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Calculating the M/G/1 busy-period density and LIFO waiting-time distribution by direct numerical transform inversion

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Abstract

It is well known that the M/G/1 busy-period density can be characterized by the Kendall functional equation for its Laplace transform. The Kendall functional equation can be solved iteratively to obtain transform values to use in numerical inversion algorithms. However, we show that the busy-period density can also be numerically inverted directly, without iterating a functional equation, exploiting a contour integral representation due to Cox and Smith (1961). The contour integral representation was originally proposed as a basis for asymptotic approximations. We derive heavy-traffic expansions for the aysmptotic parameters appearing there. We also use the integral representation to derive explicit series representations of the busy-period density for serval service-time distributions. In addition, we discuss related contour integral representations for the probability of emptiness, which is directly related to the waiting-time distribution with the LIFO discipline. The asymptotics and the numerical inversion reveal the striking difference between the waiting-time distributions for the FIFO and LIFO disciplines.

Keywords: Busy period; Probability of emptiness; M/G/1 queue; Laplace transforms; Numerical transform inversion; Last-in–first-out service discipline

1. Introduction

Consider the standard M/G/1 queue with service rate 1 and arrival rate $\rho < 1$. Let the serice-time distribution have a density $g(t)$ and let $\hat{g}(s) \equiv \int_0^\infty e^{-st} g(t)\,dt$ be its Laplace transform, where s is complex with $\mathrm{Re}(s) > 0$. Similarly, let $b(t)$ be the density of the busy period and let $\hat{b}(s) \equiv \int_0^\infty e^{-st} b(t)\,dt$ be its Laplace transform. It is well known that $b(t)$ is characterized by the *Kendall functional equation* for $\hat{b}(s)$, i.e.,

$$\hat{b}(s) = \hat{g}(s + \rho - \rho\hat{b}(s)); \tag{1}$$

e.g. see [13, Eq. (5.137), p. 212; 6, Eq. (28)].

Unfortunately, explicit representations for the transform $\hat{b}(s)$ are unavailable, except for special cases such as the M/M/1 model [3]. However, as discussed in [5], for any s (complex or real), the

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Laplace transform $\hat{b}(s)$ can easily be calculated from (1) by *iteration*, with upper and lower bounds on the cumulative distribution function (cdf) holding when we start with $\hat{b}(s)$ on the right replaced by 0 and 1. Hence, we can numerically calculate the values of the density $b(t)$ by numerically inverting $\hat{b}(s)$, using iteration to obtain the required transform values.

Here we observe that it is also possible to obtain a *direct contour integral representation* for $b(t)$, so that we can calculate $b(t)$ for any t by *directly numerically inverting a Laplace transform without performing any extra transform iteration.* The direct representation is obtained from the *series representation*

$$b(t) = \sum_{n=0}^{\infty} e^{-\rho t}(\rho t)^n g_{n+1}(t)/(n+1)!, \tag{2}$$

where $g_n(t)$ is the density of the sum of n i.i.d. service times; see [13, p. 226]. This series representation is often called the Takács series representation for the M/G/1 busy-period density, evidently due to its appearance in [18; 19, Eq. (34)], but it was obtained independently by Cox and Smith [10, Eq. (48), p. 155]. Both proofs involve a special case of the ballot theorem, which was latter discussed extensively by Takács [21]. (The series representation (2) is associated with the two-dimensional *Takács functional equation* for the joint distribution of the length of a busy period and the number of customers served in that busy period.)

We will show in Section 2 that we obtain from (2) the inversion (or contour integral) representation

$$b(t) = (\rho t)^{-1}\mathscr{L}^{-1}(\exp(-\rho t(1-\hat{g}(s)))), \tag{3}$$

where $\mathscr{L}^{-1}$ is the Laplace transform inversion operator, i.e., the Bromwich contour integral

$$f(t) \equiv \mathscr{L}^{-1}(\hat{f}(s))(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\hat{f}(s)\,ds, \tag{4}$$

with the contour being a vertical line $s = a$ such that $\hat{f}(s)$ has no singularities on or to the right of it. Algorithms to calculate $f(t)$ from $\hat{f}(s)$ via (4) are described in [4]. We note that here $b(t)$ is a density instead of a cdf or complementary cdf, and so is not bounded by 1. Hence, we do not have such a simple bound on the aliasing error in [4], but nevertheless the inversion is easily done. The possibility of constructing a contour integral representation for $b(t)$ is noted by Cox and Smith [10, p. 156], but they do not discuss numerical inversion.

