OPERATIONS RESEARCH
DOI 10.1287/opre.1070.0375ec
pp. ec1–ec19

e-companion

ONLY AVAILABLE IN ELECTRONIC FORM

Electronic Companion—"Two-Moment Approximations for Maxima" by Charles S. Crow IV, David Goldberg, and Ward Whitt, *Operations Research* 2007, 55(3) 532–548.

EC.1. Introduction

The problem is a classical one in probability theory: There are n independent and identically distributed (i.i.d.) nonnegative random variables, $Z_1, \ldots, Z_n$, each distributed as a random variable Z with a cumulative distribution function (c.d.f.) F, and we are interested in the probability distribution of the maximum $M_n \equiv \max\{Z_1, Z_2, \ldots, Z_n\}$. Given the c.d.f. F, we can easily numerically calculate the c.d.f. of M_n because

$$P(M_n \leqslant x) = F(x)^n, \quad x \geqslant 0. \tag{EC.1}$$

We also can numerically calculate the moments via

$$E[M_n^k] = \int_0^\infty k x^{k-1}[1 - F(x)^n]\, dx, \tag{EC.2}$$

e.g., see p. 150 of Feller (1971); and we can calculate quantiles ($x_{(n,q)}$ such that $P(M_n \leqslant x_{(n,q)}) = q$) by performing binary search with the c.d.f. in (EC.1).

However, suppose that we have only a partial characterization of the c.d.f. F. In particular, suppose that we know only its first two moments, $m_k \equiv E[Z^k]$ for $k = 1, 2$, or, equivalently, only its mean $m_1 \equiv E[Z]$ and its squared coefficient of variation $c^2 \equiv c_Z^2$ (SCV, variance divided by the square of the mean). What can we say about the distribution of M_n now?

In the main paper, we develop approximations for the distribution of M_n based on the mean m_1 and SCV c^2. We focus especially on the quantiles $x_{(n,q)}$. In the following sections, we give additional tables of numerical results for several underlying probability distributions. In §EC.2, we give numerical results for the hyperexponential (H_2) distribution (mixture of two exponential distributions), all of which have $c^2 > 1$. In §EC.3, we give numerical results for convolutions of exponential distributions, all of which have $c^2 < 1$. In §EC.4, we give numerical results for the gamma distribution, which has only two parameters, but covers the full range of possible SCV values: $0 < c^2 < \infty$. The gamma distribution covers the exponential distribution as a special case. In all other cases, the gamma distribution fails to have a pure-exponential tail.

EC.2. The H_2 Distribution

In this section, we assume that the underlying c.d.f. F is a mixture of two exponential distributions, and thus H_2. These distributions have $c^2 > 1$, and so tend to be more variable than the exponential distribution. We also give additional numerical results for the H_2 distribution. In successive subsections, we consider three different quantiles of the distribution of the maximum: M_n: $q = 0.25$, $q = 0.50$, and $q = 0.75$.

EC.2.1. Quantile $q = 0.25$

Tables EC.1–EC.9 feature the comparison of exact values with approximations for the $q = 0.25$ quantile of the c.d.f. of the maximum of n i.i.d. H_2 random variables with mean one and SCV $= \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

EC.2.2. Quantile $q = 0.50$

Tables EC.10–EC.18 are comparisons of exact values with approximations for the $q = 0.50$ quantile of the c.d.f. of the maximum of n i.i.d. H_2 random variables with mean one and SCV $= \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.1. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 2$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.9	1.9	2.2	−2.5	1.0	2.0	−7.2	0.6	2.0	2.0	3.2
20	2.7	3.0	3.3	0.0	2.6	3.3	−3.1	2.6	3.4	3.4	4.6
100	6.0	6.5	6.3	5.8	6.4	6.3	6.6	7.5	6.6	6.6	7.8
1,000	14.0	11.9	10.6	14.0	11.9	10.6	20.4	14.4	11.2	11.2	12.4
10,000	22.3	17.3	14.8	22.3	17.3	14.8	34.2	21.3	15.8	15.8	17.0
100,000	30.5	22.8	19.1	30.5	22.8	19.1	48.0	28.2	20.4	20.4	21.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	5.0
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	10.0
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	50.0
1,000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	500.0
10,000	698.1	2,113.2	4,031.4				416.7	1,666.7	3,750.0	3,750.0	5,000.0
100,000	6,981.0	21,132.5	40,314.4				4,166.7	16,666.7	37,500.0	37,500.0	50,000.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.2. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 4$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.7	1.6	1.9	−11.1	−0.9	1.7	−17.1	−1.6	1.6	0.5	3.7
20	2.4	2.8	4.0	−5.8	2.2	3.9	−10.2	1.8	3.9	3.3	6.4
100	6.7	9.3	9.1	6.6	9.3	9.1	5.9	9.9	9.3	9.7	12.9
1,000	24.2	19.5	16.5	24.2	19.5	16.5	28.9	21.4	17.0	18.9	22.1
10,000	41.9	29.7	23.9	41.9	29.7	23.9	51.9	32.9	24.6	28.1	31.3
100,000	59.6	39.9	31.4	59.6	39.9	31.4	75.0	44.4	32.3	37.3	40.5
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	2.5
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	5.0
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	25.0
1,000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	250.0
10,000	325.8	1,127.0	2,325.8				250.0	1,000.0	2,250.0	1,562.5	2,500.0
100,000	3,257.7	11,270.2	23,257.7				2,500.0	10,000.0	22,500.0	15,625.0	25,000.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.3. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 8$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.6	1.3	1.0	−34.0	−7.2	−0.5	−41.4	−8.2	−0.6	−5.4	1.8
20	2.2	2.0	3.8	−23.1	−1.4	3.6	−28.9	−2.0	3.5	0.1	7.3
100	4.7	12.3	13.1	2.2	12.3	13.1	0.0	12.5	13.2	13.0	20.2
1,000	38.4	31.8	26.7	38.3	31.8	26.7	41.5	33.2	27.0	31.4	38.6
10,000	74.5	51.3	40.3	74.5	51.3	40.3	82.9	53.9	40.8	49.8	57.0
100,000	110.7	70.8	53.9	110.7	70.8	53.9	124.4	74.7	54.6	68.3	75.5
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	1.3
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	2.5
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	12.5
1,000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	125.0
10,000	159.1	590.4	1,270.2				138.9	555.6	1,250.0	703.1	1,250.0
100,000	1,591.0	5,904.1	12,702.1				1,388.9	5,555.6	12,500.0	7,031.2	12,500.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.4. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 16$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.6	1.2	0.7	−91.0	−25.0	−8.2	−99.8	−26.4	−8.4	−22.9	−7.5
20	2.1	1.7	1.3	−69.0	−13.6	−0.4	−76.3	−14.6	−0.5	−11.8	3.6
100	3.8	13.0	17.7	−17.9	12.9	17.7	−21.6	12.8	17.7	14.0	29.3
1,000	55.2	50.9	43.6	55.1	50.9	43.6	56.7	51.9	43.8	50.8	66.2
10,000	128.2	88.8	69.4	128.2	88.8	69.4	135.0	91.1	69.9	87.7	103.0
100,000	201.3	126.8	95.3	201.3	126.8	95.3	213.3	130.2	96.0	124.5	139.8
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	1.3
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	6.3
1,000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	62.5
10,000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	625.0
100,000	787.8	3,033.2	6,670.2				735.3	2,941.2	6,617.6	3,320.3	6,250.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.5. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 32$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.6	1.1	0.6	−227.2	−71.4	−30.7	−237.6	−73.1	−30.9	−68.9	−37.2
20	2.1	1.5	0.9	−183.1	−48.9	−15.5	−191.9	−50.2	−15.6	−46.7	−15.0
100	3.5	4.2	20.0	−80.5	3.4	19.8	−85.6	2.9	19.8	4.8	36.5
1,000	66.3	78.2	70.3	66.3	78.2	70.3	66.3	78.9	70.5	78.5	110.1
10,000	213.1	153.0	120.7	213.1	153.0	120.7	218.3	154.9	121.1	152.2	183.8
100,000	359.8	227.8	171.2	359.8	227.8	171.2	370.3	230.9	171.8	225.9	257.5
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	3.1
1,000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	31.2
10,000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	312.5
100,000	392.2	1,538.8	3,422.5				378.8	1,515.2	3,409.1	1,611.3	3,125.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.6. Approximations for the $q = 0.25$ quantile for H_2 with SCV $= 64$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.5	1.1	0.6	−544.2	−186.0	−89.9	−556.1	−188.0	−90.2	−183.1	−118.8
20	2.1	1.4	0.8	−455.7	−141.3	−59.9	−466.0	−142.9	−60.1	−138.7	−74.4
100	3.3	2.5	10.0	−250.1	−37.5	9.7	−256.8	−38.3	9.6	−35.7	28.6
1,000	44.1	111.1	109.3	44.0	111.0	109.3	42.5	111.4	109.4	111.7	175.9
10,000	338.2	259.5	208.9	338.2	259.5	208.8	341.9	261.1	209.2	259.0	323.3
100,000	632.3	408.0	308.4	632.3	408.0	308.4	641.2	410.7	309.0	406.4	470.7
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	1.6
1,000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	15.6
10,000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	156.3
100,000	195.7	775.2	1,734.2				192.3	769.2	1,730.8	793.5	1,562.5

