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Correction

Correction note on $L = \lambda W$

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The purpose of this note is to point out and correct a careless error in the proofs of theorems 2.1 and 2.2 in Whitt [6], which also appeared in Glynn and Whitt [1–3]. In particular, as pointed out by S. Stidham, Jr., the first inequality in (2.1) in [6] is not correct unless the service discipline is first-in first-out (FIFO). Of course, the relation $L = \lambda W$ does not actually depend on the FIFO condition. To obtain what was intended, simply replace $D(t)$ in both (2.1) and the statement of theorem 2.1 of [6] by $D'(t)$, where $D'(t)$ counts the number of k such that $D'_k \leq t$ with $D'_k \equiv \max\{D_j : 1 \leq j \leq k\}$ as in (2.6) of [6]. Note that the complication of non-FIFO disciplines is accounted for in (2.6); this modification does the same for (2.1). Indeed, a variant of this modification is used in the more general setting in Glynn and Whitt [4]; see remark 5 and lemma 3 on p. 640 there. Moreover, proper modifications of (2.1) routinely appear elsewhere, such as in Stidham [5] and Wolff [7]. The use of D'_k and $D'(t)$ is a conceptually simple approach.

The indicated modification of (2.1) in [6] is also needed in the proof of theorem 2.2 in [6] (which shows that the condition on $D'(t)$ in the new statement of theorem 2.1 in [6] is actually not needed). This modification works because, first, $A_k \leq D_k \leq D'_k$ and $D'(t) \leq D(t) \leq A(t)$ and, second, $t^{-1}D'(t) \to \lambda$ as $t \to \infty$ if and only if $k^{-1}D'_k \to \lambda^{-1}$ as $k \to \infty$. (We use the fact that D'_k is nondecreasing in k here. In contrast, as noted in theorem 2(d) of [1], in general (without FIFO) the limit $k^{-1}D_k \to \lambda^{-1}$ as $k \to \infty$ implies, but is not implied by, the limit $t^{-1}D(t) \to \lambda$ as $t \to \infty$. The failure of the limit $t^{-1}D(t) \to \lambda$ as $t \to \infty$ to imply the limit $k^{-1}D_k \to \lambda^{-1}$ as $k \to \infty$ is the key reason for the asymmetry in theorem 2.2 of [6]; the second sentence of remark (2.2) in [6] confuses the issue.)

Unfortunately, even though this oversight concerning (2.1) in [6] does not appear in [4], it does appear in previous papers. This same error appears in the first inequality in theorem 1a, p. 196, of Glynn and Whitt [1], but not in the remainder term $R(t)$ there; the proof of theorem 2(f) on p. 686 of Glynn and Whitt [2]; and (4.2) in the proof of (1.16) in theorem 3 on p. 704 of Glynn and

Whitt [3]. Fortunately, the error is easily corrected by the argument above in each case.

The relation among the relevant limits seems to be well summarized by theorem 2 of [1], with the understanding that part (b) should be augmented by the equivalent limit $t^{-1}D'(t) \to \lambda$ as $t \to \infty$. However, in [1] the assumption that the limits w and q be finite should be stated. Indeed, if $w = \infty$, then (viii) does not imply (iv) there, and theorem 1(e) is incorrect. (See remark 2.3 of [6].) Moreover, the last sentence on p. 199 of [1] should read: "The implication (viii) → (ix) is provided by applying theorem 1(a) plus (b) and (d) above." Finally, $D'(t)$ should appear instead of $O(t)$ on top of p. 200, as well as in theorem 1(a) of [1].

References

[1] P.W. Glynn and W. Whitt, A central-limit-theorem version of $L = \lambda W$, Queueing Systems 2 (1986) 191–215.

[2] P.W. Glynn and W. Whitt, Ordinary CLT and WLLN versions of $L = \lambda W$, Math. Oper. Res. 13 (1988) 674–692.

[3] P.W. Glynn and W. Whitt, An LIL version of $L = \lambda W$, Math. Oper. Res. 13 (1988) 693–710.

[4] P.W. Glynn and W. Whitt, Extensions of the queueing relations $L = \lambda W$ and $H = \lambda G$, Oper. Res. 37 (1989) 634–644.

[5] S. Stidham, Jr., A last word on $L = \lambda W$, Oper. Res. 22 (1974) 417–421.

[6] W. Whitt, A review of $L = \lambda W$ and extensions, Queueing Systems 9 (1991) 235–268.

[7] R.W. Wolff, *Stochastic Modeling and the Theory of Queues* (Prentice-Hall, Englewood Cliffs, NJ, 1989).