Note that the argument of $\mathscr{L}^{-1}$ in (3) contains the variable t. Thus, the argument of $\mathscr{L}^{-1}$ in (3) is *not* the Laplace transform of the function $\rho t b(t)$, $t > 0$. Nevertheless, for each fixed t, we can calculate $b(t)$ by doing an inversion.

Cox and Smith emphasize the importance of contour integral representations such as (3) for doing asymptotics. In particular, they apply saddle-point methods as in [16, p. 121] to establish the asymptotic behavior for the busy-period density tail, i.e., they show that

$$b(t) \sim \alpha(\pi t^3)^{-1/2}e^{-t/\tau} \quad \text{as } t \to \infty, \tag{5}$$

where $f(t) \sim g(t)$ means that $f(t)/g(t) \to 1$ as $t \to \infty$. The asymptotics for $b(t)$ in (5) easily extend to the associated complementary cdf and many other M/G/1 quantities via integral (or stationary-excess) representations; see [6, Theorems 2-4]. The parameter τ in (5) is the *relaxation time*, which describes the time required for the basic queueing processes to approach steady state. Here in Section 3 we develop heavy-traffic expansions for the relaxation time τ and the *asymptotic constant* α appearing in (5), paralleling our treatment of the GI/G/1 waiting-time tail probabilities in [2]. As before, we see what aspects of the service-time distribution the relaxation time and asymptotic constant primarily depend upon.

In Section 2 we derive the inversion formula (3). In Section 3 we present the heavy-traffic expansions for the tail asymptotics in (5). In Section 4 we use formula (3) to derive explicit results in special cases. In Section 5 we discuss a representation related to (3) for the probability of emptiness due to Beneš [8]. There we also recall the remarkable fact due to Takács [20] that the M/G/1 waiting-time cdf with the last-in-first-out (LIFO) discipline can be expressed directly in terms of the M/G/1 emptiness probability function. Hence, we can calculate the M/G/1 LIFO waiting-time cdf directly via numerical transform inversion and we can establish asymptotics. The M/G/1 LIFO waiting-time distribution has asymptotics just like (5). Indeed, the associated asymptotic decay rate τ^{-1} is the *same* as for the busy-period density in (5). The asymptotic decay rate is smaller for LIFO than for FIFO. Also, with LIFO there is the $t^{-3/2}$ factor instead of the pure exponential asymptotics typically holding with FIFO [2]. We illustrate with a numerical example comparing the FIFO and LIFO M/G/1 waiting-time cdf's (for a service-time distribution with a non-rational Laplace transform).

2. Derivation of formula (3)

Note that we can rewrite (2) as

$$b(t) = h(t, \rho t), \tag{6}$$

where

$$h(t, x) = x^{-1}(1 - e^{-x}) f(t, x) \tag{7}$$

and

$$f(t, x) = (1 - e^{-x})^{-1} \sum_{n=1}^{\infty} e^{-x} x^n g_n(t)/n!, \quad t \geqslant 0. \tag{8}$$

For each $x > 0$, $f(t, x)$ as a function of t is a *probability density function*, in particular, the conditional density of a compound Poisson distribution given that there is at least one Poisson event. Its Laplace transform is

$$\hat{f}(s, x) \equiv \int_0^{\infty} e^{-st} f(t, x)\, dt = (1 - e^{-x})^{-1} \times [e^{-x(1-\hat{g}(s))} - e^{-x}]. \tag{9}$$

Hence, we can directly calculate $b(t)$ by calculating $f(t, x)$ by numerically inverting $\hat{f}(s, x)$ in (9) and then using (6) and (7).

