously, $\mathrm{adm}(\mathscr{A}, m(t_0)) \subseteq \mathrm{SF}(\mathscr{A}, t_0)$. If $\min\{\mathrm{LPM}(t_0, F)\colon F \in \mathscr{A}$, mean $(F) = \mu\}$ is strictly increasing in μ for $\mu \geqslant \mu^*$, where $\mu^* =$ mean (F) for an F attaining the minimum of $\mathrm{LPM}(t_0, F)$ over $\mathscr{A}$, then $\mathrm{adm}(\mathscr{A}, m(t_0)) = \mathrm{SF}(\mathscr{A}, t_0)$ if $\mathrm{SF}(\mathscr{A}, t_0)$ is redefined as the restricted union: $\mathrm{SF}(\mathscr{A}, t_0) =$

$\bigcup_{\mu:\mu \geqslant \mu^*} \mathrm{SF}(t_0, \mu)$. It is easy to see that $\mathscr{U}_2$-order and $\mathscr{U}^g$-order coincide in this special case. Moreover, if $t_0 < 0$ too, then $m(t_0)$-order, $\mathscr{U}_2$-order and $\mathscr{U}^g$-order all coincide. This follows from Theorems 5.2 and 5.4 because

$$s^{-1} \int_{-\infty}^{st_0+t} \Psi(x)\, dx$$

is then decreasing in s. The condition that $\min\{\mathrm{LPM}(t, F)\colon F \in \mathscr{A}, \text{mean } (F) = \mu\}$ be strictly increasing in μ for $\mu \geqslant \mu^*$ holds in a case of major interest, namely, with portfolios, because then we are minimizing a convex function over a convex set; Bawa (1978). However, to fit in with the rest of this section, the portfolio distributions must all belong to a single location-scale family. In that setting, this paper adds theoretical support for the procedure suggested by Bawa (1978).

For limit theorems in the setting of Section 2 in which $\mathscr{A}_n \to \mathscr{A}$ with $\mathscr{A}$ a location-scale family, we still need to verify the touching condition A6. Sufficient conditions are contained in:

THEOREM 5.5. *Let $\mathscr{A}$ be a location-scale family with $0 < \Psi(x) < 1$ for all x. If mean $(F_1) \neq$ mean (F_2) when $F_1 \neq F_2$ in $\mathscr{A}$, then $u(F_1) > u(F_2)$ for all (integrable) $u \in \mathscr{U}_2$ when $u(F_1) \geqslant u(F_2)$ for all (integrable) $u \in \mathscr{U}_2$* (A6 *holds*).

Proof [GW, p. 31.]

6. Two-Parameter Admissible Sets

We now want to show that the two-parameter admissible sets are close when sets of alternatives are close, where the sets of alternatives are general sets of probability measures on the real line. To treat the three cases discussed in Section 5, let MV, MLPM, and $q-r$ represent the two-parameter pairs consisting of the mean and variance, mean and lower partial moment at some t_0, and p-quantile and (p_1, p_2)-quantile range, respectively. Each of these two-parameter pairs corresponds to a $\mathscr{U}$-order for a set $\mathscr{U}$ containing two elements.

Beginning with the mean and variance, let

$$\mathscr{P}_{\mathrm{MV}} = \left\{ P \in X\colon \int |x|^{2+\delta}\, dP \leqslant M \right\},$$

where δ and M are arbitrary positive constants and X is the space of all probability measures on the real line still endowed with the topology of weak convergence. The following parallels Theorem 4.1, both in statement and proof.

