A Data-Driven Model of an Appointment-Generated Arrival Process at an Outpatient Clinic

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Abstract

In order to develop appropriate queueing models that can be applied to efficiently provide good service in a service system with arrivals by appointment, we carefully consider what stochastic arrival process models may be appropriate. We provide a guideline or template for creating stochastic arrival process models by carefully examining appointment and arrival data from an endocrinology clinic. In addition to three recognized sources of variability, (i) no-shows, (ii) extra unscheduled arrivals and (iii) deviations in the actual arrival times from the scheduled times, we find that the primary source of variability is variability in the daily schedule itself. We create detailed stochastic models that can be used to simulate the arrival process and analyze system performance. We develop a classification scheme that can be used to compare appointment systems and their performance.

Keywords: appointment-generated arrival processes; scheduled arrivals in service systems; outpatient clinics; data-driven modeling; stochastic models in healthcare; appointment scheduling; sources of variability in outpatient clinics.

Short Title: Appointment-Generated Arrival Process at an Outpatient Clinic

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1 Introduction

Many service systems, such as police, fire and hospital emergency departments, fast-food restaurants and the majority of call centers, aim to respond promptly to all demand as it arises. Even though demand is uncertain, a good response often can be achieved by identifying systematic regularity in the arrival process and the service times, which usually has at least the following three predictable components: (i) a deterministic time-varying arrival rate, possibly depending on other factors such as the day of the week, (ii) Poisson variability about that rate, and (iii) independent service times with a common distribution. Ways to respond to the uncertain demand using time-varying staffing levels, exploiting these systematic properties, are surveyed in Green et al. (2007a).

In contrast, other service systems, such as private doctors, dentists and hospital outpatient clinics, use appointment systems to manage the arrival process. Instead of responding to whatever demand arises, there is an active effort to control the arrival process. The appointment systems tend to make the actual arrivals more regular, in some cases even producing nearly deterministic evenly spaced arrivals. Nevertheless, appointment-generated arrivals are often quite complicated, as we illustrate in this paper.

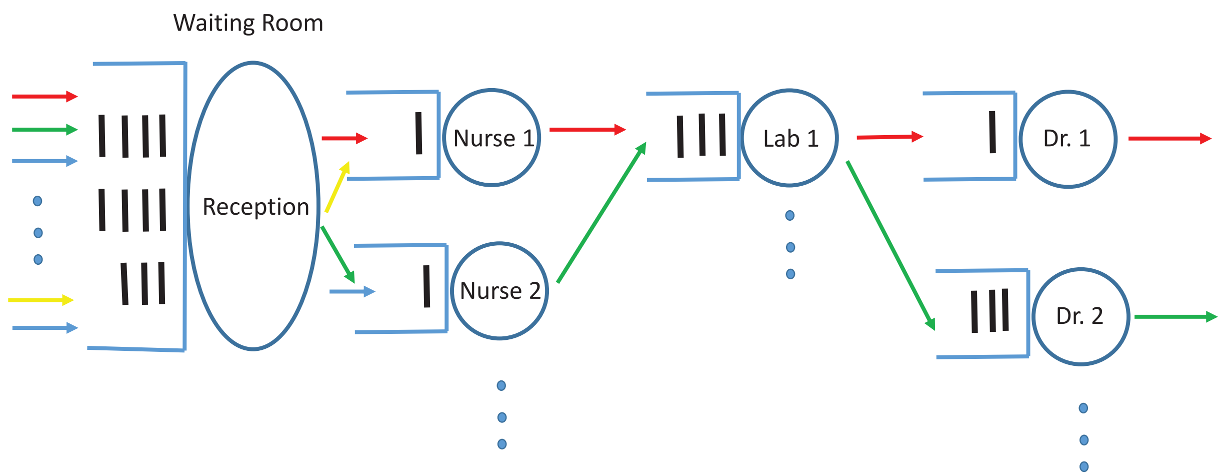
Our purpose in this paper is to develop an approach for creating more realistic stochastic models of appointment-generated arrival processes that can be used as part of a full queueing model to apply to improve operations, e.g., to improve throughput, control individual workloads, set staffing levels and allocate other resources) to efficiently provide good service in a service system with arrivals by appointment. Our goal is just as in Green et al. (2007a), but we want to find an appropriate replacement for the nonhomogeneous Poisson process used to model the arrival processes in service systems without appointments. To better understand what assumptions may be realistic, we carefully examine data from one specific setting: an endocrinology outpatient clinic of a major teaching hospital in South Korea. We aim to carry out one step in the careful performance analysis of this outpatient clinic, but not a full performance analysis of the clinic. The detailed analysis of the clinic arrival process is intended to provide a general guideline or template for carrying out similar analyses for other outpatient clinics and other service systems with arrivals by appointment.

1.1 A Long History of Modeling and Analysis of Outpatient Clinics

There is a long history of modeling and analysis of outpatient clinics and other healthcare systems, with notable early work Bailey (1952), Welch and Bailey (1952), Fetter and Thompson (1965), surveys Cayirli and Veral (2003), Gupta and Denton (2008), Jacobson et al. (2006), Jun et al. (1999) and edited reviews Hall (2006, 2012). The large literature can be divided roughly into three types, depending on their focus: (i) conducting a full analysis of an outpatient clinic to make operational improvements, (ii) designing an effective appointment system and (iii) performance analysis of queueing models capturing important properties of clinic arrival processes.

As illustrated by the seminal paper of Fetter and Thompson (1965), it is recognized that outpatient clinics can be usefully represented as a complex network of queues, with queueing associated with the reception area, nurses, labs and doctors, as depicted abstractly in Figure 1. Patients often follow different paths through

Figure 1: The representation of a hypothetical clinic as a network of queues. Patients to see all doctors go to a common waiting room; in this example patients to see Doctors 1 (red) and 2 (green) share a common lab.



the clinic, depending on many factors, such as the doctor they are scheduled to see, their medical condition and the results of medical tests. Thus, to analyze and improve the performance of a clinic, it is important to construct careful process maps or work-flow diagrams as illustrated by Figure 1 of Chand et al. (2009) and Figure 2 in Harper and Gamlin (2003). The system complexity has made simulation the dominant choice for detailed analysis of a clinic. Many successful simulation studies have been conducted, as can be seen from Chand et al. (2009), Chakraborty et al. (2010), Guo et al. (2004), Harper and Gamlin (2003), Swisher et al. (2001). There also is potential for analytical queueing network models such as in Whitt (1983), as discussed by Zonderland and Boucherie (2012).

Most outpatient clinics have a substantial portion of their arrivals scheduled in advance, i.e., generated by an appointment system. Thus, as expected, a large part of the literature is devoted to designing an effective appointment system, as can be seen from the surveys Cayirli and Veral (2003), Gupta and Denton (2008) and recent work Liu et al. (2010), Luo et al. (2012), Liu and Ziya (2014).

It is also recognized that appointment-generated arrival processes are very different from arrival processes where customers independently decide when to arrive. It is well known that the arrival process often tends to have a nearly periodic structure determined by appointment time slots, but that it also can be significantly different because of no-shows, unscheduled arrivals and earliness or lateness. Empirical studies of patient no-shows and non-punctual arrivals have been conducted, and they show that the no-show rates vary across different services and patient populations; the reported no-show rates are as low as 4.2% at a general practice outpatient clinic in United Kingdom (Neal et al. 2001) and as high as 31% at a family practice clinic (Moore et al. 2001). Thus, ever since the seminal papers by Bailey (Bailey 1952, Welch and Bailey 1952), work has been done to analyze queueing models that reflect key structural properties of appointment-generated arrival processes; e.g., see Feldman et al. (2014), Hassin and Mendel (2008), Jouini and Benjaafar (2012), Kaandorp and Koole (2007), Wang et al. (2010, 2014), Zacharias and Pinedo (2014).

1.2 Probing Deeply Into One Clinic Arrival Process

In order to provide a fresh view of outpatient clinics, in this paper we do not follow any of the three time-tested approaches. Instead, we devote this entire paper to carefully examining arrival data from an outpatient clinic appointment system. In doing so, we are primarily interested in the arrival process resulting from the schedule, so that we can construct more realistic stochastic simulation models and analytic queueing models, so we do not consider alternative scheduling algorithms, but our analysis should provide insight into possible scheduling algorithms. The arrival process resulting from the schedule plays an important role in an operational analysis of the full clinic. It is well established that modeling and analysis of clinic performance can be useful; our goal is to a better job by more faithfully modeling the arrival process resulting from the schedule.

We carefully analyze and compare the sources of variability. In addition, we develop a detailed stochastic model that can be used in simulation experiments and can serve as a starting point for developing more tractable approximations. We think that our study provides useful insight into what are appropriate appointment-generated arrival processes in full simulation studies and in analytical queueing models.

1.3 Randomness in the Appointment Schedule

Even though we find the customary deviations from an ideal deterministic pattern of arrivals - no-shows, extra unscheduled arrivals and non-punctuality, *we find that the main deviation from a regular deterministic arrival pattern is variability in the schedule itself.* But first we need to define what we mean by the schedule, because it can be defined in different ways. By “the schedule,” we mean both the number and the arrival times of all arrivals scheduled for a given day, as determined by the appointment system by the end of the previous day.

In fact, we claim that the clinic schedule for a given day often should be regarded as a vector-valued stochastic process, which evolves over time, because the schedule typically fills up dynamically over a substantial time period, over several months in our case. Thinking of the common case in which the day is divided into ν time slots, with β available spaces in each time slot, a relatively simple representation of the schedule at any time is the number of patients assigned to each available time slot. With this idea, it is natural to let this stochastic process take values in the product set $\{0, 1, \ldots, \beta\}^\nu$, which is of dimension $(\nu + 1)\beta$, but this simple view may be inadequate, because extra patients may get scheduled in any given time slot or in extra time slots outside the main time interval, as we will illustrate.

At first glance, it might be thought that viewing the schedule as random is inappropriate, because unlike call centers where arrivals are generated exogenously, an appointment-generated schedule is endogenous, at least partly controlled by management. However, filling the final appointment schedule is rarely straightforward. In service systems with low demand, there may be inadequate demand to fill the schedule, so that the resulting schedule could be far less than desired. Then it may be natural to view the final schedule as random, corresponding closely to the random demand.

On the other hand, our outpatient clinic experiences high demand, as do many other service systems. One might think that should lead to the ideal deterministic pattern, but that view does not take account of patient needs and preferences. In healthcare, patients typically differ widely in the urgency of their needs. Thus, urgent requests for service can arrive after the schedule is full. In many cases, as in our clinic, the clinic wants to respond to this important extra demand. Again, management can decide how to respond, but to understand what is actually done, and thus to understand how the arrival process will affect the performance of the clinic, it is important to look at arrival process data. Thus, we think that it is natural, and important, to regard the schedule as random, even though there are possibilities for control.

Indeed, an important managerial insight of this paper, which should be generalizable to other appointment contexts, is the idea that the schedule itself may be random and thus it may be necessary to carefully model, monitor and manage the schedule. It is evident that appointment scheduling is important, but it may not be evident that it can be important to examine the schedules resulting from the appointment system as well as adherence to that schedule. To the best of our knowledge, this is the first study of an outpatient clinic to suggest that the schedule itself should be regarded as random and characterize its stochastic structure.

1.4 Data from an Endocrinology Outpatient Clinic

We carefully examine data from one specific setting: an endocrinology outpatient clinic of the Samsung Medical Center in South Korea. The data were collected over a 13-week period from July 2013 to September 2013. Included in the data are the day and time of the appointment and when the appointment was made as well as the final disposition, i.e., whether there was a cancellation or a no-show without cancellation, and the actual time of arrival. Sixteen doctors work in this clinic, but patients make an appointment to see a particular doctor, so that each arriving patient knows which doctor they will meet. Hence, each doctor operates as a single-server system. Each doctor works in a subset of available days and shifts. There are three shifts: morning (am) shifts, roughly from 8:30 am to 12:30 pm, afternoon (pm) shifts, roughly from 12:30 pm to 4:30 pm, and full-day shifts. During the weekdays of the 13-week study period, the 16 doctors worked for a total of 228 am shifts, 220 pm shifts, 25 full-day shifts. The shifts are not evenly distributed among the doctors; the numbers ranged from 11 to 46.