We can simplify the transform expression by eliminating the constant. This corresponds to adding an atom at 0 to both sides. In particular,

$$f(t, x) + e^{-x}(1 - e^{-x})^{-1}\delta(t) = (1 - e^{-x})^{-1}\mathscr{L}^{-1}(e^{-x(1-\hat{g}(s))}), \tag{10}$$

where $\delta(t)$ is the delta function representing an atom at 0. If we restrict attention to strictly positive t, then the atom does not appear. Combining (6), (7) and (10), we obtain (3).

We have performed numerical calculations with (3) and found that the results agree with those obtained by the method of [5].

3. Heavy-traffic expansions for the asymptotic parameters

Cox and Smith [10] establish the asymptotic relation (5) using saddle-point methods; see [16, p. 121 and Eq. (49), p. 156]. They show that the asymptotic decay rate (reciprocal of the relaxation time) is

$$\tau^{-1} = \rho + \zeta - \rho\hat{g}(-\zeta), \tag{11}$$

where ζ is the unique real number u to the left of all singularities of the moment generating function $\hat{g}(-s)$ (assumed to exist) such that

$$\hat{g}'(-u) = -\rho^{-1}. \tag{12}$$

Then

$$\alpha = [2\rho^3\hat{g}''(-\zeta)]^{-1/2}. \tag{13}$$

It is interesting to compare the root ζ of (12) and the decay rate τ^{-1} in (11) with the asymptotic decay rate η of the steady-state waiting-time complementary cdf, which is discussed in [2]. The asymptotic decay rate η is the root of the equation $\hat{g}(-u) = 1 + \rho^{-1}u$, from which we see that $\tau^{-1} < \zeta < \eta$. The heavy-traffic expansions give good quantitative estimates. It turns out that $\eta = O(1 - \rho)$, while $\tau^{-1} = O((1 - \rho)^2)$ as $\rho \to 1$.

To obtain the heavy-traffic expansions for τ^{-1} and α^{-1}, we proceed as in [2], and consider the power series representation for $\hat{g}(s)$ involving moments and expand τ^{-1} and α^{-1} in powers of $(1 - \rho)$. We obtain

$$\tau^{-1} = \frac{(1-\rho)^2}{2(1 + c_s^2)}(1 + (1 - \rho)(1 - \xi) + (1 - \rho)^2[1 - \xi(2 - \tfrac{9}{4}\xi) - \gamma]) + O((1 - \rho)^5), \tag{14}$$

and

$$\alpha^{-1} = \sqrt{2(1 + c_s^2)}\,(1 - \tfrac{3}{2}(1 - \rho)(1 - \xi))) + O((1 - \rho)^2), \tag{15}$$

where m_k is the kth service-time moment (with $m_1 = 1$),

$$c_s^2 = m_2 - 1, \quad \xi = \frac{m_3}{3m_2^2}, \quad \gamma = \frac{m_4}{12m_2^3}. \tag{16}$$

We now illustrate the asymptotic approximation (5) with (11)–(13) and the associated heavy-traffic approximation using (14)–(16) by comparing them to the exact values (computed from both (3) and [2, 3]) for the case of a gamma service-time distribution with shape parameter $\frac{1}{2}$, i.e.,

$$g(t) = (2\pi t)^{-1/2} e^{-t/2}, \quad \hat{g}(s) = (1 + 2s)^{-1/2}. \tag{17}$$

The first four moments of this service-time distribution are 1, 3, 15 and 105. Let $\rho = 0.75$. In this case, the exact asymptotic parameters are $\tau^{-1} = 0.011777$ and $\alpha = 0.6996$, while the heavy-traffic approximations from (14) and (5) are $\hat{\tau}^{-1} = 0.011764$ and $\hat{\alpha} = 0.6928$. The numerical results for the density function itself are given in Table 1.

As in the M/M/1 case, which was studied by Abate and Whitt [3], the asymptotics do not produce good approximations until relatively large times. On the other hand, the heavy-traffic approximations tend to match the true asymptotics quite well when ρ is not too small.