THEOREM 6.1. *$\mathscr{P}_{\mathrm{MV}}$ is a compact subset of the metric space X such that mean(P_n) $\to$ mean(P) and Variance(P_n) $\to$ Variance(P) when $P_n \to P$ in $\mathscr{P}_{\mathrm{MV}}$.*

For the mean and lower partial moment, let

$$\mathscr{P}_{\mathrm{MLPM}} = \left\{ P \in \mathscr{P} : \int |x|^{1+\delta}\, dP(x) \leqslant M \right\},$$

where δ and M are arbitrary positive constants. By similar reasoning, we have

THEOREM 6.2. *$\mathscr{P}_{\mathrm{MLPM}}$ is a compact subset of X such that mean (P_n) $\to$ mean (P) and $\mathrm{LPM}(t_0, P_n) \to \mathrm{LPM}(t_0, P)$ when $P_n \to P$ in $\mathscr{P}_{\mathrm{MLPM}}$.*

It is interesting that with the quantiles no moment restrictions are needed. A natural way to get compactness is to assume that every probability measure is stochastically dominated above and below by fixed probability measures, i.e., work with

$$\mathscr{P}^b = \{P \in X \mid Q_1 \lesssim_1 P \lesssim_1 Q_2\},$$

where Q_1 and Q_2 are fixed probability measures in X and $P_1 \lesssim_1 P_2$ means $\int f\, dP_1 \leqslant \int f\, dP_2$ for all bounded nondecreasing real-valued functions f. Of course, if the set of possible outcomes is a compact interval, then we can let $\mathscr{P}_b = X$.

THEOREM 6.3. *$\mathscr{P}_b$ is a compact subset of X such that $F_n^{-1}(p_0) \to F^{-1}(p_0)$ and $F_n^{-1}(p_2) - F_n^{-1}(p_1) \to F^{-1}(p_2) - F^{-1}(p_1)$ when $F_n \to F$ in $\mathscr{P}_b$, provided the sets $\{t \mid F(t) = p_i\}$ each have only a single element for $i = 0, 1, 2$.*

Proof. To see that $\mathscr{P}_b$ has compact closure, note that

$$P([a, b]^c) \leqslant Q_1((-\infty, a)) + Q_2((b, \infty))$$

for any $P \in \mathscr{P}_b$ and apply Prohorov's theorem; p. 36 of Billingsley (1968). However, $\mathscr{P}_b$ is in fact closed because first-order stochastic dominance is a closed partial order: $P_{n1} \to P_1$, $P_{n2} \to P_2$ and $P_{n1} \lesssim_1 P_{n2}$ for all n implies $P_1 \lesssim_1 P_2$; Proposition 3 of Kamae *et al.* (1977). Next, convergence $P_n \to P$ is equivalent to pointwise convergence $F_n(t) \to F(t)$ for all t which are continuity points of F, which in turn is equivalent to convergence $F_n^{-1}(p) \to F^{-1}(p)$ at all continuity points p of F^{-1}. Since $\{t \mid F(t) = p_i\}$ has a single element, p_i is a continuity point of F^{-1}.

Remarks. (1) The condition on the sets $\{t \mid F(t) = p_i\}$ in Theorem 6.3 automatically holds for all p_i if F is an element of a location-scale family based on a strictly increasing Ψ.

(2) In order to compare two-parameter continuity results with the continuity in Section 4, let the set of possible probability measures be the intersection of $\mathscr{P}^g$ in Section 4 with $\mathscr{P}_{\mathrm{MV}}$, $\mathscr{P}_{\mathrm{MLPM}}$, or $\mathscr{P}_b$ here. If $g(x) = M_1 x$ in $\mathscr{U}^g$, then constraints can be chosen so that $\mathscr{P}_{\mathrm{MV}} \subseteq \mathscr{P}_{\mathrm{MLPM}} = \mathscr{P}^g$.