We have studied the data for all sixteen doctors, but in this paper we primarily focus on patient arrivals to the am shifts of one doctor. This doctor is selected among the sixteen candidate doctors because of the relatively high volume of patients; he worked for a total of 22 am shifts (12 on Tuesdays and 10 on Fridays) and 22 pm shifts (11 on Mondays, 2 on Wednesdays, and 9 on Thursdays) during our study period. The analysis results of the other doctors are in our longer more detailed study Kim et al. (2015a). We emphasize that the analysis of the arrival process for each doctor is important, because patients with appointments for different doctors tend to follow different paths through the clinic, as depicted in Figure 1. The overall arrival process for all doctors is of course important for studying the congestion in the waiting room, but the total arrival process can be directly modeled as the superposition of the arrival processes for the individual doctors.

1.5 Organization

The paper analyzes the appointment-generated arrival process in steps leading up to a full stochastic process model. We do not immediately write down the final model because we regard the process leading up to the model more important than the resulting model for the arrival process to this one doctor.

We start by focusing on what we regard as the most novel and important step: developing the model of the random schedule. In §2, we first examine the observed schedules to infer an underlying deterministic framework. Afterwards, we view the actual schedule as a random modification of that framework. We find that the main deviation from a regular deterministic arrival pattern is variability in the schedule itself.

Next, in §3, we view the actual arrivals as a random modification of the schedule and examine to what extent the actual arrivals adheres to the schedule. In §4, we study the pattern of arrivals over each day, and directly compare the arrivals to the schedule. In §5, we provide mathematical representations of the stochastic counting processes for the schedules and actual arrivals. Thus, we illustrate how a realistic arrival process model can be constructed that can be simulated or further approximated to study the performance of the appointment system and system operation.

In §5.3 we also develop a simple parsimonious model that may be a convenient substitute for mathematical analysis. The simple model is a refinement of the Gaussian-Uniform model proposed in Kim et al. (2015b), which has Gaussian random daily totals and then, conditional on the total, arrival times that are independent random variables uniformly distributed over the shift. Here we make two modifications: First, we add a component to the model for extra arrivals after the main time interval, which we find appropriate for occasional extra scheduled arrivals to meet high demand. Second, within the main interval, we again propose i.i.d. arrival times, conditional on the Gaussian daily total, but we propose a non-uniform density to respond to empirical evidence, which shows a tendency for patients to arrive early.

We conclude in §6 by providing the classification of appointment-generated arrival processes. This provides a basis for comparing the different doctors in this clinic with each other and with other similar clinics. It should help compare alternative appointment systems. From this broader perspective, we emphasize that the clinic appointment system has two properties that make careful analysis possible and useful. First, appointment systems differ in *scale*. A small-scale appointment system might have 8 or fewer arrivals over the normal business day, whereas a large-scale appointment system might have 50 or more. The scale of our outpatient clinic is relatively large, with each doctor seeing more than 60 patients per day. Second, the arrivals might or might not exhibit *significant variability*. For the clinic, there is significant variability in the arrivals, making a careful analysis worthwhile.

2 Defining and Modeling the Daily Schedule

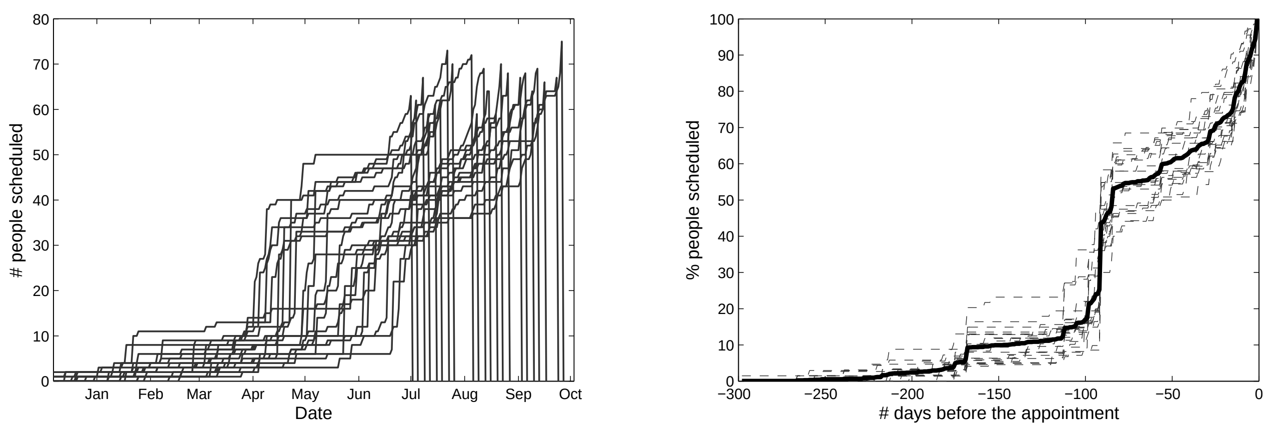
As indicated in 1.4, we examine the schedule and arrival data for one clinic doctor over his 22 morning (am) shifts. The arrivals planned for each day are given in a daily schedule, which has a specified number of arrivals in each of several evenly spaced ten-minute time intervals. Our schedule data are the 22 observed schedules for the doctor on his am shifts. Even though much can be learned from consulting the appointment manager, we try to see what can be learned directly from the data.

2.1 The Evolution of a Schedule

The actual schedule for a given day evolves over time, typically starting many weeks before the specified day. Thus, even though an appointment system and manager create the schedule, when we consider the resulting schedule, we regard the evolution of the schedule as a stochastic process, with additions and cancelations occurring randomly over time. For each day, we define the final schedule as its value at the end of the previous day.

We illustrate in the left plot of Figure 2, which shows the evolution over the year 2013 of the scheduled daily total number of patients scheduled over the previous year for the 22 days in the data set. It shows the specific appointment days as well, which are spread out between July and October.

Figure 2: The evolution of the daily number of patients scheduled over the previous year for 22 days (left) and the percentage of patients that are scheduled k days in advance for each of the 22 appointment days (right). The thick line indicates the average over the 22 appointment days.



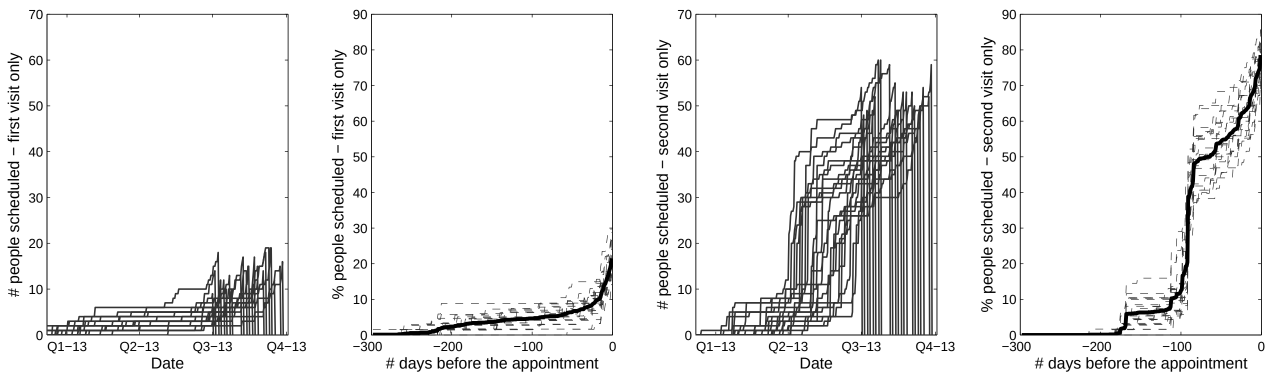
The right plot of Figure 2 presents a useful alternative view, showing the percentage of the final schedule reached k days before the appointment data, as a function of k. For all 22 days, 100% of the schedule is filled at $k = 0$. We see much less variability in the right plot than in the left plot. The percentage of the schedule reached 30 days before appears at $k = -30$. Especially revealing is the average of these 22 percentage plot, which is shown in the single heavier curve. From this average plot, we see jumps at regular intervals especially around 90 days (3 months) before the appointment date. The right plot of Figure 2 shows that about 24% of all appointments are made more than 93 days in advance, while about 30% are made between 93 and 84 days in advance (about 3 months). About 30% are made in the last month, while about 13% are made in the last week.

2.2 New and Repeat Visits

There is increasing interest in the delays from request to appointment, including how to determine a good panel size for doctors; see Green et al. (2007b), Liu and Ziya (2014), and Liu et al. (2010) and references therein. Unfortunately, our data set does not include a measure of the urgency or time-sensitivity of each appointment, so we cannot see from the data whether patients are unable to get urgent appointments quickly enough. Fortunately, the data set does specify whether or not each scheduled arrival is a repeat visit or a new visit. Since 78% of all appointments are repeat visits, we conclude that the long intervals between schedule date and appointment date do not imply that patients are failing to get urgent needs addressed promptly.

Figure 3 repeats Figure 2 for new and repeat visits. These figures show that this classification is very important. Figure 3 shows that only about half of the new patients wait for more than a week for an appointment. The median number of days between the appointment schedule date and the appointment date was 93 for repeat visits and 24 for new patients.

Figure 3: The evolution of the daily number of patients scheduled and the percentage of patients that are scheduled k days in advance for each of the 22 appointment days: new patients (left two) and repeat visits (right two). The thick line indicates the average over the 22 appointment days.



2.3 Inferring a Deterministic Framework for the Schedules

From the perspective of the eventual arrival process over each day, the evolution of the schedule should not matter much if the final schedule reaches a nearly deterministic regular form, which varies little from day to day. However, for the clinic there is considerable variability in the schedules, so that the evolution may matter.

First, we define the schedule as the daily total plus the actual scheduled arrival times of all these patients. We define the schedule as its value at the end of the previous day. We define arrivals on the same day as unscheduled arrivals. Given that definition, we next look for an underlying deterministic framework. The starting point for our data analysis is the 22 observed daily schedules. These are displayed in Table 1. Table 1 shows the number of patients scheduled for different ten-minute time slots (displayed vertically) over the am shift of 22 days (displayed horizontally). The ten-minute time slot is specified by its start time.