We conclude this section by noting that (5) does *not* hold with a long-tail service-time distribution. Then (12) does not have a solution. De Meyer and Teugels [11] establish the tail behavior of the busy-period cdf $B(t)$ in this case. Let $G(t)$ be the service-time cdf. They show that $1 - B(t) \sim (1-\rho)^{-(1+c)} t^{-c} L(t)$ as $t \to \infty$ for a slowly varying function $L(t)$ and $c \geqslant 1$ if and only if $1 - G(t) \sim t^{-c} L(t)$ as $t \to \infty$. In the case of non-integer c, their argument yields the corresponding result for the densities as well. Related results for FIFO waiting times are discussed in [1].

4. Other consequences

In this section we mention other consequences of (3). We start by giving an alternative expression to (3), namely,

$$b(t) = t^{-1} \mathscr{L}^{-1}(-\hat{g}'(s) \exp(-\rho t(1 - \hat{g}(s)))). \tag{18}$$

Formula (18) can be obtained by considering the *conditional* busy-period density $b(t, \theta)$ given that the first customer has service time θ. Cox and Smith [10] established the representation

$$b(t, \theta) = \frac{\theta e^{-\rho t}}{t} \mathscr{L}^{-1}(\exp(-s\theta + \rho t \hat{g}(s))). \tag{19}$$

To get (18) from (19), we uncondition, writing

$$b(t) = \int_0^\infty b(t, \theta) g(\theta) \, d\theta. \tag{20}$$

Combining (19) and (20) and changing the order of integration yields (18).

It is interesting that (3) is obtained from (18) by removing $-\hat{g}'(s)$ inside $\mathscr{L}^{-1}$ and adding ρ^{-1}

Table 1
A comparison of the exact and approximate formulas for the busy-period density in the case of the gamma service-time distribution in (17)

Time	Exact	Asymptotic	Heavy traffic (5), (14), (15)
0.3	0.540550	1.7	1.7
0.5	0.355860	0.78	0.78
1.0	0.180100	0.27	0.27
2.0	0.078626	0.096	0.095
5.0	0.021867	0.024	0.023
10.0	0.007573	0.0078	0.0078
20.0	0.002423	0.00246	0.00244
40.0	0.000683	0.000689	0.000682
60.0	0.000294	0.000296	0.000294
80.0	0.000151	0.000152	0.000151

outside. To relate (3) to (18), note that if $h(t) = \mathscr{L}^{-1}(e^{\alpha\hat{g}(s)})$, then

$$th(t) = \mathscr{L}^{-1}\left(-\frac{d}{ds}e^{\alpha\hat{g}(s)}\right) = \alpha\mathscr{L}^{-1}(-\hat{g}'(s)e^{\alpha\hat{g}(s)}). \tag{21}$$

In the case of an *exponential* service-time distribution, $\hat{g}(s) = (1+s)^{-1}$, so that (18) becomes

$$b(t) = t^{-1}\mathscr{L}^{-1}((1+s)^{-2}\exp(-\rho ts/(1+s)). \tag{22}$$

We can recognize that the function to be inverted in (22) belongs to the family of "Bessel" density functions

$$f_{c,k}(t) = (t/c)^{k/2}\exp(-(c+t))I_k(2\sqrt{ct}), \quad t \geqslant 0, \tag{23}$$

with transforms

$$\hat{f}_{c,k}(s) = (1+s)^{-(k+1)}\exp(-cs/(1+s)); \tag{24}$$

see [12, Eqs. (3.11) and (3.12), p. 438; 7, p. 148]. Hence,

$$b(t) = t^{-1}f_{\rho t,1}(t), \quad t \geqslant 0, \tag{25}$$

agreeing with the known result, e.g., [3, Eq. (2.8); 10, p. 148].

We now show that it is possible to derive explicit expressions in other cases. First suppose that the service-time density is *gamma* with shape parameter $\frac{1}{2}$ as in (17). Then, using transform pairs (5.66) on p. 255 and (1.27) on p. 210 of [15] and the shift rule (as exponential damping), we obtain

$$b(t) = g(t)e^{-\rho t}\int_0^\infty \sqrt{u/\rho t^3}\,e^{-u^2/2t}I_1(2\sqrt{\rho tu})\,du. \tag{26}$$