7. Portfolios

Let $\mathbf{Z} \equiv (Z_1, ..., Z_k)$ be a random vector in R^k with associated joint c.d.f. $H \equiv H(x_1, ..., x_k)$. We call $\mathbf{Z}$ the random rate of return vector associated with k investment opportunities. Let a *portfolio* be a vector $\pi \equiv (\pi_1, ..., \pi_k)$ in R^k, that is, a function π: $R^k \to R$, defined as $\pi(x_1, ..., x_k) = \pi_1 x_1 + \cdots \pi_k x_k$, where $\pi_1 + \cdots + \pi_k = 1$ and $\pi_j \geqslant 0$ for all j. The portfolio associated with a specific random rate of return vector $\mathbf{Z}$ is $\pi(\mathbf{Z}) = \pi_1 Z_1 + \cdots + \pi_k Z_k$, which has c.d.f.

$$F_\pi(t) = P(\pi(\mathbf{Z}) \leqslant t)$$
$$= \int_{-\infty}^{t} \int_{-\infty}^{t-x_k} \cdots \int_{-\infty}^{t-x_2-\cdots-x_k} H\left(\frac{dx_1}{\pi_1}, ..., \frac{dx_k}{\pi_k}\right).$$

Since π is a continuous function, we can apply the continuous mapping theorem, Theorem 5.1 of Billingsley (1968), to show that $\pi(\mathbf{Z}_n) \to \pi(\mathbf{Z})$ in law for any portfolio π if $\mathbf{Z}_n \to \mathbf{Z}$ in law.

Let Π be the set of all possible portfolios, i.e.,

$$\Pi = \{(\pi_1, ..., \pi_k): \pi_1 + \cdots + \pi_k = 1, \pi_j \geqslant 0, 1 \leqslant j \leqslant k\},$$

and $\Pi(\mathbf{Z})$ the set of all probability measures associated with a random rate of return $\mathbf{Z}$ and the set Π of all portfolios. It is easy to show that $\Pi(\mathbf{Z})$ is a compact subset of the metric space X. It is significant that $\Pi(\mathbf{Z})$ is *not* a convex subset of X even though the set of all possible portfolios Π is a convex subset of R^k. Continuity of the admissible set for portfolios follows from

THEOREM 7.1. *If* $\mathbf{Z}_n \to \mathbf{Z}$ *in law, then* $\Pi(\mathbf{Z}_n) \to \Pi(\mathbf{Z})$.

Remarks. (1) This sections shows that continuity in the sense of Section 3 holds for portfolios. For portfolios, of course, it is necessary that the joint distributions H_n converge.

(2) As a consequence of the remark following Theorem 5.4, if $\mathbf{Z}_n \to \mathbf{Z}$ in law where $\mathbf{Z}$ has a multivariate normal law or a multivariate stable law such that all portfolios have the same characteristic exponent and skewness parameter, Chapters 6 and 12 of Press (1972), then $\mathrm{adm}_{c,d}(\Pi(\mathbf{Z}), m(t_0)) = \mathrm{adm}_{c,d}(\Pi(\mathbf{Z}), \mathscr{U}_2)$, so that $\mathrm{adm}_{c,d}(\Pi(\mathbf{Z}_n), m(t_0))$ and $\mathrm{adm}_{c,d}(\Pi(\mathbf{Z}_n), \mathscr{U}_2)$ tend to be close in the sense of Theorem 3.3.