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.7. Approximations for the $q = 0.25$ quantile for H_2 with $\text{SCV} = 128$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.5	1.0	0.5	−1,267.1	−459.1	−237.6	−1,280.5	−461.4	−237.9	−455.9	−326.3
20	2.0	1.4	0.7	−1,089.8	−370.1	−178.0	−1,101.7	−372.0	−178.3	−367.1	−237.6
100	3.3	2.3	1.3	−678.2	−163.3	−39.7	−686.5	−164.4	−39.9	−161.1	−31.6
1,000	5.8	132.7	158.2	−89.3	132.6	158.1	−92.4	132.6	158.2	133.6	263.1
10,000	499.5	428.5	355.9	499.5	428.5	355.9	501.7	429.7	356.2	428.3	557.9
100,000	1,088.4	724.4	553.7	1,088.4	724.4	553.7	1,095.7	726.7	554.2	723.1	852.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	0.8
1,000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	7.8
10,000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	78.1
100,000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	789.3

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.8. Approximations for the $q = 0.25$ quantile for H_2 with $\text{SCV} = 256$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.5	1.0	0.5	−2,890.4	−1,093.8	−591.6	−2,905.4	−1,096.4	−592.0	−1,090.2	−830.1
20	2.0	1.4	0.7	−2,535.7	−916.0	−472.9	−2,549.1	−918.3	−473.3	−912.7	−652.7
100	3.2	2.2	1.2	−1,712.1	−503.2	−197.3	−1,721.9	−504.7	−197.5	−500.7	−240.6
1,000	5.3	87.6	197.1	−533.8	87.4	197.0	−538.4	87.1	197.0	88.8	348.8
10,000	644.6	678.1	591.3	644.6	678.0	591.3	645.2	678.9	591.5	678.2	938.3
100,000	1,822.9	1,268.7	985.6	1,822.9	1,268.6	985.6	1,828.7	1,270.6	986.0	1,267.7	1,527.7
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	0.4
1,000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	3.9
10,000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	39.1
100,000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	390.6

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.9. Approximations for the $q = 0.25$ quantile for H_2 with $\text{SCV} = 512$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	1.5	1.0	0.5	−6,492.2	−2,540.2	−1,417.7	−6,508.7	−2,543.2	−1,418.1	−2,536.2	−2,015.1
20	2.0	1.4	0.7	−5,782.6	−2,185.0	−1,180.7	−5,797.5	−2,187.6	−1,181.1	−2,181.3	−1,660.2
100	3.2	2.2	1.1	−4,134.9	−1,360.1	−630.4	−4,146.3	−1,362.0	−630.6	−1,357.3	−836.2
1,000	5.1	3.9	157.1	−1,777.6	−180.1	156.9	−1,783.8	−180.7	156.9	−178.4	342.7
10,000	579.7	1,000.1	944.2	579.6	1,000.0	944.2	578.7	1,000.5	944.3	1,000.5	1,521.7
100,000	2,936.9	2,180.1	1,731.5	2,936.9	2,180.1	1,731.5	2,941.1	2,181.7	1,731.8	2,179.5	2,700.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
1,000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	1.9
10,000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	19.5
100,000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	195.3

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

TABLE EC.10. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 2.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.7	3.0	3.3	0.0	2.6	3.3	−3.1	2.6	3.4	3.4	3.2
20	3.7	4.4	4.6	2.5	4.3	4.6	1.1	4.7	4.8	4.8	4.6
100	8.3	8.1	7.6	8.3	8.1	7.6	10.8	9.5	8.0	8.0	7.8
1,000	16.5	13.5	11.8	16.5	13.5	11.8	24.6	16.4	12.6	12.6	12.4
10,000	24.8	19.0	16.1	24.8	19.0	16.1	38.4	23.4	17.2	17.2	17.0
100,000	33.0	24.4	20.4	33.0	24.4	20.4	52.2	30.3	21.8	21.8	21.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	5.0
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	10.0
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	50.0
1,000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	500.0
10,000	698.1	2,113.2	4,031.4				416.7	1,666.7	3,750.0	3,750.0	5,000.0
100,000	6,981.0	21,132.5	40,314.4				4,166.7	16,666.7	37,500.0	37,500.0	50,000.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.11. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 4.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.4	2.8	4.0	−5.8	2.2	3.9	−10.2	1.8	3.9	3.3	3.7
20	3.3	5.3	6.2	−0.5	5.2	6.1	−3.3	5.3	6.2	6.0	6.4
100	11.9	12.4	11.3	11.9	12.4	11.3	12.8	13.3	11.6	12.5	12.9
1,000	29.5	22.6	18.8	29.5	22.6	18.8	35.9	24.9	19.3	21.7	22.1
10,000	47.2	32.8	26.2	47.2	32.8	26.2	58.9	36.4	27.0	30.9	31.3
100,000	64.9	43.0	33.6	64.9	43.0	33.6	81.9	47.9	34.6	40.1	40.5
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	2.5
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	5.0
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	25.0
1,000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	250.0
10,000	325.8	1,127.0	2,325.8				250.0	1,000.0	2,250.0	1,562.5	2,500.0
100,000	3,257.7	11,270.2	23,257.7				2,500.0	10,000.0	22,500.0	15,625.0	25,000.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.12. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 8.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.2	2.0	3.8	−23.1	−1.4	3.6	−28.9	−2.0	3.5	0.1	1.8
20	2.9	4.7	7.8	−12.2	4.5	7.7	−16.5	4.2	7.7	5.7	7.3
100	13.1	18.2	17.2	13.1	18.1	17.2	12.5	18.7	17.4	18.5	20.2
1,000	49.2	37.6	30.8	49.2	37.6	30.8	54.0	39.5	31.2	37.0	38.6
10,000	85.4	57.1	44.4	85.4	57.1	44.4	95.4	60.2	45.0	55.4	57.0
100,000	121.6	76.6	58.0	121.6	76.6	58.0	136.8	80.9	58.8	73.8	75.5
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	1.3
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	2.5
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	12.5
1,000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	125.0
10,000	159.1	590.4	1,270.2				138.9	555.6	1,250.0	703.1	1,250.0
100,000	1,591.0	5,904.1	12,702.1				1,388.9	5,555.6	12,500.0	7,031.2	12,500.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.13. Approximations for the $q = 0.50$ quantile for H_2 with SCV $= 16$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.1	1.7	1.3	−69.0	−13.6	−0.4	−76.3	−14.6	−0.5	−11.8	−7.5
20	2.7	2.5	7.6	−47.0	−2.2	7.4	−52.7	−2.8	7.3	−0.7	3.6
100	6.0	24.4	25.5	4.1	24.3	25.5	2.0	24.6	25.6	25.1	29.3
1,000	77.1	62.3	51.4	77.1	62.3	51.3	80.3	63.7	51.7	61.9	66.2
10,000	150.2	100.2	77.2	150.2	100.2	77.2	158.6	102.9	77.8	98.7	103.0
100,000	223.3	138.2	103.1	223.3	138.2	103.1	236.9	142.0	103.9	135.6	139.8