For small t, we can use the relation

$$\sqrt{y}I_1(2\sqrt{y}) = \sum_{n=1}^{\infty} y^n/(n-1)!n!, \tag{27}$$

and integrate term by term to get

$$b(t) = g(t)e^{-\rho t}\sum_{n=1}^{\infty}\frac{(\sqrt{2}\rho t^{3/2})^{n-1}\Gamma((n+1)/2)}{(n-1)!n!}. \tag{28}$$

Next consider the *Erlang* (E_2) service-time distribution with

$$g(t) = 4te^{-2t}, \quad \hat{g}(s) = (1+(s/2))^{-2}. \tag{29}$$

From (3), we obtain the representation

$$b(t) = g(t)e^{-\rho t}\sum_{n=1}^{\infty}(4\rho t^3)^{n-1}/n!(2n-1)!. \tag{30}$$

Finally, consider the *uniform* service-time distribution on [0, 2] with transform $\hat{g}(s) = (1-e^{-2s})/2s$. From (3), we obtain

$$b(t) = \frac{e^{-\rho t}}{t\sqrt{2\rho}}I_1(t\sqrt{2\rho}) + e^{-\rho t}\sum_{n=1}^{\lfloor t/2\rfloor}\frac{(-1)^n}{2^n n!}(t\sqrt{2\rho(1-[2n/t])})^{n-1} \times I_{n-1}(t\sqrt{2\rho(1-[2n/t])}). \tag{31}$$

Formula (31) has only a few terms for small t. For example,

$$b(t) = \frac{e^{-\rho t}}{t\sqrt{2\rho}}I_1(t\sqrt{2\rho}), \quad 0 \leqslant t < 2, \tag{32}$$

and

$$b(t) = \frac{e^{-\rho t}}{t\sqrt{2\rho}}I_1(t\sqrt{2\rho}) - \frac{e^{-\rho t}}{2}I_0(t\sqrt{2\rho(1-[2/t])}), \quad 2 < t < 4. \tag{33}$$

5. The probability of emptiness and the LIFO CDF

A representation similar to (3) for the probability of emptiness was derived by Beneš [8, Eq. (47), p. 155]; see also [9, Eq. (21), p. 25; 24, Eq. (1), p. 443]. Let $P_{00}(t)$ be the probability of having an empty system at time t given that is started out empty at time 0. Beneš's result is

$$P_{00}(t) = t^{-1}\mathscr{L}^{-1}(s^{-2}\exp(-\rho t(1-\hat{g}(s)))). \tag{34}$$

Just as with the busy-period density, the Laplace transform $\hat{P}_{00}(s) \equiv \int_0^\infty e^{-st}P_{00}(t)\,dt$ can be obtained by iteratively solving a functional equation, either via the busy-period density as in (2) or directly as in (37) of [6]. Given that $P_{00}(t)$ is so closely related to $b(t)$ (see [6, Eqs. (36) and (37)], having (34) as well as (3) should not be be considered surprising. Interestingly, on pp. 27–30 of [9] Beneš gives a proof of (34) exploiting the ballot theorem.

We also note that a result similar to (34) holds for the complementary cdf of the busy period in the GI/M/1 queue because of duality; see [21, Eq. (77), pp. 123], namely,

$$B^c(t) = t^{-1}\mathscr{L}^{-1}\{s^{-2}\exp(-t(1-\hat{a}(s)))\}, \tag{35}$$

where $a(t)$ is the pdf of the interarrival time with mean ρ^{-1}.

It is significant that the LIFO waiting-time complementary cdf $1 - W_L(t)$ is just

$$1 - W_L(t) = P_{00}(t) - (1-\rho). \tag{36}$$

We note that the conditional cdf given that the server is busy, $W_L(t)/\rho$, coincides with the server-occupancy cdf H_0 in (23) of [6]. Formula (36) is easy to verify from standard expressions in the literature, e.g., [14, Eq. (3.24), p. 119], but the specific form does not seem well known. The basic LIFO result is due to Vaulot [22] (see also [23, 17]), but formula (36) itself is due to Takács [20, Eq. (78), p. 500].