Most appointment schedules today are designed and managed to fit into a deterministic framework, usually using a computerized appointment management system. However, it seems prudent to look at the actual schedules and infer the realized framework from the data. Not all variability occurs because of adherence to

Table 1: The number of patients scheduled in each 10-minute time slot (displayed vertically) during 22 morning shifts (displayed horizontally).

time slot	22 days in July-October 2013																						Avg	Var	Var/Avg
7:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:00	0	0	0	0	0	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	1	1	0.32	0.23	0.71
8:10	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
8:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:50	3	4	5	4	4	4	4	4	4	1	3	2	1	4	2	4	4	2	4	5	4	3	3.41	1.30	0.38
9:00	3	4	2	3	3	2	3	3	3	3	3	2	2	2	2	3	4	3	2	3	4	2	2.77	0.47	0.17
9:10	3	3	3	2	2	2	4	2	2	3	2	3	2	3	3	3	2	2	3	2	3	3	2.59	0.35	0.13
9:20	2	2	4	2	3	2	3	2	2	3	3	3	2	3	2	3	3	3	3	2	3	2	2.59	0.35	0.13
9:30	3	2	3	4	3	3	4	3	3	3	3	3	1	3	2	2	2	2	3	3	3	3	2.77	0.47	0.17
9:40	3	3	3	2	2	2	2	3	3	2	2	3	2	3	2	2	2	2	3	2	2	2	2.36	0.24	0.10
9:50	3	3	3	3	2	3	3	3	3	3	3	2	2	3	3	3	3	3	2	2	3	3	2.77	0.18	0.07
10:00	3	2	3	3	2	3	2	3	2	3	3	3	3	3	3	3	4	4	3	3	3	3	2.91	0.28	0.10
10:10	3	3	3	3	3	3	3	3	2	2	3	3	3	3	3	3	3	3	3	3	3	3	2.91	0.09	0.03
10:20	2	3	3	3	3	3	3	2	3	3	2	3	3	3	3	2	3	2	3	4	3	3	2.82	0.25	0.09
10:30	3	2	3	3	3	2	4	2	3	2	3	3	3	3	3	2	3	3	2	4	3	3	2.82	0.35	0.12
10:40	3	1	3	3	3	1	3	2	3	2	3	3	2	3	2	1	3	2	3	3	3	2	2.45	0.55	0.22
10:50	2	3	3	3	1	2	3	2	3	3	3	2	3	3	3	3	3	3	2	3	3	3	2.68	0.32	0.12
11:00	3	2	3	2	3	2	3	2	2	4	4	4	2	3	3	3	3	3	3	4	3	4	2.95	0.52	0.18
11:10	3	3	3	1	3	3	3	3	2	3	3	2	3	2	1	3	2	3	3	3	3	3	2.64	0.43	0.16
11:20	2	3	3	3	3	3	3	3	3	3	3	3	3	2	2	3	3	3	3	3	3	4	2.91	0.18	0.06
11:30	3	3	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	3	2	2.77	0.18	0.07
11:40	3	2	3	3	2	3	3	3	3	1	2	3	3	2	3	3	3	3	3	3	2	3	2.68	0.32	0.12
11:50	3	3	3	3	3	2	2	3	3	2	3	2	4	3	3	3	2	2	3	3	1	3	2.68	0.42	0.16
12:00	2	3	3	2	3	3	4	3	3	2	3	3	3	3	3	3	3	3	2	2	3	4	2.86	0.31	0.11
12:10	3	3	3	2	3	3	2	3	2	3	3	2	3	3	4	3	1	2	3	2	3	3	2.68	0.42	0.16
12:20	2	4	3	2	3	3	3	3	4	3	3	3	3	2	2	3	1	3	1	4	3	3	2.77	0.66	0.24
12:30	2	1	0	0	0	3	3	3	3	2	2	2	2	3	3	3	2	4	3	1	2	3	2.14	1.27	0.59
12:40	0	0	0	0	0	2	2	4	3	0	3	2	1	2	3	3	4	2	3	0	0	3	1.68	2.13	1.27
12:50	0	0	0	0	0	0	0	1	4	0	0	0	0	3	4	0	2	0	4	0	0	4	1.00	2.67	2.67
13:00	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
Daily Total	63	62	67	59	61	62	73	70	72	59	69	64	59	70	68	67	68	64	69	66	67	75	66.09	21.32	0.32
[8:50, 12:20] Total	60	61	67	59	60	57	67	60	61	57	63	60	56	62	57	61	60	58	59	65	64	64	60.82	9.77	0.16
All slot avg	2.0	2.0	2.2	1.9	2.0	2.0	2.4	2.3	2.3	1.9	2.2	2.1	1.9	2.3	2.2	2.2	2.2	2.1	2.2	2.1	2.2	2.4	2.07	1.73	0.84
All slot var	1.5	1.9	2.2	1.9	1.8	1.5	1.8	1.3	1.5	1.7	1.5	1.5	1.6	1.5	1.3	1.7	1.8	1.6	1.6	2.2	1.8	1.6	(across all days)		
All slot var/avg	0.7	1.0	1.0	1.0	0.9	0.8	0.8	0.6	0.6	0.9	0.7	0.7	0.8	0.6	0.6	0.8	0.8	0.8	0.7	1.1	0.8	0.7			
[8:50, 12:20] avg	2.7	2.8	3.0	2.7	2.7	2.6	3.0	2.7	2.8	2.6	2.9	2.7	2.5	2.8	2.6	2.8	2.7	2.6	2.7	3.0	2.9	2.9	2.76	0.42	0.15
[8:50, 12:20] var	0.2	0.6	0.3	0.5	0.4	0.4	0.4	0.3	0.4	0.5	0.2	0.3	0.5	0.3	0.4	0.4	0.7	0.3	0.4	0.7	0.4	0.4	(across all days)		
[8:50, 12:20] var/avg	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.2	0.1	0.3	0.1	0.2	0.2	0.1	0.1			

the schedule. The schedules show that there is substantial variability in the schedule itself.

We next define what we mean by a deterministic framework for the appointment schedule. A general deterministic framework has batches of size β_j customers arriving at intervals τ_j after an initial time 0, for $1 \leq j \leq \nu$. Thus, the associated arrival times are

$$\psi_j \equiv \sum_{i=1}^{j-1} \tau_i \quad \text{for} \quad 1 \leq j \leq \nu \quad \text{and} \quad \psi_1 \equiv 0. \tag{2.1}$$

The framework has a total targeted number N_F and time T_F defined by

$$N_F = \sum_{j=1}^{\nu} \beta_j \quad \text{and} \quad T_F = \sum_{j=1}^{\nu-1} \tau_j = \psi_{\nu-1}. \tag{2.2}$$

A principal case is the *stationary framework* with $\beta_j = \beta$ and $\tau_j = \tau$ for all j, which makes $N_F = \beta\nu$ and $T_F = (\nu - 1)\tau$, leaving the target parameter triple (β, τ, ν), but there often are variations in practice. The more general model is important to consider alternative nonstationary schedules that might be used, or contemplated, to improve various measures of performance.

From Table 1, we infer that the deterministic framework above is approximately valid with $\tau = 10$ minutes. However, the scheduled arrivals in each time slot are not constant over different days or different times on each day. Table 1 indicates that, for the am shifts of the doctor in the endocrinology clinic, the stationary framework is roughly valid as an idealized model with $\beta = 3$, $\tau = 10$ minutes and $\nu = 22$, starting at 8:50 and ending at 12:20 (including the intervals [8:50, 9:00) and [12:20, 12:30), closed on the left and open on the right), which we refer to as the interval $[8{:}50, 12{:}20]$. However, some shifts start as early as 8:00 am, while some shifts end as late as 13:00 (including the interval [13:00, 13:10)). The daily total for the stationary framework is $22 \times 3 = 66$, which matches the average daily total for the 22 days, even though the schedule is otherwise more variable.

Upon closer examination, we can see consistent structure in the schedule variability. First, we see that some days have higher daily totals, evidently because an effort is being made to respond to high demand. Second, we see random batch sizes in the slots over the entire shift. We discuss each of these features in turn.

2.4 High-Demand Service with Overloaded and At-Capacity Schedules

In general, it seems useful to classify service systems with arrivals by appointment into two categories. First, there are the low-demand service systems, for which it is challenging to fill a target schedule. For such service systems the randomness in the schedule is due to the random level of demand. We then might focus on the extent to which demand is adequate to fill the target schedule.

Second, there are the high-demand service systems, for which there is almost always ample demand, and often there is excess demand. In that case, the system may or may not actually respond to the excess demand, i.e., schedule more than the normal workload in order to meet that excess demand. Of course, there can be more complicated scenarios in which a service system oscillates between the low-demand and high-demand modes.

From Table 1, we infer first that the doctor operates as a high-demand service system and that indeed he responds to excess demand on some days but not all days. We deduce from Table 1 that the appointment schedule is at capacity (AC) on some days but overloaded (OL) on other days. If we identify the deterministic framework for the am shifts to be the 22 ten-minute time slots in the interval $[8{:}50, 12{:}20]$, then we observe that the daily totals within this interval are remarkably stable, having mean 60.82 and variance 9.77. In contrast, the full daily totals for the entire am shifts are much more highly variable, having a variance of 21.19. Moreover, the data shows that the overload portion of the schedule (at the beginning and the end) were consistently made far earlier than the other appointments.

This conclusion is further confirmed by the observation that the extra patients tend to be scheduled outside (after) the main am shift interval $[8{:}50, 12{:}20]$. In particular, we regard days with 5 or more appointments outside of (after the) the main interval as overloaded. By this definition, we see 12 overloaded days (OL) and

10 at-capacity (AC) days among the 22 am shifts. Table 2 shows the distribution of the number of scheduled patients in these outside intervals, N_o, among AC and OL days.

Table 2: Estimated distribution of the batch sizes (B_s) within the main interval $[8{:}50, 12{:}20]$ and estimated distribution of the total number of scheduled arrivals after the main interval (N_o) on the 10 at-capacity (AC) days, on the 12 overloaded (OL) days, and on all days

	$\hat{P}(B_s = k)$					$\hat{P}(N_o = k)$										
number k	1	2	3	4	5	0	1	2	3	4	5	6	7	8	9	10
10 at-capacity days	0.04	0.25	0.63	0.07	0.01	0.30	0.20	0.30	0.10	0.10						
12 overloaded days	0.02	0.27	0.63	0.08							0.25	0.17		0.25		0.33
All days	0.03	0.26	0.63	0.08	0.004	0.14	0.09	0.14	0.05	0.05	0.14	0.09		0.14		0.18

Table 1 shows that the overflows happen without any empty slots in between on overloaded days. Furthermore, we observe the possibility of dependence over successive appointment days because overloaded days are often followed by overloaded days. As further confirmation of the idea that overload appears outside the main time interval, we also see higher numbers in the first shift at 8:50 (the interval $[8{:}50, 9{:}00)$); this suggests that at least some of the patients scheduled in the first interval at 8:50 are in response to pressure to provide service to more patients than the usual number. We note that it too might be regarded as an overload period as well, though we choose not to do so. Moreover, the data shows that the overload portion of the schedule (at the beginning and the end) were consistently made far earlier than the other appointments.

When we next consider random batch sizes for the slots, we see that they are very consistent inside the main interval on AC and OL days, further supporting the inference that arrivals outside the main interval primarily occur because of an effort to respond to excess demand.

2.5 Random Batch Sizes

Table 1 clearly indicates that the number of patients scheduled for each 10-minute time slot is variable. This distribution becomes quite consistent over the days and the time slots if we focus on the main time interval $[8{:}50, 12{:}20]$. Table 2 shows the distribution of the schedule within each time slot within the main interval $[8{:}50, 12{:}20]$ for the AC days, the OL days and for all days. The conclusions of §2.4 are supported by the fact that the estimated distributions for AC and OL days are very similar. Those days tend to differ only to the extent that they have extra scheduled arrivals outside the main interval.

From Table 2, we conclude that it is reasonable to assume that the batch sizes in each of the time slots of the main time interval can be regarded as realizations of a random variable B_s assuming values in the set $\{1, 2, 3, 4\}$ for any j. (We omit the value 5 because the frequency is so low, and we could also possibly omit the value 1 for the same reason.) In particular, we estimate the distribution as

$$P(B_s = k) = 0.03,\ 0.26,\ 0.63, 0.08 \quad \text{for} \quad k = 1,\ 2,\ 3, 4, \quad \text{respectively,} \tag{2.3}$$

so that

$$E[B_s] = 2.76, \quad E[B_s^2] = 8.02, \quad Var(B_s) = 0.402 \quad \text{and} \quad SD(B_s) = 0.634, \tag{2.4}$$

respectively, for all j. The variance is considerably less than the mean, so we can conclude that the distribution of B_s is much less variable than Poisson. The squared coefficient of variation (scv, variance divided by the square of the mean) is remarkably low as well, being $c_B^2 = 0.053$.

2.6 Independence or Dependence Among Batch Sizes During Each Day

In §2.5 we focused on the distribution of the batch size of the scheduled arrivals in any time slot within the main time interval on any day. We now consider the joint distribution of the batch sizes over successive time slots on the same day.