	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	1.3
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	6.3
1,000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	62.5
10,000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	625.0
100,000	787.8	3,033.2	6,670.2				735.3	2,941.2	6,617.6	3,320.3	6,250.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.14. Approximations for the $q = 0.50$ quantile for H_2 with SCV $= 32$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.1	1.5	0.9	−183.1	−48.9	−15.5	−191.9	−50.2	−15.6	−46.7	−37.2
20	2.6	2.0	1.6	−138.9	−26.4	−0.3	−146.1	−27.3	−0.4	−24.5	−15.0
100	4.3	26.0	35.1	−36.3	25.9	35.0	−39.9	25.8	35.0	27.0	36.5
1,000	110.5	100.7	85.5	110.5	100.7	85.5	112.1	101.8	85.7	100.7	110.1
10,000	257.2	175.5	135.9	257.2	175.5	135.9	264.1	177.8	136.4	174.4	183.8
100,000	404.0	250.4	186.4	404.0	250.4	186.4	416.0	253.8	187.0	248.0	257.5

	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	3.1
1,000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	31.3
10,000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	312.5
100,000	392.2	1,538.8	3,422.5				378.8	1,515.2	3,409.1	1,611.3	3,125.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.15. Approximations for the $q = 0.50$ quantile for H_2 with SCV $= 64$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.1	1.4	0.8	−455.7	−141.3	−59.9	−466.0	−142.9	−60.1	−138.7	−118.8
20	2.6	1.8	1.0	−367.2	−96.6	−29.9	−375.9	−97.8	−30.1	−94.4	−74.4
100	4.0	7.4	39.8	−161.6	7.2	39.7	−166.7	6.8	39.7	8.7	28.6
1,000	132.6	155.7	139.3	132.6	155.7	139.2	132.7	156.4	139.4	156.0	175.9
10,000	426.7	304.2	238.8	426.7	304.2	238.8	432.0	306.1	239.2	303.4	323.3
100,000	720.9	452.7	338.4	720.9	452.7	338.4	731.3	455.8	339.0	450.7	470.7

	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	1.6
1,000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	15.6
10,000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	156.3
100,000	195.7	775.2	1,734.2				192.3	769.2	1,730.8	793.5	1,562.5

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.16. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 128.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.0	1.4	0.7	−1,089.8	−370.1	−178.0	−1,101.7	−372.0	−178.3	−367.1	−326.3
20	2.6	1.8	0.9	−912.6	−281.0	−118.5	−922.9	−282.6	−118.7	−278.4	−237.6
100	3.8	2.9	20.1	−501.0	−74.2	19.8	−507.6	−75.0	19.8	−72.4	−31.6
1,000	88.0	221.7	217.7	87.9	221.7	217.6	86.4	222.0	217.8	222.3	263.1
10,000	676.8	517.6	415.5	676.8	517.6	415.5	680.5	519.1	415.8	517.1	557.9
100,000	1,265.7	813.4	613.3	1,265.7	813.4	613.3	1,274.6	816.1	613.8	811.8	852.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	0.8
1,000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	7.8
10,000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	78.1
100,000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	781.3

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.17. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 256.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.0	1.4	0.7	−2,535.7	−916.0	−472.9	−2,549.1	−918.3	−473.3	−912.7	−830.1
20	2.5	1.7	0.9	−2,181.0	−738.2	−354.2	−2,192.9	−740.2	−354.5	−735.3	−652.7
100	3.8	2.7	1.5	−1,357.4	−325.4	−78.6	−1,365.6	−326.5	−78.7	−323.3	−240.6
1,000	6.4	265.3	315.8	−179.0	265.2	315.7	−182.1	265.2	315.8	266.2	348.8
10,000	999.3	855.8	710.0	999.3	855.8	710.0	1,001.4	857.0	710.3	855.7	938.3
100,000	2,177.7	1,446.4	1,104.3	2,177.7	1,446.4	1,104.3	2,185.0	1,448.8	1,104.8	1,445.1	1,527.7
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	0.4
1,000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	3.9
10,000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	39.1
100,000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	390.6

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.18. Approximations for the $q = 0.50$ quantile for H_2 with SCV = 512.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.0	1.4	0.7	−5,782.6	−2,185.0	−1,180.7	−5,797.5	−2,187.6	−1,181.1	−2,181.3	−2,015.1
20	2.5	1.7	0.9	−5,073.0	−1,829.7	−943.7	−5,086.4	−1,832.0	−944.0	−1,826.4	−1,660.2
100	3.8	2.6	1.3	−3,425.3	−1,004.9	−393.4	−3,435.1	−1,006.4	−393.6	−1,002.4	−836.2
1,000	5.8	175.4	394.0	−1,068.0	175.2	393.9	−1,072.6	174.9	393.9	176.5	342.7
10,000	1,289.3	1,355.3	1,181.2	1,289.2	1,355.3	1,181.2	1,289.8	1,356.1	1,181.4	1,355.4	1,521.7
100,000	3,646.5	2,535.3	1,968.5	3,646.5	2,535.3	1,968.5	3,652.3	2,537.3	1,968.9	2,534.4	2,700.6
	Exact np						Approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
1,000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	1.9
10,000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	19.5
100,000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	195.3

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

TABLE EC.19. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 2$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	4.1	4.8	4.9	3.2	4.7	4.9	2.2	5.3	5.1	5.1	3.2
20	5.9	6.4	6.2	5.7	6.4	6.2	6.4	7.3	6.5	6.5	4.6
100	11.4	10.2	9.2	11.4	10.2	9.2	16.0	12.2	9.7	9.7	7.8
1,000	19.7	15.6	13.5	19.7	15.6	13.5	29.9	19.1	14.3	14.3	12.4
10,000	27.9	21.1	17.8	27.9	21.1	17.8	43.7	26.0	19.0	19.0	17.0
100,000	36.2	26.5	22.0	36.2	26.5	22.0	57.5	32.9	23.6	23.6	21.6