Since the exponential terms in (34) and (3) coincide, the asymptotics for $P_{00}(t)$ and $b(t)$ are closely related. Indeed, $P_{00}(t)$ has the *same* asymptotic decay rate τ^{-1} in (11). In particular,

$$P_{00}(t) - (1-\rho) \sim \omega(\pi t^3)^{-1/2}e^{-t/\tau} \quad \text{as } t \to \infty, \tag{37}$$

where

$$\omega = \rho\alpha/\zeta^2; \tag{38}$$

ζ is the root of (12) and α is the busy-period asymptotic constant in (13).

As in the case for $b(t)$, (37) does *not* hold with a long-tail service-time distribution. Asmussen and Teugels [7] establish the tail behavior in this case. They show that

$$P_{00}(t) - (1-\rho) \sim \rho(1-\rho)^{1-c}(c-1)^{-1} \times t^{-(c-1)}L(t) \quad \text{as } t \to \infty \tag{39}$$

for a slowly varying function $L(t)$ and $c > 1$ if and only if the service-time cdf has the tail behavior $1 - G(t) \sim t^{-c}L(t)$ as $t \to \infty$.

We can also derive the asymptotics for $1 - W_L(t)$ by recognizing that the steady-state LIFO wait coincides with the busy period generated by the stationary excess of a service-time distribution. Then we see that the asymptotics for $1 - W_L(t)$ is more closely related to the asymptotics for the busy-period complementary cdf $1 - B(t)$ rather than for the density $b(t)$. Since the asymptotic constant for $1 - B(t)$ is $\alpha\tau$ instead of α in (5), we can think of the asymptotic constant ω in (38) as $\omega = (\alpha\tau)(\rho/\tau\zeta^2)$. We also obtain the heavy-traffic expansion for ω. It is

$$\omega = \left(\frac{\tau}{2}\right)\sqrt{\frac{1+c_s^2}{2}}\,(1 - \tfrac{1}{2}(1-\rho)(1-\xi)) + O((1-\rho)^2) \tag{40}$$

as $\rho \to 1$ for τ in (11) and ξ in (16).

However, we know that the asymptotics do not provide very good approximations, because we have already examined the M/M/1 special case in considerable detail; see [3, pp. 167, 168]. Note that the LIFO asymptotics here with the $t^{-3/2}$ term is quite different from the FIFO waiting-time asymptotics considered in [2]. Also note that the asymptotics here for $P_{00}(t)$ yields asymptotics for the M/G/1 workload moment cdfs $H_j(t)$ for $0 \leqslant j \leqslant 4$ by Theorem 2 of [6].

We conclude by doing a numerical example. We compare the M/G/1 steady-state waiting-time tail probabilities with the FIFO and LIFO disciplines. For both disciplines, we apply the Euler algorithm in [4]. For FIFO we use the Pollaczek–Khintchine transform, while for LIFO we use (34) and (36). We consider the service-time Laplace transform

$$(1 + \tfrac{2}{3}r(s))e^{-r(s)} \tag{41}$$

where

$$r(s) = \tfrac{1}{2}(\sqrt{1+12s} - 1). \tag{42}$$

This is a generalized inverse Gaussian distribution, normalized to have mean 1. The density is

$$g(t) = \sqrt{3/4\pi t^5}\exp(-((t-3)^2/12t)), \quad t > 0. \tag{43}$$

The first three moments are 1, 3 and 27. The waiting-time results are given in Table 2.

To interpret Table 2, recall that the probability that the server is busy is ρ with *both* disciplines. Moreover, the mean waiting time is

$$EW = \int_0^\infty P(W > t)\,dt = \frac{\rho(c_s^2+1)}{2(1-\rho)} = 4.5$$

Table 2
A comparison of the steady-state waiting-time tail probabilities in the M/G/1 queue with LIFO and FIFO disciplines

Time	$P(W > t)$	
t	LIFO	FIFO
0.0	0.750	0.750
0.5	0.478	0.653
1.0	0.366	0.580
5.0	0.158	0.291
10.0	0.097	0.144
20.0	0.0532	0.0408
40.0	0.02455	0.00400
80.0	0.00853	0.00005
160.0	0.00182	0.00000

for *both* disciplines. However, consistent with intuition, with LIFO longer waits are more likely. With numerical transform inversion, we can easily quantify this effect.

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