8. Conclusions and Extensions

We first briefly review how the various results in this paper can be combined to at least partially answer the question originally posed. For this purpose, let $\mathscr{U}_2$ be the set of risk-averse utility functions defined in Section 4 and let $\mathscr{U}_{\mathrm{MV}}$ be the two-parameter set of evaluation functions containing the mean and variance defined in Section 5. Let $\mathscr{A}_1$ be a location-scale family of probability measures on the real line with uniformly bounded second moment. By Theorem 6.2, $\operatorname{adm}(\mathscr{A}_1, \mathscr{U}_{\mathrm{MV}}) = \operatorname{adm}(\mathscr{A}_1, \mathscr{U}_2)$. Suppose $\mathscr{A}_1 \sim \mathscr{A}_2$, where $\sim$ means approximately equal. At the outset we asked the question: is $\operatorname{adm}(\mathscr{A}_2, \mathscr{U}_{\mathrm{MV}}) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}_2)$? In this paper we have answered this question affirmatively based on several specific interpretations for the approximation relation $\sim$. First, for any pair $(\mathscr{A}, \mathscr{U})$ of these $\mathscr{A}$ and $\mathscr{U}$ sets we regard $\operatorname{adm}_{c,d}(\mathscr{A}, \mathscr{U}) \sim \operatorname{adm}_{c+,d+}(\mathscr{A}, \mathscr{U})$ for appropriate c and d. In other words, we assume that the convergence involving (c, d)-dominance in Theorem 3.3 adequately represents the relation $\sim$. Then Theorems 3.3 and 4.1 imply that $\operatorname{adm}(\mathscr{A}_1, \mathscr{U}^g) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}^g)$ for any g consistent with the uniformly bounded second moment. Next Theorems 3.4 and 4.2 imply that $\operatorname{adm}(\mathscr{A}_1, \mathscr{U}^g) \sim \operatorname{adm}(\mathscr{A}_1, \mathscr{U}_2)$ and $\operatorname{adm}(\mathscr{A}_2, \mathscr{U}^g) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}_2)$ for appropriate g within the class above. As a consequence, $\operatorname{adm}(\mathscr{A}_1, \mathscr{U}_2) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}_2)$. By Theorems 3.3 and 6.1, $\operatorname{adm}(\mathscr{A}_1, \mathscr{U}_{\mathrm{MV}}) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}_{\mathrm{MV}})$. Finally, by transitivity, we have the desired relation: $\operatorname{adm}(\mathscr{A}_2, \mathscr{U}_{\mathrm{MV}}) \sim \operatorname{adm}(\mathscr{A}_2, \mathscr{U}_2)$.

The relation $\sim$ is established in each case above by a limit theorem. When the spaces of alternatives and evaluation functions are subsets of compact metric spaces, the convergence can be expressed in the Hausdorff metric h defined in Section 1. However, from the decision making point of view, one might wonder whether the Hausdorff metric measures distance in a way that is consistent with preferences. We now mention one way that is does.

Consider a fixed set $\mathscr{U}$ of test functions and the pseudometric

$$m(P, Q) = \sup_{u \in \mathscr{U}} |u(P) - u(Q)|.$$

In many applications m will be a metric on X, but clearly $m(P, Q) = 0$ need not imply $P = Q$ if the set $\mathscr{U}$ is small. Since order is defined on $\mathscr{A}$ via $\mathscr{U}$, it is natural to define the topology on $\mathscr{A}$ via $\mathscr{U}$ too, which is what m does. Suppose $\mathscr{U}$ is a compact metric space and $u_n(P) \to u(P)$ whenever $u_n \to u$ in $\mathscr{U}$. Then it is easy to see that if the sequence $\{P_n\}$ converges to P in any metric d on X such that A3 holds, then $m(P_n, P) \to 0$. Thus, m is an appropriate distance on X.

Let $B^{\varepsilon(m)}$ be the ε-ball about B in X using the pseudometric m instead of the original metric d. Let

$$h_m(\mathscr{B}_1, \mathscr{B}_2) = \inf\{\varepsilon > 0 \mid \mathscr{B}_1 \subseteq \mathscr{B}_2^{\varepsilon(m)} \text{ and } \mathscr{B}_2 \subseteq \mathscr{B}_1^{\varepsilon(m)}\}$$

be the associated Hausdorff metric. We now define another distance between sets of alternatives, which measures the difference in maximum expected utility, namely,

$$k(\mathscr{A}_1, \mathscr{A}_2) = \sup_{u \in \mathscr{U}} | \sup_{P \in \mathscr{A}_1} u(P) - \sup_{P \in \mathscr{A}_2} u(P)|.$$

The idea is that decision makers should like the distance k. However, it is not difficult to see that $k(\mathscr{A}_1, \mathscr{A}_2) \leqslant h_m(\mathscr{A}_1, \mathscr{A}_2)$; p. 15 of GW. Combining the remarks above, we obtain the following justification for our mode of convergence.

THEOREM 8.1. *Suppose* A1–A3 *hold and* X *and* Y *are compact. If* $\mathscr{A}_n \to \mathscr{A}$, *then* $k(\mathscr{A}_n, \mathscr{A}) \to 0$.