	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	5.0
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	10.0
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	50.0
1,000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	500.0
10,000	698.1	2,113.2	4,031.4				416.7	1,666.7	3,750.0	3,750.0	5,000.0
100,000	6,981.0	21,132.5	40,314.4				4,166.7	16,666.7	37,500.0	37,500.0	50,000.0

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.20. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 4$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	3.7	6.1	6.8	1.0	6.1	6.7	−1.4	6.2	6.9	6.8	3.7
20	6.5	9.2	9.0	6.3	9.1	9.0	5.5	9.7	9.2	9.5	6.4
100	18.6	16.3	14.2	18.6	16.3	14.2	21.6	17.7	14.5	16.0	12.9
1,000	36.3	26.5	21.6	36.3	26.5	21.6	44.6	29.3	22.2	25.2	22.1
10,000	54.0	36.7	29.0	54.0	36.7	29.0	67.7	40.8	29.9	34.4	31.3
100,000	71.6	46.9	36.4	71.6	46.9	36.4	90.7	52.3	37.6	43.6	40.5

	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	2.5
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	5.0
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	25.0
1,000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	250.0
10,000	325.8	1,127.0	2,325.8				250.0	1,000.0	2,250.0	1,562.5	2,500.0
100,000	3,257.7	11,270.2	23,257.7				2,500.0	10,000.0	22,500.0	15,625.0	25,000.0

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.21. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 8$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	3.2	6.2	8.9	−9.3	6.1	8.8	−13.1	5.9	8.8	7.1	1.8
20	4.6	12.0	12.9	1.6	12.0	12.9	−0.6	12.2	13.0	12.7	7.3
100	26.9	25.6	22.4	26.9	25.6	22.4	28.3	26.6	22.6	25.6	20.2
1,000	63.1	45.1	36.0	63.1	45.1	36.0	69.8	47.4	36.4	44.0	38.6
10,000	99.2	64.6	49.6	99.2	64.6	49.6	111.2	68.1	50.3	62.4	57.0
100,000	135.4	84.1	63.2	135.4	84.1	63.2	152.7	88.8	64.1	80.8	75.5

	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	1.3
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	2.5
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	12.5
1,000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	125.0
10,000	159.1	590.4	1,270.2				138.9	555.6	1,250.0	703.1	1,250.0
100,000	1,591.0	5,904.1	12,702.1				1,388.9	5,555.6	12,500.0	7,031.2	12,500.0

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.22. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 16$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.9	3.0	9.6	−41.1	0.9	9.5	−46.4	0.4	9.4	2.3	−7.5
20	3.7	12.4	17.3	−19.1	12.3	17.2	−22.8	12.2	17.3	13.4	3.6
100	32.0	38.9	35.4	32.0	38.8	35.3	31.9	39.5	35.5	39.1	29.3
1,000	105.0	76.8	61.2	105.0	76.8	61.2	110.2	78.7	61.6	76.0	66.2
10,000	178.1	114.7	87.1	178.1	114.7	87.1	188.5	117.8	87.7	112.8	103.0
100,000	251.2	152.7	113.0	251.2	152.7	113.0	266.8	157.0	113.8	149.7	139.8
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	1.3
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	6.3
1,000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	62.5
10,000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	625.0
100,000	787.8	3,033.2	6,670.2				735.3	2,941.2	6,617.6	3,320.3	6,250.0

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.23. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 32$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.8	2.2	4.1	−127.0	−20.3	3.8	−133.8	−21.2	3.7	−18.5	−37.2
20	3.4	3.8	19.2	−82.8	2.2	19.0	−88.1	1.7	19.0	3.6	−15.0
100	19.8	54.5	54.3	19.8	54.5	54.3	18.2	54.8	54.4	55.1	36.5
1,000	166.5	129.3	104.7	166.5	129.3	104.7	170.1	130.8	105.0	128.8	110.1
10,000	313.3	204.1	155.2	313.3	204.1	155.2	322.1	206.8	155.7	202.5	183.8
100,000	460.1	278.9	205.6	460.1	278.9	205.6	474.1	282.8	206.4	276.2	257.5
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	0.6
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	3.1
1,000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	81.2
10,000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	812.5
100,000	392.2	1,538.8	3,422.5				378.8	1,515.2	3,409.1	1,611.3	8,125.0

Notes. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.24. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 64$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.7	1.9	1.1	−343.4	−84.6	−21.9	−351.7	−85.7	−22.0	−82.4	−118.8
20	3.3	2.5	8.4	−254.8	−39.9	8.1	−261.6	−40.7	8.0	−38.1	−74.4
100	5.2	64.0	77.8	−49.2	63.9	77.7	−52.4	63.9	77.8	64.9	28.6
1,000	244.9	212.5	177.3	244.9	212.4	177.3	247.0	213.6	177.5	212.3	175.9
10,000	539.1	361.0	276.9	539.1	361.0	276.9	546.3	363.3	277.3	359.7	323.3
100,000	833.2	509.5	376.4	833.2	509.5	376.4	845.6	512.9	377.1	507.0	470.7
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	0.3
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	1.6
1,000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	15.6
10,000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	156.3
100,000	195.7	775.2	1,734.2				192.3	769.2	1,730.8	793.5	1,562.5

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.25. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 128$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.7	1.9	1.0	−864.9	−257.1	−102.5	−874.8	−258.6	−102.6	−254.6	−326.3
20	3.2	2.3	1.3	−687.7	−168.0	−42.9	−696.0	−169.2	−43.0	−165.9	−237.6
100	4.7	39.0	95.5	−276.1	38.8	95.4	−280.8	38.5	95.4	40.2	−31.6
1,000	312.9	334.7	293.2	312.8	334.7	293.2	313.3	335.5	293.4	334.9	263.1
10,000	901.7	630.6	491.0	901.7	630.6	491.0	907.4	632.5	491.4	629.6	557.9
100,000	1,490.6	926.4	688.9	1,490.6	926.4	688.9	1,501.4	929.6	689.4	924.3	852.6
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	0.8
1,000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	7.8
10,000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	78.1
100,000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	781.3

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.26. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 256$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.7	1.8	0.9	−2,085.7	−690.4	−322.3	−2,097.1	−692.3	−322.6	−687.6	−830.1
20	3.2	2.2	1.2	−1,731.0	−512.6	−203.6	−1,740.9	−514.2	−203.8	−510.2	−652.7
100	4.5	3.5	72.2	−907.4	−99.8	72.0	−913.6	−100.5	71.9	−98.1	−240.6
1,000	271.1	490.8	466.3	271.0	490.8	466.3	269.9	491.2	466.4	491.3	348.8
10,000	1,449.3	1,081.4	860.6	1,449.3	1,081.4	860.6	1,453.4	1,083.0	860.9	1,080.8	938.3
100,000	2,627.7	1,672.0	1,254.9	2,627.7	1,672.0	1,254.9	2,637.0	1,674.8	1,255.5	1,670.2	1,527.7
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	0.1
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	0.4
1,000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	3.9
10,000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	39.1
100,000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	390.6

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

TABLE EC.27. Approximations for the $q = 0.75$ quantile for H_2 with SCV $= 512$.