Let $B_{s,j}$ be the scheduled batch size in slot j, $1 \le j \le 22$, on a given day. For simplicity from a stochastic modeling perspective, it is natural to assume that the batch variables $B_{s,j}$ in successive slots j are independent, which corresponds to appointments being made independently for specific slots. On the other hand, it may be more realistic to assume that the appointments are primarily for days, and that the actual appointments are distributed approximated evenly over the day, with the person or system creating the schedule only partly responding to the patient requests. Alternatively, appointments may overflow to nearby slots, which should also create positive correlation. In any context, then, it is interesting to ask about the dependence among the scheduled batch sizes $B_{s,j}$ on each day.

To illustrate, let N_S be the daily total of the schedule (focusing on the main interval [8:50, 12:20] with $\nu = 22$ slots) and consider the case in which the distribution of B_s is independent of j. If the batch sizes are mutually independent, then

$$Var(N_S) = \nu Var(B_s). \tag{2.5}$$

On the other hand, if we assume that the daily total is random and if we distribute it evenly among the slots, then we might have

$$B_s \approx \frac{N_S}{\nu} \quad \text{so that} \quad Var(N_S) = \nu^2 Var(B_s). \tag{2.6}$$

More generally, the dependence among the batch sizes might be usefully summarized by the correlations

$$\rho_{j_1,j_2} \equiv corr(B_{s,j_1}, B_{s,j_2}) = \frac{cov(B_{s,j_1}, B_{s,j_2})}{\sqrt{Var(B_{s,j_1})Var(B_{s,j_2})}}. \tag{2.7}$$

We propose a model that enables us to incorporate a range of possibilities in a parsimonious manner. We assume that

$$\rho_{j_1,j_2} = \rho_S, \quad -1 \le \rho_S \le 1 \quad \text{for all} \quad j_1 \neq j_2. \tag{2.8}$$

We can then estimate the single pairwise correlation parameter ρ_S empirically in any given appointment setting.

Under assumption (2.8), we have

$$\sigma_S^2 \equiv Var(N_S) \;\; = \;\; \sum_{j=1}^{\nu}\sum_{k=1}^{\nu} Cov(B_{s,j}, B_{s,k}) = \nu Var(B_s)[1 + (\nu - 1)\rho_S]. \tag{2.9}$$

We thus estimate the correlation ρ_S in (2.8) by

$$\rho_S \equiv \frac{Var(N_S) - \nu Var(B_s)}{\nu Var(B_s)(\nu - 1)}, \tag{2.10}$$

where we use our estimates of $Var(N_S)$ and $Var(B_s)$. From Table 1, our estimate of $Var(N_S)$ is 9.77; from (2.4), our estimate of $\nu Var(B_s)$ is $22 \times 0.402 = 8.80$. We thus estimate that ρ_S is $0.97/185 = 0.0052$, which is quite small. It is sufficiently small that we consider the i.i.d. model reasonable.

2.7 Outside the Main Time Interval

It remains to specify scheduled arrivals outside the main time interval. Since the average total outside is only about 10% of the full daily total, and since we do not have a great amount of data overall, we do not try for a high-fidelity model. Based on the limited data provided by Tables 1 and 2, we classify a typical day as AC with probability $10/22$ and OL with probability $12/22$. For days of each type we allocate the total number of scheduled arrivals outside (after) the main interval according to the appropriate distributions specified for each day in Table 2. If the total number is 7 or fewer, then we divide the number into two parts, putting the larger or equal number in the first slot and the smaller or equal number in the second slot. If the total number is 8 or more, we divide the total into three parts, as evenly as possible and put the numbers in decreasing order in the first three slots after the main interval.

2.8 Summary of the Schedule Model

In summary, the clinic data clearly indicates a well defined structured framework provided that we identify and focus on a main time interval $[8{:}50, 12{:}20]$ containing 22 slots. The scheduled numbers in these slots can be regarded as i.i.d. random variables distributed as the random variable B_s, as in (2.3). Our analysis in §2.6 supports regarding these slot numbers as mutually independent.

Our doctor evidently experiences high demand. The data indicate that some days are overloaded while others are at capacity. Based on the empirical data, we would say that a typical day is overloaded with probability $12/22$, but at-capacity with probability $10/22$. The distributions of batch sizes within the slots are the same for these two kinds of days. In contrast, the number of extra scheduled arrivals after the main interval does depend on this classification. As stipulated in §2.7, we allocate the totals randomly according to the distributions in Table 2, and we distribute them in a balanced decreasing order over the outside intervals. Since the numbers outside are smaller, we devote less effort to developing a high-fidelity model for that part.

Only about 10% of the mean 66 in the daily totals is due to the scheduled arrivals outside the main interval (the mean inside is 60.8), while the variance in the daily totals of 21.2 is primarily due to the random occurrence of scheduled arrivals outside the main interval, because the variance inside is 9.77. (See equation (3.6) for a more precise statement.) Thus, we tentatively conclude that the greatest contribution to the overall variability of the schedules for the doctor in our study is the inconsistent response to extra demand. By examining the scheduled and the realized arrivals to the other 15 doctors in the clinic, we found that this conclusion applies to all the other doctors as well: see Figures 1-3 and Figures 4-11 in our longer more detailed study (Kim et al. 2015a). To draw a firm conclusion, we would want to see data on the original demand, i.e., requests for appointments, including ones that were not satisfied or moved to another day.

3 Adherence to the Schedule: No-Shows and Unscheduled Arrivals

We now come to the question of adherence to the schedule. The level of adherence converts the schedule into the actual arrival process. We identify three familiar forms of additional randomness in the model: (i) no-shows, (ii) extra unscheduled arrivals and (iii) lateness or earliness. We first focus on the no-shows and the unscheduled arrivals, which together determine how the scheduled daily number of arrivals is translated into the actual daily total number of arrivals. In §4 we focus on the the lateness or earliness, which has a significant impact on the pattern of actual arrivals over the day.

3.1 No-Shows

The no-shows are the scheduled arrivals that do not actually arrive. Instead of the number of actual arrivals in time slot j on a given day, which we denote by $B_{a,j}$, we are now focusing on the number from among the $B_{s,j}$ arrivals that were scheduled to arrive in slot j on that day that arrived *at some time on that appointment day*, which we denote by $B_{a|s,j}$, which necessarily satisfies the inequalities

$$0 \le B_{a|s,j} \le B_{s,j} \quad \text{for all} \quad j. \tag{3.1}$$

The no-shows in slot j are thus defined as

$$B_{n,j} \equiv B_{s,j} - B_{a|s,j}. \tag{3.2}$$

These are shown in Table 3.

Table 3: The number of no-shows ($B_{n,j} \equiv B_{s,j} - B_{a|s,j}$) for each 10-minute time slot j (displayed vertically) during 22 morning shifts (displayed horizontally).

time slot	22 days in July-October 2013																						Avg	Var	Var/Avg
7:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:00	0	0	0	0	0	0	1	1	1	0	1	0	0	0	1	0	0	0	0	0	1	1	0.32	0.23	0.71
8:10	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
8:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:50	1	0	0	1	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0.23	0.28	1.23
9:00	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0.18	0.16	0.86
9:10	0	0	0	1	0	0	2	0	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0.27	0.30	1.11
9:20	0	0	0	0	0	0	0	0	0	0	0	1	0	0	2	0	0	1	0	0	0	0	0.18	0.25	1.38
9:30	0	0	0	2	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.18	0.25	1.38
9:40	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0.18	0.25	1.38
9:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
10:00	0	0	0	0	0	0	0	0	0	1	0	2	1	0	0	0	1	0	1	0	0	0	0.27	0.30	1.11
10:10	0	0	0	1	0	0	0	1	0	0	1	1	0	0	0	0	1	0	0	0	2	0	0.32	0.32	1.01
10:20	0	0	1	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	0	0	0.23	0.18	0.81
10:30	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0.18	0.16	0.86
10:40	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1	0	1	0.23	0.18	0.81
10:50	0	0	1	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2	0.27	0.30	1.11
11:00	0	0	0	1	1	0	0	0	0	1	1	0	1	0	0	1	0	0	0	1	0	0	0.32	0.23	0.71
11:10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0.18	0.16	0.86
11:20	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	1	0	0	1	0	0	0.23	0.18	0.81
11:30	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	0	0.23	0.18	0.81
11:40	0	0	1	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0.23	0.18	0.81
11:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0.14	0.12	0.90
12:00	0	0	1	0	0	0	1	0	1	1	0	1	2	0	0	1	2	0	0	0	1	1	0.55	0.45	0.83
12:10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0.05	0.05	1.00
12:20	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
12:30	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0.14	0.12	0.90
12:40	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0.09	0.09	0.95
12:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0.05	0.05	1.00
13:00	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
Daily Total	3	2	6	8	2	2	10	7	5	4	6	10	6	2	5	5	6	5	4	7	5	10	5.45	6.35	1.17
[8:50, 12:20] Total	2	2	6	8	1	2	8	4	4	4	4	10	6	1	4	5	6	5	4	7	4	7	4.73	5.64	1.19
All slot avg	2.0	2.0	2.2	1.9	2.0	2.0	2.4	2.3	2.3	1.9	2.2	2.1	1.9	2.3	2.2	2.2	2.2	2.1	2.2	2.1	2.2	2.4	0.17	0.17	1.00
All slot var	1.5	1.9	2.2	1.9	1.8	1.5	1.8	1.3	1.5	1.7	1.5	1.5	1.6	1.5	1.3	1.7	1.8	1.6	1.6	2.2	1.8	1.6	(across all days)		
All slot var/avg	0.7	1.0	1.0	1.0	0.9	0.8	0.8	0.6	0.6	0.9	0.7	0.7	0.8	0.6	0.6	0.8	0.8	0.8	0.7	1.1	0.8	0.7			
[8:50, 12:20] avg	2.7	2.8	3.0	2.7	2.7	2.6	3.0	2.7	2.8	2.6	2.9	2.7	2.5	2.8	2.6	2.8	2.7	2.6	2.7	3.0	2.9	2.9	0.21	0.21	0.98
[8:50, 12:20] var	0.2	0.6	0.3	0.5	0.4	0.4	0.4	0.3	0.4	0.5	0.2	0.3	0.5	0.3	0.4	0.4	0.7	0.3	0.4	0.7	0.4	0.4	(across all days)		
[8:50, 12:20] var/avg	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.2	0.1	0.3	0.1	0.2	0.2	0.1	0.1			

Table 3 shows that no-shows are relatively rare, compared to many appointment systems; the no-shows range from 2 to 10 per day, with an average of 5.45 per day. The overall proportion of no-shows is $5.45/66.09$ or 8.2%.

In general, we might try to model the no-shows quite carefully, as we did the schedule batch sizes $B_{s,j}$, but here we simply assume that each scheduled patient fails to arrive in each slot on each day with probability $\delta = 0.082$, independently of all others. Overall, in the model, the number of total no-shows would be a binomial distribution with parameters equal to the total number, say n, of scheduled patients over all days and probability $p = \delta = 0.082$, which would make the distribution approximately Poisson with variance slightly less than the mean. Table 3 shows that the observed sample variance of the average number of no-shows is 6.35, only slightly greater than the overall average 5.45. Hence, we conclude that the model with i.i.d. Bernoulli no-shows is quite well supported by the data.

3.2 Additional Unscheduled Arrivals

Some medical services have significant proportions of both unscheduled and scheduled arrivals. However, there are relatively few unscheduled arrivals at the clinic. As indicated before, these are defined as scheduled arrivals that are scheduled on the same day (after the end of the previous day). On average, there were 2.18 unscheduled patients per day, of which 1.95 arrived. In the Appendix, Table 7 shows all additional unscheduled arrivals, while Table 8 shows the additional unscheduled arrivals that actually arrived. The total number of unscheduled arrivals (that arrived) on all 22 days was 43. Table 8 shows that the total daily number of unscheduled arrivals exceeds 3 on only two days, then having values of 4 and 7. The one exceptional day is evidently responsible for the variance for all days, 2.43, being somewhat larger than the mean. The unscheduled arrivals are somewhat more likely to be outside the main time interval, but that is consistent with our interpretation of that being a place for overload.