The relation $\sim$ is established in each case above by a limit theorem. The conclusion means the $\operatorname{adm}(\mathscr{A}_2, \mathscr{A}_{\mathrm{MV}})$ is approximately equal to $\operatorname{adm}(\mathscr{A}_2, \mathscr{U}_2)$ if $\mathscr{A}_2$ is close enough to $\mathscr{A}_1$, but we have not specified what "closed enough" is. For that we need estimates on the rate of convergence. We now describe a result in this direction. Again we use the pseudometric m on X.

THEOREM 8.2. (a) *If* $\mathscr{A}_1 \subseteq \mathscr{A}_2^\delta$, *then* $(\operatorname{adm}_c \mathscr{A}_2)^\delta \cap \mathscr{A}_1 \subset \operatorname{adm}_{c+2\delta} \mathscr{A}_1$.

(b) *If* $\mathscr{A}_1 \subseteq \mathscr{A}_2^\delta$ *and* $\mathscr{A}_2 \subseteq \mathscr{A}_1^\delta$, *then* $\operatorname{adm}_{c-2\delta} \mathscr{A}_2 \subseteq (\operatorname{adm}_c \mathscr{A}_1)^\delta \subseteq (\operatorname{adm}_{c+2\delta} \mathscr{A}_2)^{2\delta}$.

Proof [GW, p. 14].

Obviously the condition in Theorem 8.2(b) can be expressed with the Hausdorff metric, but unfortunately the conclusion cannot.

Our application of the general continuity properties has been to second-order stochastic dominance, where the order is determined by $\mathscr{U}_2$. We now briefly discuss the application to first-order stochastic dominance, where the order is determined by $\mathscr{U}_1$, the set of all bounded nondecreasing real-valued functions on the real-line; Lehmann (1955) and Kamae *et al.* 1977). Without loss of generality, we assume the functions in $\mathscr{U}_1$ are bounded below by zero and above by one. Let $\mathscr{U}^r$ be the subset of right-continuous functions in $\mathscr{U}_1$. The set $\mathscr{U}^r$ can be identified with the set of all (possibly defective) distribution functions, which is known to be a compact metric space using the Lévy metric; pp. 267, 285 of Feller (1971).

Since the elements of $\mathscr{U}^r$ need not be continuous, we need a stronger topology on the space X of probability measures than weak convergence. One approach is to assume that $\mathscr{P}^r$ is the set of all probability distributions that are absolutely continuous with respect to Lebesque measure with a density f which is of bounded variation, p. 99 of Royden (1968):

$$\mathscr{P}^r = \left\{ P \in X \mid P(A) = \int_A f(t)\, dt \text{ for all } A \right\},$$

where $f = g_1 - g_2$ with g_1 and g_2 both nondecreasing, right-continuous and $0 \leqslant g_1(t), g_2(t) \leqslant h(t)$ for all t, $\int_{-\infty}^{\infty} h(t)\, dt \leqslant M$. Note that the decomposition $f = g_1 - g_2$ is unique for each f. Just as with $\mathscr{U}^r$, $\mathscr{P}^r$ is á compact metric space with the Levy metric, here applied to g_1 and g_2 separately, i.e.,

$$\lambda'(f_1, f_2) = \lambda(g_{11}, g_{21}) + \lambda(g_{12}, g_{22}).$$

THEOREM 8.3. *$\mathscr{U}^r$ and $\mathscr{P}^r$ as defined here are compact metric space in which $u_n(P_n) \to u(P)$ if $u_n \to u$ in $\mathscr{U}^r$ and $P_n \to P$ in $\mathscr{P}^r$.*

Proof [GW, p. 39].

Other settings for which continuity in first-order stochastic dominance holds can be obtained from Proposition 18 on p. 232 of Royden (1968). Joint continuity $u_n(P_n) \to u(P)$ holds if $P_n \to P$ setwise and $u_n \to u$ pointwise for $\{u_n\}$ appropriately dominated.

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