	Exact (1.1)			Asymp. (3.6)			Product (4.13)				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	Simple (1.9)	Crude (1.8)
10	2.7	1.8	0.9	−4,882.3	−1,734.3	−880.0	−4,895.3	−1,736.5	−880.3	−1,731.1	−2,015.1
20	3.2	2.2	1.1	−4,172.7	−1,379.0	−643.0	−4,184.1	−1,380.9	−643.3	−1,376.2	−1,660.2
100	4.5	3.1	1.8	−2,525.0	−554.2	−92.7	−2,532.8	−555.2	−92.8	−552.2	−836.2
1,000	7.5	625.9	694.6	−167.8	625.9	694.6	−170.4	626.0	694.7	626.8	342.7
10,000	2,189.5	1,805.9	1,481.9	2,189.5	1,805.9	1,481.9	2,192.1	1,807.2	1,482.1	1,805.7	1,521.7
100,000	4,546.8	2,986.0	2,269.2	4,546.8	2,986.0	2,269.2	4,554.5	2,988.4	2,269.6	2,984.6	2,700.6
	Exact *np*						Approx. *np*				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	In (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	0.0
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	0.2
1,000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	1.9
10,000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	19.5
100,000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	195.3

Notes. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

EC.2.3. Quantile $q = 0.75$

Tables EC.19–EC.27 feature the comparison of exact values with approximations for the $q = 0.75$ quantile of the c.d.f. of the maximum of n i.i.d. H_2 random variables with mean one and SCV $= \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for *np*, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

EC.3. Convolutions of Exponential Distributions

In this section, we assume that the underlying c.d.f. F is the convolution of two or more exponential distributions, all with different means. These distributions have $c^2 < 1$ and so tend to be less variable than the exponential distribution. These distributions also all have a pure-exponential tail.

We can achieve any SCV value between $1/n$ and 1 with a convolution of n exponentials, e.g., see Aldous and Shepp (1987). If we restrict attention to $n = 2$, there are only two parameters, which we can match to the mean and the SCV. We get the equations

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \qquad \text{and} \qquad \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = c^2, \tag{EC.3}$$

with the constraints $\lambda_1 < \lambda_2$ and $0.5 < c^2 < 1$. That yields a quadratic equation for λ_1^{-1} in terms of c^2.

In the first subsection, we consider the convolution of two exponential distributions, that have SCV values $0.5 < c^2 < 1$. In the second subsection, we consider convolutions of four exponentials, that yield SCV values $0.25 < c^2 < 1$.

EC.3.1. Convolutions of Two Exponentials: $0.50 < \text{SCV} < 1$

We present numerical results for a wide range of c^2 in Tables EC.28–EC.36 in particular, for c^2 in $\{0.99, 0.9, 0.8, 0.7, 0.6, 0.55, 0.51, 0.501, 0.5001\}$. We include the cases with $c^2 \leqslant 0.51$ to deliberately include difficult cases, in which λ_2^{-1} is close to λ_1^{-1}. In this case, the extreme-value approximations are not good initially, but eventually improve.

Except in the relatively pathological cases with c^2 close to 0.5, the extreme-value asymptotics, i.e., (3.6), are spectacular for all values of n. However, the asymptotic approximation begins to perform poorly for smaller n as c^2 decreases toward 0.5. The simple rough approximation in (1.9) is also spectacular for all n when c^2 is high, e.g., above 0.7. For lower c^2, (1.9) produces an error for very large n. For $10 \leqslant n \leqslant 1{,}000$, (1.9) is consistently good for all c^2, being much better than the asymptotic approximation in (3.6) for the last "pathological" cases with c^2 very close to 0.5.

EC.3.2. Convolutions of Four Exponentials: $0.25 < \text{SCV} < 1.0$

To obtain values of c^2 less than 0.5, we need to consider convolutions of more exponential distributions (sums of more independent exponential random variables). To illustrate, below in Tables EC.37–EC.45 we give results for the sum of four independent exponential random variables. As a base case in Table EC.37 we use means 0.4, 0.3, 0.2, and 0.1. Here the SCV is $c^2 = 0.3$, and the mean is again one. (The mean is the sum of the means, while the variance is the sum of the variances, where the variance of an exponential is the square of its mean.) The other tables show SCV values ranging from $c^2 = 0.4038$ to $c^2 = 0.2500$.

As before, the asymptotic extreme-value results are excellent, although there is about 5% error for small n. The simple rough approximation is good for smaller n, but begins to deviate for larger n. There is a 13% error for $n = 1{,}000$. Overall, the simple rough approximation in (1.9) is reasonable.

EC.4. The Gamma Distribution

In this final section, we consider the gamma distribution, which covers the full range of possible SCV values: $0 < c^2 < \infty$. We consider the cases $c^2 > 1$ and $c^2 < 1$ in turn in the following subsections.

TABLE EC.28. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.99 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.69	2.66	2.66	3.55	3.54	3.54	2.29
20	3.37	3.35	3.35	4.23	4.23	4.23	2.98
100	4.96	4.95	4.95	5.83	5.83	5.83	4.57
1,000	7.24	7.24	7.24	8.12	8.12	8.12	6.85
100,000	11.82	11.82	11.82	12.70	12.70	12.70	11.41
1,000,000	14.12	14.12	14.12	14.99	14.99	14.99	13.69

Note. The distribution parameters are $\lambda_1^{-1} = 0.9950$, $\lambda_2^{-1} = 0.0050$, $C_{1,2} = 1.0051$, and $C_{2,2} = -0.0051$.

TABLE EC.29. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.90 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.62	2.58	2.58	3.43	3.42	3.42	2.17
20	3.26	3.24	3.24	4.08	4.07	4.08	2.79
100	4.77	4.76	4.77	5.60	5.60	5.60	4.24
1,000	6.94	6.94	6.95	7.78	7.78	7.79	6.31
100,000	11.31	11.31	11.32	12.14	12.14	12.16	10.46
1,000,000	13.49	13.49	13.51	14.32	14.32	14.34	12.53

Note. The distribution parameters are $\lambda_1^{-1} = 0.9472$, $\lambda_2^{-1} = 0.0528$, $C_{1,2} = 1.0590$, and $C_{2,2} = -0.0590$.

TABLE EC.30. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.8 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.52	2.49	2.49	3.28	3.27	3.28	2.02
20	3.12	3.10	3.11	3.89	3.88	3.90	2.58
100	4.54	4.53	4.55	5.31	5.31	5.34	3.86
1,000	6.58	6.57	6.61	7.36	7.36	7.40	5.70
100,000	10.66	10.66	10.73	11.44	11.44	11.52	9.39
1,000,000	12.70	12.70	12.79	13.48	13.48	13.58	11.23

Note. The distribution parameters are $\lambda_1^{-1} = 0.8873$, $\lambda_2^{-1} = 0.1127$, $C_{1,2} = 1.1455$, and $C_{2,2} = -0.1455$.

TABLE EC.31. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.7 for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.81623$, $\lambda_2^{-1} = 0.18377$, $C_{1,2} = 1.2906$, and $C_{2,2} = -0.29057$.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.42	2.39	2.40	3.12	3.10	3.13	1.86
20	2.97	2.95	2.98	3.68	3.67	3.71	2.35
100	4.27	4.27	4.32	4.99	4.98	5.06	3.47
1,000	6.15	6.15	6.25	6.86	6.86	6.99	5.09
100,000	9.90	9.90	10.10	10.62	10.62	10.84	8.31
1,000,000	11.78	11.78	12.03	12.50	12.50	12.76	9.92

Note. The distribution parameters are $\lambda_1^{-1} = 0.8873$, $\lambda_2^{-1} = 0.1127$, $C_{1,2} = 1.1455$, and $C_{2,2} = -0.1455$.