Paralleling our previous modeling, we would represent the daily total number of unscheduled arrivals within the main time interval as Poisson with mean 1.55 and outside the main interval as Poisson with mean 0.40. We would then distribute those randomly (uniformly) within the respective time periods. With larger numbers, we might try more careful modeling. However, in general, some sort of Poisson process is natural for unscheduled arrivals, because they are likely to be a result of individual people making decisions independently.

3.3 Daily Totals: Adding No-Shows and Unscheduled Arrivals to the Schedule

We now examine the impact of no-shows and unscheduled arrivals on the actual daily totals of arrivals. Let N_A, N_S, N_N, and N_U be the random daily total numbers of actual arrivals, scheduled arrivals, no-shows and unscheduled arrivals, respectively. In general, we have the basic flow conservation formula

$$N_A = N_S - N_N + N_U. \tag{3.3}$$

Combining the summary data from Tables 1, 3 and 8, we see that the means are

$$E[N_A] = E[N_S] - E[N_N] + E[N_U] = 66.1 - 5.5 + 2.0 = 62.6 \tag{3.4}$$

We see that the final mean daily number of arrivals $E[N_A] = 62.6$ is only about 5% less than the mean scheduled daily number $E[N_S] = 66.1$. Hence, from the perspective of the daily totals, their is strong adherence to the schedule.

Moreover, we see that the variability of the daily number of arrivals N_A is primarily due to the variability of the schedule. Indeed, the sample variances of the four daily numbers were:

$$Var(N_A) = 17.4, \quad Var(N_S) = 21.3, \quad Var(N_N) = 6.4 \quad \text{and} \quad Var(N_U) = 2.4. \tag{3.5}$$

Note that the estimated variances are ordered by $Var(N_A) < Var(N_S)$. The dispersions (sample variance divided by the sample mean) are ordered as well:

$$Var(N_A)/E[N_A] = 17.4/62.6 = 0.278 < 0.322 = 21.3/66.1 = Var(N_S)/E[N_S]. \quad (3.6)$$

We also find that the data show a significant correlation between N_S and N_N. From further analysis, we find that $Var(N_S - N_N) = Var(N_A - N_U) = 18.91$. Since we necessarily have

$$Var(N_S - N_N) = Var(N_S) + Var(N_N) - 2Cov(N_S, N_N) = 21.32 + 6.35 - 2Cov(N_S, N_N) = 18.91,$$

we estimate the covariance $Cov(N_S, N_N)$ and the associated correlation $Cor(N_S, N_N)$ by

$$\begin{aligned} Cov(N_S, N_N) &= (27.67 - 18.91)/2 = 4.38 \quad \text{and} \\ Cor(N_S, N_N) &\equiv \frac{Cov(N_S, N_N)}{\sqrt{Var(N_S)Var(N_N)}} = \frac{4.38}{11.64} = 0.376 \end{aligned} \quad (3.7)$$

which is quite high. However, notice that two of the three largest no-show values of 10 occur on days 7 and 22, which have the largest two daily totals, 73 and 75. It remains to determine why the variables N_S and N_N should be positively correlated.

4 The Arrival Pattern Over the Day

We now shift our attention to the pattern of arrivals over each day, given the daily totals. This primarily means whether each patient arrives before or after the appointment time (earliness or lateness), but it might also mean systematic time-dependence of the schedule, the no-shows or the unscheduled arrivals over the day.

4.1 The Big Picture of the Daily Pattern

Table 4 provides the details yielding the big picture for the time interval [8:50,12:20]. The first four columns of Table 4 show the average numbers scheduled, percentage of no-shows, percentage late and percentage late by more than 15 minutes by half-hour intervals over the am shift, while the first four columns of Table 5 shows the same summary statistics for new and repeat patients, separately, which are significantly different. Table 4 shows that the scheduled numbers and the no-shows are remarkably stable across time. As we have observed in previous sections, the main irregularity in the schedule occurs due to occasional overload scheduled outside these time intervals.

However, we see a different pattern in the lateness or earliness, shown in the last four columns of Table 4. Specifically, Table 4 shows the percentage of patients that arrive late, the average of the lateness X^+ among those patients who are late, the average of the earliness X^- among those patients who are early and the overall

Table 4: Average numbers for each 30-minute interval within the main 3.5-hour time interval of scheduled arrivals, proportions of no-shows and lateness, and the average earliness (X^-), lateness (X^+) and overall deviation (X), plus 95% confidence intervals.

Interval	Avg # Scheduled	% No-show	% Late	% (Late>15 min)	Avg(X^+)	Avg(X^-)	Avg(X)
[8:50, 9:20)	8.8±0.7	7.9±4.8	21.2±6.9	12.3±5.5	35.8±18.7	-25.8±2.7	-11.4±6.9
[9:20, 9:50)	7.7±0.5	6.9±4.6	16.7±6.1	4.8±3.4	24.1±25.4	-35.7±5.6	-25.7±5.2
[9:50, 10:20)	8.6±0.4	6.8±4.4	15.0±6.7	6.4±3.4	20.3±10.9	-38.8±5.2	-30.2±6.5
[10:20, 10:50)	8.1±0.6	7.9±3.2	17.6±5.0	3.3±2.9	9.7±4.9	-45.0±7.3	-34.5±6.0
[10:50, 11:20)	8.3±0.5	9.0±3.9	13.6±4.4	5.4±3.9	18.4±11.2	-48.6±9.1	-39.2±8.7
[11:20, 11:50)	8.4±0.3	7.9±3.7	10.4±4.7	3.9±3.0	16.0±6.4	-61.2±9.1	-53.3±9.4
[11:50, 12:20)	8.2±0.5	9.3±4.1	9.5±5.4	3.8±3.5	12.7±6.6	-58.2±9.5	-51.7±9.8
[8:50, 12:20)	58.0±1.3	8.0±1.7	15.0±1.5	5.8±1.6	21.3±5.6	-44.9±3.0	-34.9±2.9

Table 5: Average numbers for new and repeat patients for the main interval and outside of the interval, proportions of no-shows and lateness, and the average earliness (X^-), lateness (X^+) and overall deviation (X), plus 95% confidence intervals.

Interval	Avg # Scheduled	% No-show	% Late	% (Late>15 min)	Avg(X^+)	Avg(X^-)	Avg(X)
New	14.2±1.3	5.5±2.4	22.2±4.4	7.7±3.4	23.2±9.7	-34.2±4.4	-21.2±3.9
New - [8:50, 12:30)	13.7±1.3	5.7±2.5	23.1±4.6	8.0±3.5	23.2±9.7	-33.9±4.7	-20.5±4.1
New - outside	0.5±0.3	0	0	0		-42.0±27.9	-42.0±27.9
Repeat	51.9±2.1	8.8±1.8	11.8±1.7	4.7±1.8	18.7±5.8	-49.3±3.3	-41.2±3.6
Repeat - [8:50, 12:30)	47.1±1.7	8.3±1.9	12.1±1.7	4.7±1.8	18.8±5.8	-48.7±2.9	-40.4±3.1
Repeat - outside	4.8±1.5	16.9±11.6	7.0±6.8	4.0±5.7	14.6±12.9	-62.6±24.1	-59.9±25.3

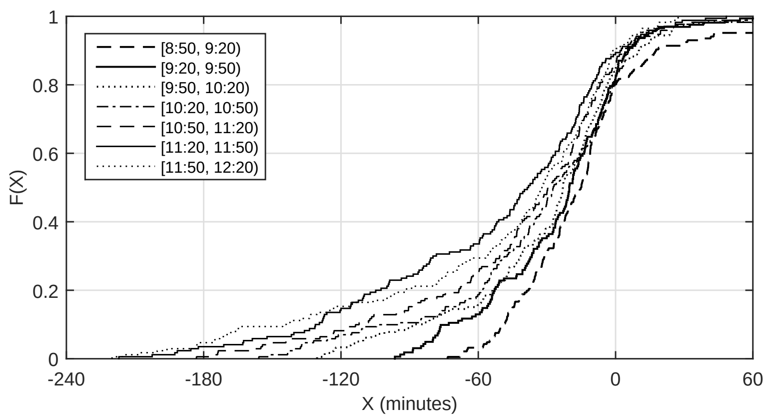
average lateness X (whose values are negative when the patient is early). Table 4 shows that the likelihood of lateness and the expected value of that lateness tend to decrease over the day. In particular, we see that, on average, 15% of the patients are late (arrive after the appointment time) each day and have an average lateness of $E[X^+] = 21$ minutes, but the percentage decreases over the day from 21.2% in the first half hour to 9.5% in the last half hour, while the average amount of lateness among these late patients, $E[X^+]$, decreases from 35.8 minutes to 12.7 minutes. In general, Table 4 shows that patients tend to arrive early rather than late. This again reflects strong adherence to the schedule.

4.2 Toward a Model of the Deviations

We now look closer into the deviations of the actual arrival times from the scheduled arrival times. Figure 4 shows the *empirical cumulative distribution functions* (ecdf's) of the lateness for each of the half hour time slots in Table 4. Figure 4 shows that the lateness consistently decreases over the day in the strong sense that each successive ecdf is stochastically less than the one before; see §9.1 of Ross (1996). (One ecdf is stochastically less than or equal to another if the entire ecdf lies *above* the other, e.g., the stochastically largest ecdf (with the most lateness) falls below all others, and occurs in the first half hour.)

We continue to create a model of patient lateness (or earliness). The model has each scheduled arrival

Figure 4: The lateness empirical cdfs in each of the 30-minute intervals



arrive at a random deviation from its scheduled arrival time. Let the arrivals scheduled to arrive at each time be labeled in some determined order, independent of the actual arrival time. We let the k^{th} arrival among the scheduled arrivals in time slot j (at time ψ_j in (2.1)) actually occur at time

$$A_{j,k} = \psi_j + X_{j,k} = \sum_{i=1}^{j-1} \tau_i + X_{j,k}, \tag{4.1}$$

where $X_{j,k}$ are mutually independent random variables, independent of the schedule (assuming arrivals are acting independently), where $X_{j,k}$ is distributed as the random variable X_j with *cumulative distribution function* (cdf)

$$F_j(x) \equiv P(X_j \leq x), \quad -\infty < x < +\infty. \tag{4.2}$$

We allow X_j to assume both positive and negative values, representing arriving late and arriving early.

The ecdf's in Figure 4 can be regarded as estimates of the cdf's F_j, where we use the same cdf F_j for all three ten-minute time slots j in the specified half hour. For a simple model, we might want a single cdf F, but Table 4 and Figure 4 presents strong evidence that F_j should be allowed to depend on j, at least to some extent.

Finally, we note that it may be deemed useful to incorporate constraints on the arrival times at the beginning and the end of the time period. We might replace $A_{j,k}$ with the constrained version

$$A^c_{j,k} \equiv \max\{0, \min\{T_F, A_{j,k}\}\}. \tag{4.3}$$

As concrete stochastic models, we suggest fitting $P(X_j > 0)$ to the observed proportion of lateness in the half hour containing j, and then fitting distributions to the observed values of lateness X^+ or earliness X^-

separately. The lateness probability estimates are given directly in Table 4. Similarly, we can use the ecdf's, denoted by $\hat{F}(x)$, to generate the model cdf's of X^+ and X^-, letting

$$F_{X^+}(x) \equiv P(X \le x | X > 0) \equiv \frac{\hat{F}_j(x) - \hat{F}_j(0)}{1 - \hat{F}_j(0)} \quad \text{and} \quad F_{X^-}(x) \equiv P(X \le -x | X \le 0) \equiv \frac{\hat{F}_j(-x)}{\hat{F}_j(0)}, \quad x \ge 0. \tag{4.4}$$

From Figure 4 it appears that it should also be possible to use more elementary parametric models. We show the results of fitting exponential cdfs to X^+ and X^- over hours to the sample means in Figure 5. Figure 5 shows that the estimated scv c^2 is less than 1 for X^- and greater than 1 for X^+. Given the limited data, the exponential fit for X^- might be judged adequate, but we might want to allow for greater variability in the lateness. We provide for that by considering a two-moment hyperexponential (mixture of two exponentials, with $c^2 > 1$ and balanced means, as on p. 137 of Whitt (1982)), in Figure 6.