TABLE EC.32. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.6 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.30	2.28	2.29	2.93	2.92	2.97	1.69
20	2.79	2.78	2.83	3.42	3.42	3.51	2.10
100	3.95	3.95	4.08	4.58	4.58	4.76	3.07
1,000	5.61	5.61	5.86	6.25	6.25	6.54	4.45
100,000	8.94	8.94	9.43	9.58	9.58	10.11	7.21
1,000,000	10.61	10.61	11.21	11.25	11.25	11.89	8.60

Note. The distribution parameters are $\lambda_1^{-1} = 0.7236$, $\lambda_2^{-1} = 0.2764$, $C_{1,2} = 1.6180$, and $C_{2,2} = -0.6180$.

TABLE EC.33. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.55 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.25	2.24	2.24	2.82	2.82	2.89	1.60
20	2.70	2.70	2.75	3.27	3.27	3.40	1.98
100	3.75	3.75	3.95	4.33	4.33	4.60	2.86
1,000	5.27	5.27	5.65	5.85	5.85	6.31	4.13
100,000	8.30	8.30	9.07	8.88	8.88	9.72	6.66
1,000,000	9.82	9.82	10.78	10.39	10.39	11.43	7.93

Note. The distribution parameters are $\lambda_1^{-1} = 0.65811$, $\lambda_2^{-1} = 0.34189$, $C_{1,2} = 2.0811$ and $C_{2,2} = -1.0811$.

TABLE EC.34. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.51 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.20	2.32	2.19	2.73	2.82	2.82	1.52
20	2.62	2.72	2.69	3.15	3.22	3.31	1.87
100	3.58	3.63	3.84	4.09	4.14	4.46	2.69
1,000	4.92	4.95	5.48	5.43	5.45	6.11	3.87
100,000	7.57	7.58	8.77	8.07	8.08	9.40	6.21
1,000,000	8.89	8.89	10.41	9.39	9.39	11.04	7.39

Note. The distribution parameters are $\lambda_1^{-1} = 0.5707$, $\lambda_2^{-1} = 0.4293$, $C_{1,2} = 4.0355$, and $C_{2,2} = -3.0355$.

TABLE EC.35. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV = 0.501 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.1949	2.6781	2.1814	2.7135	3.1375	2.8039	1.4999
20	2.6045	3.0402	2.6720	3.1165	3.4996	3.2945	1.8471
100	3.5361	3.8809	3.8112	4.0347	4.3403	4.4337	2.6535
1,000	4.8303	5.0837	5.4410	5.3161	5.5431	6.0635	3.8071
100,000	7.3424	7.4893	8.7006	7.8151	7.9486	9.3231	6.1142
1,000,000	8.5779	8.6920	10.3304	9.0473	9.1514	10.9529	7.2678

Note. The distribution parameters are $\lambda_1^{-1} = 0.52236$, $\lambda_2^{-1} = 0.47764$, $C_{1,2} = 11.680$, and $C_{2,2} = -10.680$.

TABLE EC.36. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of two exponential distributions, having overall mean one and SCV $= 0.5001$ for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.1940	3.1685	2.1803	2.7114	3.6144	2.8022	1.4981
20	2.6027	3.5200	2.6705	3.1134	3.9659	3.2924	1.8447
100	3.5315	4.3361	3.8087	4.0286	4.7820	4.4306	2.6496
1,000	4.8205	5.5036	5.4370	5.3040	5.9495	6.0589	3.8011
100,000	7.3172	7.8388	8.6937	7.7866	8.2847	9.3156	6.1042
1,000,000	8.5424	9.0064	10.3220	9.0076	9.4523	10.9439	7.2557

Note. The distribution parameters are $\lambda_1^{-1} = 0.50707$, $\lambda_2^{-1} = 0.49293$, $C_{1,2} = 35.855$, and $C_{2,2} = -34.855$.

TABLE EC.37. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.4, 0.3, 0.2, and 0.1, having overall mean one and SCV $= 0.30$ for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.91	2.01	1.91	2.29	2.37	2.40	1.05
20	2.21	2.29	2.29	2.58	2.64	2.78	1.26
100	2.89	2.94	3.18	3.25	3.29	3.66	1.74
1,000	3.84	3.86	4.44	4.19	4.21	4.92	2.43
100,000	5.69	5.70	6.96	6.05	6.05	7.44	3.82
1,000,000	6.62	6.62	8.22	6.97	6.97	8.70	4.51

Note. The remaining asymptotic parameter is $p = C_{1,4} = 10.667$.

TABLE EC.38. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.4980, 0.3740, 0.1260, and 0.0020 (from $w = 0.1240$) having overall mean 1.00 and SCV $= 0.4038$ for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	2.07	2.17	2.06	2.53	2.61	2.62	1.30
20	2.43	2.51	2.50	2.89	2.95	3.06	1.58
100	3.27	3.32	3.52	3.72	3.75	4.08	2.23
1,000	4.44	4.46	4.99	4.88	4.90	5.55	3.16
100,000	6.75	6.76	7.91	7.19	7.19	8.47	5.01
1,000,000	7.90	7.90	9.38	8.34	8.34	9.93	5.94

Note. The remaining asymptotic parameter is $p = C_{1,4} = 5.3981$.

TABLE EC.39. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.4500, 0.3500, 0.1500, and 0.0500 (from $w = 0.1000$) having overall mean 1.00 and $SCV = 0.3500$ for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.99	2.11	1.99	2.41	2.51	2.51	1.17
20	2.32	2.43	2.40	2.74	2.82	2.92	1.42
100	3.09	3.15	3.35	3.50	3.55	3.87	1.98
1,000	4.15	4.19	4.71	4.56	4.58	5.23	2.79
100,000	6.25	6.26	7.44	6.65	6.65	7.96	4.40
1,000,000	7.29	7.29	8.80	7.69	7.69	9.32	5.20

Note. The remaining asymptotic parameter is $p = C_{1,4} = 7.5938$.

TABLE EC.40. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.4100, 0.3300, 0.1700, and 0.0900 (from $w = 0.0800$) having overall mean 1.00 and SCV = 0.3140 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.93	2.09	1.94	2.33	2.45	2.43	1.09
20	2.24	2.37	2.32	2.63	2.73	2.82	1.30
100	2.95	3.03	3.23	3.33	3.39	3.72	1.81
1,000	3.93	3.97	4.52	4.30	4.33	5.01	2.53
100,000	5.85	5.86	7.10	6.21	6.22	7.59	3.98
1,000,000	6.80	6.81	8.39	7.16	7.17	8.88	4.70

Note. The remaining asymptotic parameter is $p = C_{1,4} = 1.1218e{+}01$.

TABLE EC.41. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.3700, 0.3100, 0.1900, and 0.1300 (from $w = 0.0600$) having overall mean 1.00 and SCV = 0.2860 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.89	2.09	1.89	2.25	2.41	2.36	1.02
20	2.18	2.34	2.26	2.53	2.67	2.73	1.21
100	2.83	2.94	3.12	3.17	3.26	3.59	1.68
1,000	3.72	3.79	4.36	4.06	4.12	4.83	2.33
100,000	5.47	5.50	6.82	5.80	5.82	7.29	3.65
1,000,000	6.33	6.35	8.05	6.66	6.67	8.52	4.31

Note. The remaining asymptotic parameter is $p = C_{1,4} = 1.9542e{+}01$.

TABLE EC.42. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.3300, 0.2900, 0.2100, and 0.1700 (from $w = 0.0400$) having overall mean 1.00 and SCV = 0.2660 for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.85	2.15	1.86	2.19	2.44	2.31	0.96
20	2.12	2.38	2.22	2.46	2.67	2.67	1.15
100	2.73	2.91	3.05	3.04	3.20	3.50	1.58
1,000	3.55	3.67	4.24	3.86	3.96	4.69	2.19
100,000	5.13	5.19	6.61	5.43	5.48	7.06	3.41
1,000,000	5.91	5.95	7.80	6.20	6.24	8.25	4.03

Note. The remaining asymptotic parameter is $p = C_{1,4} = 4.6793e{+}01$.

TABLE EC.43. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.2900, 0.2700, 0.2300, and 0.2100 (from $w = 0.0200$) having overall mean 1.00 and SCV = 0.2540, for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.83	2.38	1.84	2.16	2.63	2.28	0.93
20	2.09	2.58	2.19	2.41	2.84	2.63	1.11
100	2.66	3.05	3.00	2.96	3.30	3.44	1.52
1,000	3.43	3.72	4.16	3.71	3.97	4.61	2.10
100,000	4.88	5.05	6.48	5.15	5.31	6.93	3.27
1,000,000	5.58	5.72	7.64	5.85	5.97	8.09	3.86

Note. The remaining asymptotic parameter is $p = C_{1,4} = 2.505e{+}02$.

TABLE EC.44. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.2700, 0.2600, 0.2400, and 0.2300, (from $w = 0.0100$) having overall mean 1.00 and $SCV = 0.2510$, for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.83	2.72	1.84	2.15	2.96	2.28	0.92
20	2.08	2.91	2.18	2.39	3.14	2.62	1.10
100	2.64	3.34	2.99	2.93	3.58	3.43	1.50
1,000	3.39	3.96	4.14	3.67	4.20	4.58	2.08
100,000	4.80	5.21	6.45	5.06	5.44	6.89	3.24
1,000,000	5.48	5.83	7.60	5.74	6.07	8.04	3.81

Note. The remaining asymptotic parameter is $p = C_{1,4} = 1.6402e{+}03$.

TABLE EC.45. A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. random variables, each distributed as the convolution of four exponential distributions with individual means 0.2520, 0.2510, 0.2490, and 0.2480 (from $w = 0.0010$) having overall mean 1.00 and SCV $= 0.2500$ for six values of n.

	$q = 0.50$			$q = 0.75$			
n	Exact	Asymp. (3.6)	(1.9)	Exact	Asymp. (3.6)	(1.9)	(1.8)
10	1.83	4.23	1.83	2.15	4.45	2.27	0.92
20	2.08	4.40	2.18	2.39	4.62	2.62	1.10
100	2.64	4.81	2.99	2.93	5.03	3.43	1.50
1,000	3.38	5.39	4.14	3.66	5.61	4.58	2.07
100,000	4.77	6.55	6.44	5.03	6.77	6.88	3.22
1,000,000	5.44	7.13	7.59	5.70	7.35	8.03	3.80

Note. The remaining asymptotic parameter is $p = C_{1,4} = 1.3336e{+}06$.

TABLE EC.46. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 2$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	3.4	3.4	3.4	3.0	4.8	5.1	5.1	4.8	3.2
20	4.5	4.5	4.8	4.4	6.0	6.2	6.5	6.4	4.6
100	7.3	7.3	8.0	8.1	8.9	9.0	9.7	10.2	7.8
1,000	11.5	11.5	12.6	13.5	13.1	13.2	14.3	15.6	12.4
100,000	20.2	20.2	21.8	24.4	21.9	21.9	23.6	26.5	21.6
100,000,000	33.6	33.5	35.6	40.8	35.3	35.3	37.4	42.9	35.5

TABLE EC.47. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 4$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	4.0	3.0	3.3	2.8	6.5	6.5	6.8	6.1	3.7
20	5.9	5.0	6.0	5.3	8.6	8.5	9.5	9.2	6.4
100	10.9	10.2	12.5	12.4	13.9	13.7	16.0	16.3	12.9
1,000	18.8	18.1	21.7	22.6	21.9	21.7	25.2	26.5	22.1
100,000	35.5	35.0	40.1	43.0	38.8	38.6	43.6	46.9	40.5
100,000,000	61.6	61.3	67.7	73.7	65.0	64.8	71.2	77.6	68.1

TABLE EC.48. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 8$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	4.4	−0.6	0.1	2.0	8.4	6.4	7.1	6.2	1.8
20	7.5	3.1	5.7	4.7	12.1	10.1	12.7	12.0	7.3
100	16.4	12.9	18.5	18.2	21.8	20.0	25.6	25.6	20.2
1,000	31.1	28.5	37.0	37.6	37.1	35.5	44.0	45.1	38.6
100,000	63.6	61.8	73.8	76.6	70.0	68.8	80.8	84.1	75.5
100,000,000	115.0	113.7	129.1	135.1	121.7	120.8	136.1	142.6	130.7

TABLE EC.49. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 16$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	3.9	−13.6	−11.8	1.7	9.9	0.4	2.3	3.0	−7.5
20	8.4	−6.5	−0.7	2.5	16.2	7.6	13.4	12.4	3.6
100	23.8	12.8	25.1	24.4	33.9	26.9	39.1	38.9	29.3
1,000	51.6	43.6	61.9	62.3	63.1	57.6	76.0	76.8	66.2
100,000	114.9	109.6	135.6	138.2	127.5	123.7	149.7	152.7	139.8
100,000,000	216.7	213.1	246.1	252.1	229.9	227.1	260.2	266.6	250.4

TABLE EC.50. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 32$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	2.1	−50.8	−46.7	1.5	9.6	−22.7	−18.5	2.2	−37.2
20	7.5	−36.8	−24.5	2.0	19.4	−8.6	3.6	3.8	−15.0
100	32.5	1.4	27.0	26.0	50.7	29.5	55.1	54.5	36.5
1,000	84.2	62.5	100.7	100.7	106.4	90.7	128.8	129.3	110.1
100,000	207.8	194.0	248.0	250.4	232.8	222.2	276.2	278.9	257.5
100,000,000	409.6	400.5	469.1	474.8	435.9	428.7	497.2	503.4	478.6

TABLE EC.51. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 64$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(1.9)	H_2 ex.	Exact	Asymp.	(1.9)	H_2 ex.	(1.8)
10	0.4	−147.3	−138.7	1.4	6.4	−91.0	−82.4	1.9	−118.8
20	4.2	−119.5	−94.4	1.8	19.0	−63.3	−38.1	2.5	−74.4
100	39.6	−43.6	8.7	7.4	71.5	12.7	64.9	64.0	28.6
1,000	133.7	78.2	156.0	155.7	176.2	134.5	212.3	212.5	175.9
100,000	374.5	340.7	450.7	452.7	423.9	397.0	507.0	509.5	470.7
100,000,000	775.0	753.2	892.8	898.3	827.4	809.5	949.1	955.0	912.8