Figure 5: Earliness (X^-) and Lateness (X^+) histograms and associated exponential fits. Top to bottom: scheduled arrivals in [9,10), [10,11), and [11,12).

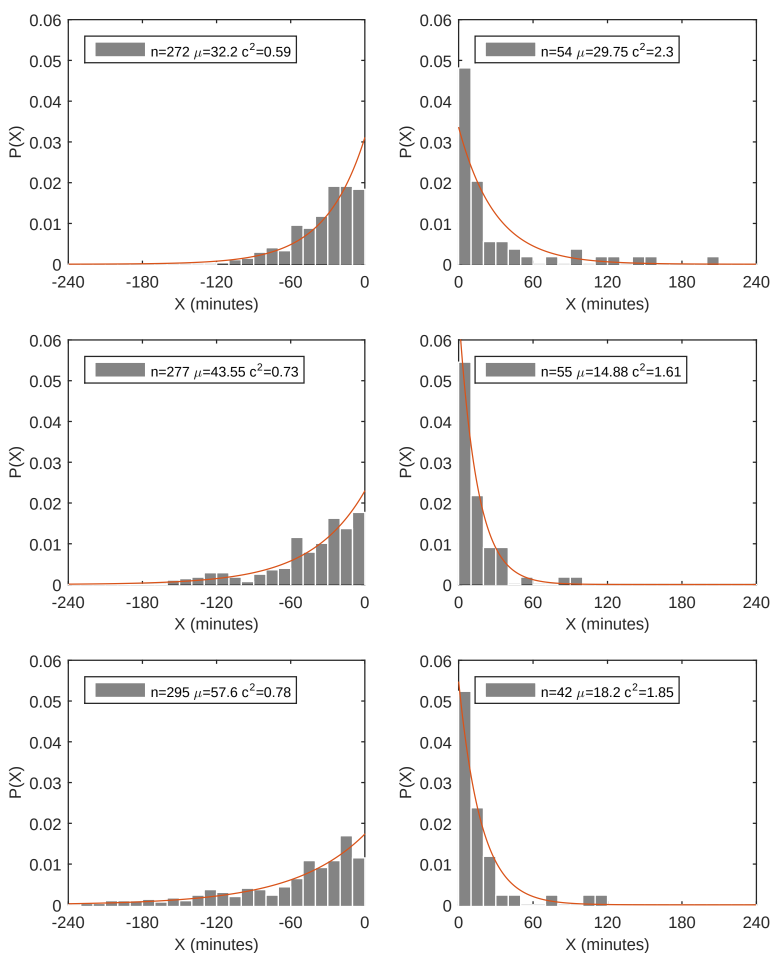
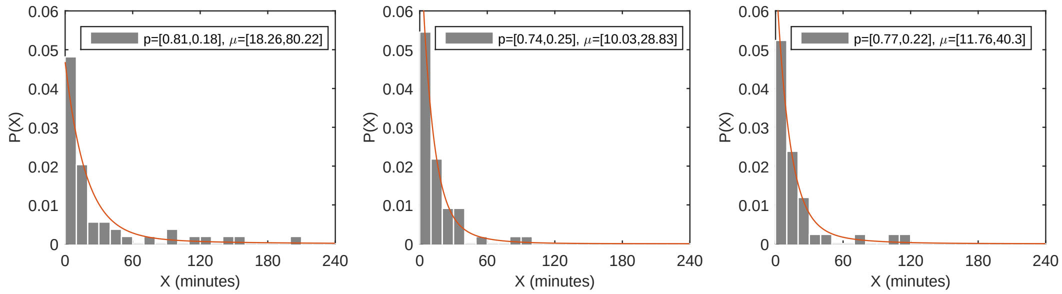


Figure 6: Lateness (X^+) histograms and associated hyperexponential (H_2) fits. Left to right: scheduled arrivals in [9,10), [10,11), and [11,12).



Given that we have specified the cdf's F_j, we have completed construction of a full stochastic model of the arrival process, which can be used to simulate arrivals to the clinic.

4.3 Direct Comparison of the Arrivals to the Schedule

We now directly compare the realized arrivals to the schedule. Table 9 in the appendix shows for each time slot during the 22 days the difference between the numbers scheduled for the slot and the numbers that arrived in that slot. The difference is often large, which we have seen must be primarily due to deviations from the scheduled arrival times, especially earliness. Figures 7 and 8 provide summary views.

Let $S(t)$ and $A(t)$ count the number of scheduled and actual arrivals up to time t in the am shift. Figure 7 shows the histograms of the 22 observed values of the counting processes $S(t)$ and $A(t)$ for a few values of t: 10 am, 11 am, 12 pm and 1 pm. Figure 7 exposes systematic effects and shows the variability. Figure 7 shows that the arrivals tend to occur before the schedule at 10 am and 11 am, but then they are about even at 12 pm and then fall slightly behind at 1 pm. We have seen that this is caused by the earliness of patient arrivals.

Figure 8 summarizes by plotting the average numbers of scheduled and actual arrivals for each of the ten-minute time slots within the 22 am shifts. Figure 8 also shows linear rate functions fit by least squares to the 22 averages of the scheduled and actual arrivals for each of the 22 ten-minute time slots within the main time interval (the solid lines). As should be expected, we see that the estimated rate function for the schedule within the main time interval is constant, but the estimated rate function of the actual arrivals is decreasing, because of the tendency for patients to arrive early.

Finally, Figure 8 shows an additional continuous piecewise-linear estimated arrival-rate function (the dotted lines) for the arrivals over the three intervals of the am shift. This dotted line has an extra linear piece before the main interval to account for the earliness. We will use this construction as the arrival rate resulting from the schedule in the main interval in a simple model constructed in §5.3.

Figure 7: Histograms of the counting processes $S(t)$ and $A(t)$ at four different times. From left to right: 10am, 11am, 12pm, and 1pm.

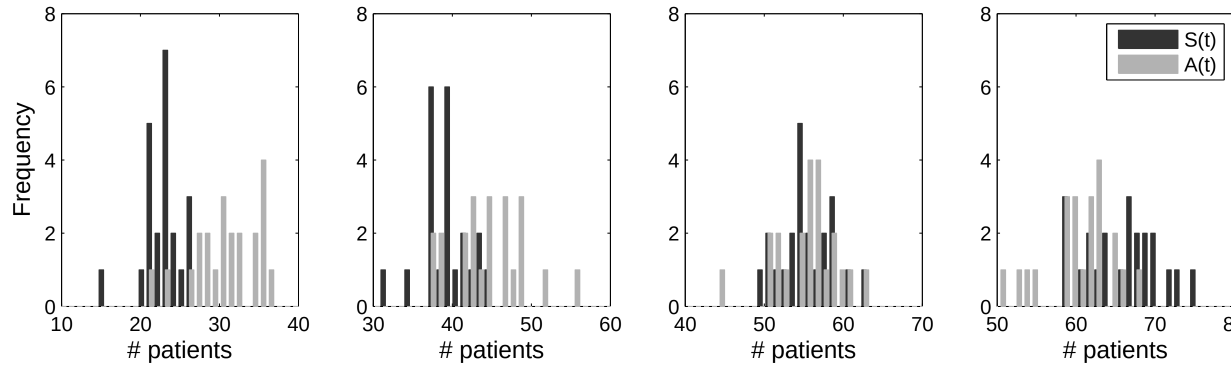
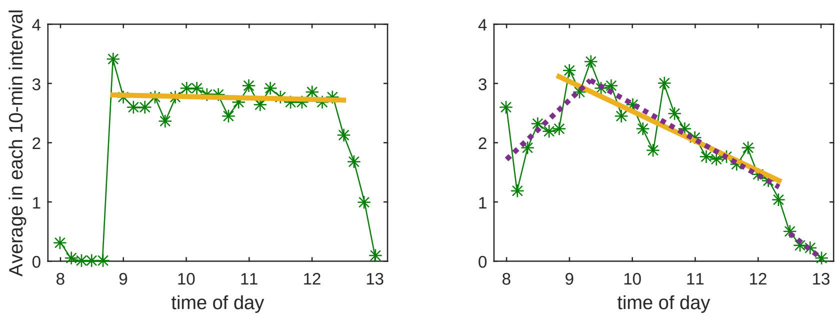


Figure 8: Plots of the average numbers of scheduled (left) and actual (right) arrivals in each of the 22 10-minute intervals in the interval $[8{:}50, 12{:}20]$ and their fitted lines.



5 Mathematical Models of the Counting Processes

In this section we give concise mathematical representations for the stochastic counting processes $S(t)$ and $A(t)$, counting the number of scheduled and actual arrivals up to time t in the am shift, defined in terms of the model elements developed in previous sections.

First, the number of scheduled arrivals up to time t can be expressed as the sum

$$S(t) = \sum_{j=1}^{k} B_{s,j}, \quad \psi_k \le t < \psi_{k+1}, \quad k \ge 0, \quad \text{for all} \quad t, \tag{5.1}$$

for ψ in (2.1) and the batch sizes $B_{s,j}$. According to the model in §2, $B_{s,j}$ should be i.i.d. random variables with distribution in (2.3) inside the main time interval and distributed outside according to §2.7.

Let $A_S(t)$ count the number of scheduled arrivals to actually arrive up to time t. To define it, let the scheduled arrivals in each arrival epoch j (at time ψ_j) be ordered in some definite manner, not having to do with their actual arrival time. Let $I_{j,k} = 1$ if scheduled arrival k at time ψ_j actually arrives on that day and let $X_{j,k}$ be the deviation of the actual arrival time from the scheduled time; if $X_{j,k} > 0$, the arrival is late, otherwise the arrival is early. (For simplicity in labeling, we have variables $X_{j,k}$ even when $I_{j,k} = 0$, but they will play no role.) We combine these two random features with the indicator random variable $I_{j,k}(t)$, defined by

$$I_{j,k}(t) \equiv 1_{\{I_{j,k}=1,\, X_{j,k} \le t\}}, \quad -\infty < t < \infty,\ 1 \le k \le B_{s,j},\ j \ge 0. \tag{5.2}$$

Given the definitions above, we can write

$$A_S(t) = \sum_{j=1}^{\infty} \sum_{k=1}^{B_{s,j}} 1_{\{I_{j,k}=1,\, X_{j,k} \le t-\psi_j\}} = \sum_{j=1}^{\infty} \sum_{k=1}^{B_{s,j}} I_{j,k}(t-\psi_j), \tag{5.3}$$

for $-\infty < t < +\infty$, where ψ_j is defined in (2.1). We may have $A_S(t) > 0$ for $t < 0$ because of early arrivals.

Let $A_U(t)$ ($A(t)$) count the number of unscheduled (all) arrivals by time t. Then we have

$$A(t) = A_S(t) + A_U(t), \quad \text{for all} \quad t. \tag{5.4}$$

From §3.2, for the clinic A_U would be independent of A_S, having two independent Poisson-based components, one for inside the main time interval and another outside.

5.1 Conditional Means and Variances Given the Schedule

Now suppose that the schedule is known, i.e., we know $B_{s,j}$ for all j, as would be the case at the end of the previous day in the clinic. Let the information about the schedule ($B_{s,j}$ for all j) be denoted by $\mathcal{S}$.