TABLE EC.52. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/2$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	2.19	1.75	1.48	2.18	2.71	2.19	1.92	2.80	1.50
20	2.60	2.23	2.09	2.67	3.11	2.67	2.52	3.29	1.84
100	3.53	3.25	3.32	3.81	4.03	3.69	3.76	4.43	2.65
1,000	4.82	4.60	4.88	5.44	5.30	5.04	5.32	6.06	3.80
100,000	7.31	7.16	7.69	8.69	7.78	7.60	8.13	9.31	6.10
100,000,000	10.96	10.85	11.61	13.58	11.42	11.29	12.05	14.20	9.56
1.000000e+10	13.36	13.26	14.14	16.83	13.81	13.70	14.58	17.46	11.86
1.000000e+12	15.74	15.66	16.62	20.09	16.19	16.10	17.06	20.71	14.16

TABLE EC.53. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/4$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	1.83	0.84	0.12	1.83	2.15	1.06	0.33	2.27	0.92
20	2.08	1.22	0.55	2.18	2.39	1.44	0.77	2.62	1.10
100	2.64	1.94	1.38	2.99	2.93	2.16	1.60	3.43	1.50
1,000	3.38	2.82	2.36	4.14	3.66	3.04	2.58	4.58	2.07
100,000	4.77	4.35	4.03	6.44	5.03	4.57	4.25	6.88	3.22
100,000,000	6.75	6.43	6.22	9.89	6.99	6.65	6.44	10.33	4.95
1.000000e+10	8.03	7.75	7.60	12.20	8.27	7.97	7.82	12.64	6.10
1.000000e+12	9.28	9.04	8.93	14.50	9.52	9.26	9.15	14.94	7.25

TABLE EC.54. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/8$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	1.57	−0.00	−0.91	1.59	1.77	0.11	−0.80	1.90	0.55
20	1.73	0.31	−0.56	1.84	1.92	0.42	−0.45	2.15	0.63
100	2.08	0.89	0.07	2.40	2.25	1.00	0.18	2.72	0.84
1,000	2.52	1.53	0.76	3.22	2.68	1.64	0.87	3.53	1.12
100,000	3.33	2.56	1.85	4.85	3.47	2.67	1.96	5.16	1.70
100,000,000	4.43	3.83	3.18	7.29	4.57	3.94	3.29	7.60	2.56
1.000000e+10	5.13	4.60	3.98	8.92	5.26	4.71	4.09	9.23	3.14
1.000000e+12	5.81	5.34	4.74	10.55	5.94	5.45	4.85	10.86	3.71

TABLE EC.55. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/16 = 0.0625$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	1.40	−0.79	−1.77	1.42	1.53	−0.74	−1.72	1.64	0.32
20	1.50	−0.50	−1.47	1.59	1.62	−0.45	−1.41	1.81	0.36
100	1.72	−0.00	−0.93	1.99	1.83	0.05	−0.88	2.21	0.46
1,000	1.99	0.52	−0.39	2.57	2.09	0.58	−0.33	2.79	0.61
100,000	2.48	1.29	0.41	3.72	2.56	1.34	0.47	3.94	0.89
100,000,000	3.12	2.16	1.32	5.45	3.19	2.22	1.37	5.67	1.32
1.000000e+10	3.51	2.66	1.83	6.60	3.59	2.71	1.88	6.82	1.61
1.000000e+12	3.90	3.12	2.30	7.75	3.97	3.17	2.35	7.97	1.90

TABLE EC.56. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/32 = 0.03125$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	1.28	−1.55	−2.55	1.30	1.36	−1.52	−2.52	1.45	0.18
20	1.35	−1.27	−2.26	1.42	1.43	−1.24	−2.24	1.57	0.20
100	1.49	−0.81	−1.78	1.70	1.56	−0.78	−1.76	1.86	0.25
1,000	1.66	−0.34	−1.31	2.11	1.72	−0.31	−1.28	2.26	0.32
100,000	1.96	0.30	−0.65	2.92	2.01	0.33	−0.62	3.08	0.47
100,000,000	2.35	0.97	0.03	4.14	2.39	1.00	0.06	4.30	0.68
1.000000e+10	2.58	1.33	0.40	4.96	2.62	1.36	0.43	5.11	0.83
1.000000e+12	2.80	1.65	0.73	5.77	2.85	1.68	0.76	5.93	0.97

TABLE EC.57. Approximations for the $q = 0.50$ and $q = 0.75$ quantile for gamma with SCV $= 1/64 = 0.015625$.

	$q = 0.50$				$q = 0.75$				
n	Exact	Asymp.	(7.10)	(1.9)	Exact	Asymp.	(7.10)	(1.9)	(1.8)
10	1.19	−2.28	−3.28	1.21	1.25	−2.26	−3.27	1.32	0.10
20	1.24	−2.01	−3.01	1.30	1.29	−1.99	−3.00	1.41	0.11
100	1.33	−1.56	−2.55	1.50	1.38	−1.55	−2.54	1.61	0.14
1,000	1.45	−1.12	−2.11	1.78	1.49	−1.11	−2.10	1.89	0.17
100,000	1.64	−0.55	−1.53	2.36	1.67	−0.54	−1.52	2.47	0.24
100,000,000	1.88	0.02	−0.95	3.22	1.91	0.03	−0.94	3.33	0.35
1.000000e+10	2.02	0.31	−0.66	3.80	2.05	0.33	−0.64	3.91	0.42
1.000000e+12	2.16	0.56	−0.40	4.37	2.18	0.58	−0.39	4.48	0.50

EC.4.1. SCV > 1

Tables EC.46–EC.51 provide a comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. gamma random variables with mean one and SCV $= \{2, 4, 6, 8, 16, 32, 64\}$ for six values of n. The approximations are the asymptotic approximation in (7.5), the associated simple rough approximation in (1.9), the exact values for H_2 with $r = 0.5$, and the crude approximation in (1.8).

From the results, we see that the various approximations all perform quite well for smaller SCV values such as $c^2 = 2.0$, but problems arise as c^2 increases. For large c^2, all the approximations based on asymptotics produce negative values for the smaller values of n. When it is large enough, the asymptotic approximation becomes reasonable, but it is never spectacular, presumably because the gamma distribution fails to have a pure-exponential tail. Fortunately, the H_2-exact approximation is roughly reasonable for all values of n.

EC.4.2. $SCV < 1$

We now turn to the less variable gamma distributions, having $0 < c^2 < 1$. Tables EC.52–EC.57 provide a comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the c.d.f. of the maximum of n i.i.d. gamma random variables with mean one and SCV $= \{1/2, 1/4, 1/8, 1/16, 1/32, 1/64\}$ for five values of n. The approximations are the asymptotic approximation in (7.5), the associated simple approximation (7.10), the simple rough approximation based on the shifted exponential in (1.9), and the crude approximation in (1.8).

Except for the first case $c^2 = 0.5$, the asymptotic approximation does not do well for the smaller values of n. Just as for $c^2 > 1$, when c^2 gets very small, the asymptotic approximation produces negative estimates for smaller values of n. Again, the asymptotic approximation is not even spectacularly accurate for extremely large n, presumably because the gamma distribution fails to have a pure-exponential tail. On the other hand, the simple rough approximation (1.9) is consistently good for smaller values of n, e.g., $10 \leqslant n \leqslant 1{,}000$.