Since the ordering on k for each j is totally arbitrary, it is natural to assume that the joint distribution of $(I_{j,k}, X_{j,k})$ is independent of k for each j, and we make that assumption. Then the conditional cumulative arrival rate function for the scheduled arrivals given the schedule is simply the conditional expected value

$$\Lambda_S(t|\mathcal{S}) \equiv E[A_S(t)|\mathcal{S}] = \sum_{j=1}^{\infty} B_{s,j} p_j(t), \quad -\infty < t < +\infty, \tag{5.5}$$

where

$$p_j(t) \equiv E[I_{j,k}(t-\psi_j)] = P(I_{j,k} = 1,\, X_{j,k} \le t-\psi_j) = (1-\delta)F_j(t-\psi_j), \tag{5.6}$$

where $F_j(t) \equiv P(X_{j,k} \le t)$, which is independent of k. As usual, the associated arrival rate function $\lambda_S(t|\mathcal{S})$ is the derivative with respect to t of the cumulative arrival rate function $\Lambda_S(t|\mathcal{S})$, i.e.,

$$\lambda_S(t|\mathcal{S}) = \sum_{j=1}^{\infty} B_{s,j}(1-\delta) f_j(t-\psi_j), \tag{5.7}$$

where f_j is the probability density function (pdf) associated with the cdf F_j. The associated conditional variance is

$$V_S(t|\mathcal{S}) \equiv Var(A_S(t)|\mathcal{S}) = \sum_{j=1}^{\infty} B_{s,j}^2 p_j(t)(1-p_j(t)), \tag{5.8}$$

for $p_j(t)$ in (5.6).

5.2 The Total Mean and Variance of $A(t)$

The total arrival rate function is then

$$\begin{aligned}\Lambda(t) &\equiv E[A(t)] = E[A_S(t)] + E[A_U(t)] = E[\Lambda_S(t|\mathcal{S})] + E[A_U(t)] \\ &= \sum_{j=1}^{\infty} E[B_{s,j}](1-\delta) f_j(t-\psi_j) + E[A_U(t)]. \end{aligned} \tag{5.9}$$

Applying the conditional variance formula, assuming that the random variables $B_{s,j}$ are mutually independent, the associated variance is

$$\begin{aligned}Var(A(t)) &= Var(A_S(t)) + Var(A_U(t)) = Var(E[A_S(t|\mathcal{S})]) + E[Var(A_S(t)|\mathcal{S})] + Var(A_U(t)) \\ &= \sum_{j=1}^{\infty} Var(B_{s,j})[(1-\delta) f_j(t-\psi_j)]^2 + \sum_{j=1}^{\infty} B_{s,j}^2 p_j(t)(1-p_j(t)) + Var(A_U(t)). \end{aligned} \tag{5.10}$$

5.3 A Parsimonious Simplified Arrival Process Model

As before, we classify each day as AC or OL, but in our model we make them random with OL occurring with probability $12/22$ and AC otherwise, coinciding with the observed frequencies among the 22 days. We divide the overall time interval [8:00,13:00] into two parts, before and after 12:30. We let D_F be the daily total during the final interval [12:30,13:00]. We let it be conditional on whether the day is OL or AC. For each kind of day, we let the daily totals be distributed as in Table 2. That makes the mean number for an OL day 7.60 and the mean number of an AC day 1.50. Thus, the overall mean number in [12:30,13:00] is 4.82. We then let those be distributed among the intervals, as indicated in §2.7.

We let D_I be the random daily total for the initial interval [8:00,12:20], where we treat all days the same. We let $E[D_I] = 66.1 - 4.8 = 61.3$, making it coincide with the observed average total of 66.1 in Table 1. We let the variance coincide roughly with the variance of the schedule inside in Table 1, so that $Var(D_I) = 10.0$. We can use a Gaussian distribution (rounded to the nearest integer) with this estimated mean and variance. Alternatively, we can fit a binomial distribution with parameter pair (n, p) to this mean and variance getting two equations in two unknowns: $E[D_I] = np = 61.3$ and $Var(D_I) = np(1-p) = 10$, so that $(1-p) = 10/61.3 = 0.163$ and $n = 61.3/0.837 = 73.2$, rounded to 73. Hence we regard D_I as binomial $(n, p) = (73, 0.837)$.

Given D_I, the daily total in the initial interval, we let these arrivals be i.i.d. over the initial interval [8:00,12:20] with probability density function proportional to the continuous two-piece arrival rate function in Figure 8, i.e., with pdf equal to the arrival rate function divided by its integral over the interval.

In Kim et al. (2015b), binomial-uniform and Gaussian-uniform models were proposed. Our model here differs in two respects. First, we treat the final subinterval [12:30,13:00] separately, accounting for whether or not the day is OL or AC. Second, we treat the initial interval similarly, but our more careful analysis here suggests a non-uniform density of the individual arrivals. We propose a scaled version of the continuous piecewise-linear curve on the right in Figure 8, which should fit the actual arrival rate better.

6 Conclusions

The Principal Source of Variability Is the Schedule. In this paper we have examined an appointment-generated arrival process for one doctor in an endocrinology clinic. As a consequence of the appointment system, the arrival process tends to be much less variable than a Poisson process, but then not nearly a regular deterministic arrival process. The dispersion (variance-to-mean ratio) is about 0.3. As others have observed before, some variability is due to no-shows, extra unscheduled arrivals and deviations of the actual arrival times from the scheduled appointment times, but §3.3 shows that the dominant source of variability in the arrival process is the schedule itself. In particular, the inequality in (3.6) shows that the dispersion of the daily schedule is actually greater than the dispersion of the daily arrivals itself!

New Stochastic Arrival Process Models. Our data analysis has culminated in both a detailed stochastic model in §2.8 and §3 and a simplified stochastic model in §5.3 that can be used to simulate the arrival process of patients to see the doctor in the clinic. The fitting process should be useful for the other doctors in this clinic and in other applications. Simulation experiments can be used to evaluate operational procedures in the clinic.

What is Generalizable? (i) Variations of the specific arrival process stochastic models developed here may be useful for other outpatient clinics, but what we think is widely generalizable is *the data-analysis process rather than the model*. Consistent with earlier work, we advocate carefully examining no-shows, extra unscheduled arrivals and punctuality. However, before doing those steps, we advocate looking at randomness in the schedule. It may even be important to view the schedule as a stochastic process. We did not have data on the original demand, but we would also advocate collecting information on the requests for appointments, including ones that were not scheduled or moved to alternate days and times. We would also want to see how the schedule relates to the original demand.

(ii) The specific arrival process models may also be useful more widely. Especially promising is the parsimonious model with Gaussian daily totals and, given those daily totals, i.i.d. arrival times within the day with a non-uniform probability density that takes account of the earliness and lateness of the patients. It is reasonable to anticipate that the earliness or lateness will alter the arrival rate during the day as we have discovered.

(iii) Even more broadly, it is important to recognize that appointment-generated arrival processes are likely to be neither deterministic and evenly spaced nor Poisson. Many systems will have variability in between those two extremes, just as we have seen.

What is the practical relevance? In this paper we have not performed a complete performance analysis of the endocrinology outpatient clinic, so we have not yet improved the performance of that clinic. However, from the long history of modeling and analysis of outpatient clinics briefly surveyed in §1.1, it is understood that modeling and analysis can improve system performance. Thus, we did this work with the conviction that improved arrival process models can produce improved performance.

We see two principal ways that the stochastic models of the appointment system can be used to improve the performance of the clinic, and similar stochastic models can be used to improve performance in other appointment system applications. First, the model provides a basis for analyzing the performance of the clinic with the given arrival process by conducting standard performance (queueing) analyses, after incorporating an additional detailed analysis of the patient processing and flow after arrival, which we do not consider here. Second, the model can be used to consider alternative scheduling strategies to achieve various objectives, such as reducing variability of the schedule and thus reducing the variability in the doctor workloads, and ensuring that patients with urgent needs have limited delays for getting an appointment.

Classification of Appointment-Generated Arrival Processes. In addition to gaining a better understanding of the appointment-generated arrival process in the endocrinology clinic, we have learned how to think about appointment-generated arrival processes more generally. While diverse appointment systems should have much in common, there also can be important differences. A useful first step when considering appointment systems and appointment-generated arrival processes is to first classify the system. Our analysis of the clinic helps, summarized in Table 6, show how that can be done. There are three main steps, which we explain in detail below. For any new appointment system to be considered, we recommend seeking this information. After evaluating both the schedule and the adherence to the schedule, comparing them to what is desired, one could consider ways to improve both the schedule and the adherence.

Step 1. General Classification We should first identify the *time frame*, which we take to be a day. However, there are two different perspectives: first, the times when the arrivals occur and, second, the times when the appointments are actually made. We are primarily focusing on the times when the arrivals occur, aiming to understand variability over the day.

However, as in the clinic, the appointments may have been made over a much longer time frame, weeks or even months before the appointment day, so that the delay in getting an appointment may be in a longer scale.

Table 6: Steps to classify an appointment-generated arrival process and their application to the arrivals for the doctor at the clinic

Category	Issue	For the doctor at the endocrinology outpatient clinic
General	time frame for arrivals	one morning shift on a single day
	time from schedule to appointment	mostly 1-4 months
	time sensitivity (urgency) of appointment	not known
	repeat versus new	78% of visits are repeat
	scale	moderately large, average daily total 66
	variability of arrival process	significant but less than Poisson, dispersion $V/M = 0.3$ for daily totals
Schedule	variability of the schedule	significant but less than Poisson, dispersion $V/M = 0.3$ for daily totals
	deterministic framework	identifiable as 22 ten-minute intervals with batches of size 3
	primary deviation from the framework	extra scheduled arrivals outside the main interval
	high or low demand	high demand
	extent of overload	12 of 22 days overloaded, overload produces 10% of daily totals
	manifestation of overload	overload occurs outside, usually after, the main interval
	distribution of the main schedule	the data support i.i.d batches with mean 2.76 in all time slots
Adherence	no–shows	relatively few no-shows, about 8.5%
	unscheduled arrivals	relatively few unscheduled arrivals about 2 per day (3%)
	deviations (lateness or earliness)	significant deviations of about 60 minutes, but mostly early, about 15% late with average conditional lateness given late of about 20 minutes

With such long delays between the date the schedule is made and the appointment date, we have observed that it is important to consider whether arrivals represent, perhaps routine, *repeat visits* or are new requests. Especially in healthcare, an important question is whether the system can respond well to urgent requests for service. Unfortunately, the time-sensitivity or urgency was not part of the clinic arrival data, but we were able to identify repeat visits, which were 78% of all visits. It is important to recognize that long delays do not necessarily mean that patients with urgent problems are having excessive delays before their needs can be addressed. For healthcare appointment systems, it would be good to have information of the *delay sensitivity or urgency* of the service to be provided.

We next would focus on the *scale*, determined by the typical daily totals. Is the scale large or small? The clinic doctors operate in a fairly large scale, with our doctor seeing about 66 patients in each shift (am or pm).

Assuming that our goal is to understand the arrival process over a single day and possibly make improvements in it, the next question is the *level of variability in the appointment-generated arrivals*. Are the arrivals highly regular or not? Our analysis is devoted to the case in which the arrivals exhibit significant variability. An initial rough classification of the variability is the *dispersion* or variance-to-mean ratio V/M of the daily totals.

The remaining classification is aimed at exposing the primary sources of the variability observed in the arrivals. Then careful analysis is devoted to identifying and quantifying the important sources of that variability. It is natural to start with the schedule.

Step 2. The Schedule Given that the actual arrivals are irregular, we ask if the scheduled arrivals are also irregular, also exhibiting a significant level of variability. For our doctor in the clinic, we found that the schedule is indeed quite irregular, exhibiting significant variability. It too can be roughly quantified by the dispersion of the daily totals. In fact, we conclude that the primary source of variability in the arrivals is the variability in the schedule. This is supported by the fact that both the variance and the dispersion of the scheduled daily totals are greater than for the actual arrivals.

Whether the schedule is regular or not, we want to identify the fundamental deterministic framework, if possible. In general, a first step in analyzing the schedule is to infer this framework. An orderly framework might be communicated by system managers, but it is important to see data showing what actually happened. From our examination of the schedule on the 22 am shifts, we were able to identify a stationary framework involving small batches of arrivals at ten minute intervals during the time interval $[8{:}50, 12{:}20]$.

We should then ask what are the major deviations from this framework. We found that the batch sizes in each time slot are variable, but the largest deviation from that framework were extra scheduled arrivals outside the main time interval.

In general, it is evidently important to determine whether the service system is a *high-demand or a low-demand system*. Is the variability due to the uncertain ability to fill the schedule in the presence of low demand or because of uncertain response to pressures to meet high demand? Or do we see a combination of these? We concluded that our doctor in the clinic consistently operates as a high-demand system, with significant response being made to the high demand. In particular, 12 of the 22 days are overloaded and the remaining 10 are at-capacity.

We then come to the distribution of the scheduled arrivals in the main interval. We concluded in §2.5 and §2.6 that the scheduled arrivals in the 22 time slots of the main interval each day can be regarded as i.i.d. random variables with the distribution in (2.3), which has mean 2.76. We found relatively low variability in the scheduled arrivals within this main interval.

Step 3. Adherence to the Schedule We next shift attention to the adherence to the schedule. We focused on three ways that the arrivals might not adhere to the schedule: (i) no-shows, (ii) extra unscheduled arrivals and (iii) deviations in actual arrival times from the scheduled times. Since our clinic data included cancellations, no-shows were easily identifiable as scheduled arrivals that never came. Given that all arrivals were in our clinic data, and our definition of the schedule by its value at the end of the previous day, we defined unscheduled arrivals as arrivals that were scheduled and arrived on the current day.

It is well known that no-shows and unscheduled arrivals can be quite high in appointment-generated arrival processes. However, in the clinic there was a relatively low percentage of no-shows and unscheduled arrivals. The average percentage of no-shows for our doctor in the clinic was about 8.5%. This level was fairly constant over the day except was somewhat larger during the first intervals of the am shift. The average number of unscheduled arrivals in the clinic was only about 2 per day, which is 3% of the daily total. About half of those occurred outside the main interval, again indicating effort to respond to extra high demand.

Significant deviations in the actual arrival time from the scheduled arrival times were observed, of approximately 60 minutes, but most were due to early arrivals. Only about 6% of the arrivals were late by more than 15 minutes. Overall, we conclude that the adherence to the schedule was very good, relative to other appointment systems.

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APPENDIX

This is an appendix to the main paper. We display three additional tables.

Table 7: The extra unscheduled arrivals, i.e., the same-day arrivals $B_{u,j}$ scheduled for slot j, on each of the 22 days.

time slot	22 days in July-October 2013																						Avg	Var	Var/Avg
7:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0.05	0.05	1.00
8:10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:50	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0.14	0.12	0.90
9:00	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0.09	0.09	0.95
9:10	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0.14	0.12	0.90
9:20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
9:30	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
9:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0.09	0.09	0.95
9:50	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
10:00	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
10:10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
10:20	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0.14	0.12	0.90
10:30	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0.09	0.09	0.95
10:40	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0.14	0.12	0.90
10:50	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
11:00	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
11:10	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0.09	0.09	0.95
11:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0.05	0.05	1.00
11:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
11:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
11:50	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0.09	0.09	0.95
12:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
12:10	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0.14	0.12	0.90
12:20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
12:30	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0.14	0.12	0.90
12:40	0	0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0.14	0.22	1.60
12:50	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0.09	0.09	0.95
13:00	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0.09	0.09	0.95
Daily Total	2	3	0	1	1	1	1	3	2	1	3	8	3	2	2	2	3	4	0	2	3	1	2.18	2.82	1.29
[8:50, 12:20] Total	2	3	0	1	1	1	1	1	2	1	2	5	2	1	2	2	3	3	0	1	2	1	1.68	1.27	0.76

Table 8: The unscheduled arrivals that actually arrived ($B_{a|u,j}$) for slot j, on each of the 22 days.

Slot	Different Days																						Avg	Var	Var/Avg
7:50	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
8:50	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0.14	0.12	0.90
9:00	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0.09	0.09	0.95
9:10	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0.14	0.12	0.90
9:20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
9:30	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
9:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0.09	0.09	0.95
9:50	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
10:00	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
10:10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
10:20	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0.14	0.12	0.90
10:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0.05	0.05	1.00
10:40	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0.14	0.12	0.90
10:50	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
11:00	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0.09	0.09	0.95
11:10	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
11:20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0.05	0.05	1.00
11:30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
11:40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
11:50	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0.09	0.09	0.95
12:00	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.00		
12:10	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	0.14	0.12	0.90
12:20	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.05	0.05	1.00
12:30	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0.09	0.09	0.95
12:40	0	0	0	0	0	0	0	0	0	0	1	2	0	0	0	0	0	0	0	0	0	0	0.14	0.22	1.60
12:50	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0.09	0.09	0.95
13:00	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0.09	0.09	0.95
Daily Total	2	3	0	1	1	1	1	2	2	0	3	7	3	1	2	2	3	4	0	2	2	1	1.95	2.43	1.24
[8:50, 12:20] Total	2	3	0	1	1	1	1	1	2	0	2	4	2	0	2	2	3	3	0	1	2	1	1.55	1.21	0.78

Table 9: The difference between the number of patients scheduled to arrive at slot j ($B_{s,j}$) and the number of patients that actually arrived at slot j ($B_{a,j}$), on each of the 22 days. Summary statistics are based on absolute values.

Slot	Different Days																						Avg	Var	Var/Avg
7:50	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-3	0	0	0	0.18	0.44	2.43
8:00	0	-2	-3	-1	-7	-2	-2	-4	1	-2	-4	-4	-2	-5	-1	-3	-2	-4	0	-1	-2	0	2.36	3.10	1.31
8:10	-3	0	0	0	-2	-4	-1	1	-3	0	0	-4	0	0	-2	-3	-2	0	0	-2	0	0	1.23	2.09	1.70
8:20	-2	-1	-3	-3	-1	-2	-2	-5	-3	-1	0	0	-3	0	-1	-3	-4	0	-4	0	-2	-2	1.91	2.18	1.14
8:30	0	-2	-3	-1	-2	-2	-1	-1	-2	0	-3	-2	-2	-2	-3	-3	-3	-2	-1	-4	-5	-7	2.32	2.51	1.08
8:40	-3	-5	-4	-5	-3	-3	-3	-3	-1	-2	0	-2	0	-3	-1	-1	-1	-1	-3	0	-2	-2	2.18	2.16	0.99
8:50	2	2	4	3	3	2	1	3	2	-3	2	1	0	0	1	0	1	-6	3	2	2	1	2.00	2.00	1.00
9:00	-1	-2	1	0	1	-1	-2	-1	-3	-2	2	0	0	-1	-3	0	1	-1	0	0	0	2	1.09	0.94	0.87
9:10	0	1	1	0	-3	0	1	0	-1	0	-3	1	-2	3	1	-1	-2	1	0	-2	1	-2	1.18	1.01	0.86
9:20	0	-1	-1	-2	-1	-4	-1	-2	-3	-2	-1	1	1	-1	1	1	-1	2	-2	0	1	-2	1.41	0.82	0.59
9:30	-2	1	0	-1	1	2	1	1	1	-3	-4	1	-1	0	1	-3	0	2	1	1	-2	0	1.32	1.08	0.82
9:40	2	2	-1	0	-2	-2	0	2	0	2	-3	-3	1	1	-1	0	-4	-2	-1	-2	-1	-1	1.50	1.12	0.75
9:50	1	2	-2	2	0	-1	-2	0	-1	1	2	-1	-1	1	2	0	2	1	0	-3	1	3	1.32	0.80	0.61
10:00	-4	0	1	0	-1	3	-4	1	1	2	0	1	-1	-2	0	1	3	2	0	0	2	1	1.36	1.58	1.16
10:10	0	-1	2	3	1	0	1	2	2	-1	1	3	-1	-2	-2	2	0	1	1	0	2	1	1.32	0.80	0.61
10:20	1	1	1	2	0	2	1	1	2	0	-2	3	2	1	2	-1	2	0	0	2	1	0	1.23	0.76	0.62
10:30	0	-3	-3	0	1	1	0	-1	1	0	-1	1	2	-3	0	1	1	-2	1	1	-1	0	1.09	0.94	0.87
10:40	2	0	0	1	2	-2	-1	1	0	0	0	0	-1	0	0	0	2	0	-4	0	-1	0	0.77	1.14	1.47
10:50	0	-3	2	2	-2	0	0	-1	1	0	2	1	0	0	3	2	-1	1	1	-1	2	1	1.18	0.92	0.78
11:00	0	1	1	1	-1	-2	3	1	-1	4	3	1	1	2	2	0	2	1	0	3	-2	-1	1.50	1.12	0.75
11:10	-2	0	2	-1	3	2	2	1	1	0	1	0	1	1	-1	2	1	1	1	1	0	3	1.23	0.76	0.62
11:20	1	3	-1	3	3	3	1	2	2	1	0	1	-3	1	-2	2	3	0	2	1	2	1	1.73	0.97	0.56
11:30	0	0	2	1	2	1	1	-1	1	3	0	3	2	2	-2	2	2	-1	1	1	1	1	1.36	0.72	0.53
11:40	3	1	2	2	0	0	3	3	1	0	1	-2	1	-1	2	1	1	0	0	3	1	1	1.32	1.08	0.82
11:50	-1	1	3	2	3	0	0	-2	-1	-1	1	1	1	2	2	-1	2	1	2	2	-1	1	1.41	0.63	0.45
12:00	1	3	1	0	2	3	4	0	0	2	2	2	2	0	-2	3	0	2	1	1	3	1	1.59	1.40	0.88
12:10	3	2	1	-1	2	3	2	2	0	2	2	1	3	2	1	1	-1	2	-1	1	2	0	1.59	0.73	0.46
12:20	2	2	3	2	2	1	3	1	2	2	3	2	3	1	1	2	0	1	-1	2	2	2	1.82	0.63	0.35
12:30	2	1	0	-1	0	2	3	2	0	2	2	2	2	3	3	1	0	4	3	1	2	2	1.73	1.26	0.73
12:40	0	0	0	0	0	2	2	3	3	0	3	2	1	0	2	3	3	2	3	0	0	2	1.41	1.68	1.19
12:50	0	0	0	0	0	0	0	1	4	0	0	0	0	2	2	0	1	0	4	0	-1	4	0.86	2.03	2.35
13:00	1	0	0	0	1	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.14	0.12	0.90
Daily Total	39	44	48	40	52	52	48	49	45	38	48	46	40	42	47	43	48	43	44	37	45	44	44.64	17.67	0.40
[8:50, 12:20] Total	28	32	35	29	36	35	34	29	27	31	36	30	30	27	32	26	32	30	23	29	31	25	30.32	12.61	0.42
All slot avg	2.0	2.0	2.2	1.9	2.0	2.0	2.4	2.3	2.3	1.9	2.2	2.1	1.9	2.3	2.2	2.2	2.2	2.1	2.2	2.1	2.2	2.4	1.39	1.42	1.02
All slot var	1.5	1.9	2.2	1.9	1.8	1.5	1.8	1.3	1.5	1.7	1.5	1.5	1.6	1.5	1.3	1.7	1.8	1.6	1.6	2.2	1.8	1.6		(across all days)	
All slot var/avg	0.7	1.0	1.0	1.0	0.9	0.8	0.8	0.6	0.6	0.9	0.7	0.7	0.8	0.6	0.6	0.8	0.8	0.8	0.7	1.1	0.8	0.7			
[8:50, 12:20] avg	2.7	2.8	3.0	2.7	2.7	2.6	3.0	2.7	2.8	2.6	2.9	2.7	2.5	2.8	2.6	2.8	2.7	2.6	2.7	3.0	2.9	2.9	1.38	1.02	0.74
[8:50, 12:20] var	0.2	0.6	0.3	0.5	0.4	0.4	0.4	0.3	0.4	0.5	0.2	0.3	0.5	0.3	0.4	0.4	0.7	0.3	0.4	0.7	0.4	0.4		(across all days)	
[8:50, 12:20] var/avg	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.2	0.1	0.3	0.1	0.2	0.2	0.1	0.1			