A Review of Basic FCLT's

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Abstract

We briefly review Donsker's functional central limit theorem (FCLT), which is a generalization of the classic central limit theorem (CLT). We then review FCLT's for renewal processes and renewal-reward processes and their generalization to other counting processes and random sums. We show how to apply the continuous mapping and generalizations that preserve convergence in the function space $\mathcal{D}$. These methods have played an important role in establishing heavy-traffic limits for queueing models.

1 Fundamental Concepts

1.1 Donsker's Functional Central Limit Theorem (FCLT)

Donsker's FCLT is a generalization of the classical central limit theorem (CLT). To state the CLT, let X_k, $k \geq 1$, be independent and identically distributed (i.i.d.) random variables with finite mean m and variance $\sigma^2 > 0$. Let $S_k \equiv X_1 + \cdots + X_k$, $k \geq 1$, be the partial sums of the first k random variables X_k with $S_0 \equiv 0$, where $\equiv$ denotes equality by definition. Let $N(m, \sigma^2)$ be a random variable with a normal (Gaussian) distribution having mean m and variance σ^2. The CLT states that

$$n^{-1/2}(S_n - nm) \Rightarrow N(0, \sigma^2) \stackrel{\mathrm{d}}{=} \sigma N(0,1) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad n \to \infty, \tag{1.1}$$

where $\Rightarrow$ denotes convergence in distribution of random variables taking values in the real line $\mathbb{R}$ and $\stackrel{\mathrm{d}}{=}$ means "equal in distribution."

Donsker's FCLT is a limit for the entire sequence $\{S_k\}$ instead of one term. It is expressed in terms of special functions of a continuous argument t in the function space $\mathcal{D} \equiv \mathcal{D}([0,\infty))$ of all right-continuous real-valued functions on $[0,\infty)$ with left limits everywhere in $(0,\infty)$ or in the subset $\mathcal{C}$ of all continuous functions in $\mathcal{D}$. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x (the floor function). Let the meaning of $\Rightarrow$ be appropriately generalized, as in §3.2 and §11.3 of Whitt [2002a]. Let

$$\mathbf{S}_n \equiv \mathbf{S}_n(t) \quad \equiv \quad n^{-1/2}(S_{\lfloor nt \rfloor} - mnt), \quad t \geq 0, \tag{1.2}$$

be the n^{th} scaled function in $\mathcal{D}$ induced by the sequence $\{S_k\}$, $n \geq 1$. Notice that function $\mathbf{S}_n$ in (1.2) has time scaling by n and space scaling by $\sqrt{n}$, just as in the CLT in (1.1).

Theorem 1.1 (*Donsker's FCLT, Theorem 4.3.2 of* Whitt [2002a]) *Under the independence and moment conditions above,*

$$\mathbf{S}_n \Rightarrow \sigma \mathbf{B} \quad \mathit{in} \quad \mathcal{D} \quad \mathit{as} \quad n \to \infty, \tag{1.3}$$

where $\mathbf{B}$ *is standard (zero drift, unit variance) Brownian motion (BM).*

We make a few remarks:

(i) Several important concepts deserve further discussion: (i) convergence of distribution of real-valued random variables in $\mathbb{R}$ in (1.1), (ii) convergence in distribution of random elements of a more general topological space, (iii) the function space $\mathcal{D}$ as a topological space, and (iv) convergence in distribution of random elements of $\mathcal{D}$ in (1.3). Chapters 1-4 of Whitt [2002a] is devoted to these topics, with additional details in later chapters. That in turn draws heavily on the basic book, Billingsley [1968, 1999].

(ii) Convergence in distribution in $\mathbb{R}$ can be defined as follows: $Z_n \to Z$ in $\mathbb{R}$ if either (i) $P(Z_n \leq x) \to P(Z \leq x)$ for all x or (ii) $E[f(Z_n)] \to E[f(Z)]$ for all continuous and bounded real-valued functions f. Clearly the first version does not extend to general spaces, but fortunately the second does, and is used.

(iii) In the setting of Theorem 1.1, where the limit process has continuous sample paths w.p.1 (as with BM), the mode of convergence in $\mathcal{D}$ is equivalent to uniform convergence on bounded intervals. It is important to remember that the mode is stronger than only pointwise convergence for each t but also weaker than uniform convergence over the entire real line. That last difference means that we cannot extract limits for a sequence of steady-state distributions from associated limits for a sequence of stochastic processes without doing a lot more work; e.g. see Gamarnik and Zeevi [2006].

1.2 Why Should We care?

Donsker's FCLT in Theorem 1.1 is an important extension to the CLT in (1.1), because it can be used as a tool to obtain additional limits of interest, such as heavy-traffic limits for queueing models. These extensions primarily follow from the continuous mapping theorem (CMT) and generalizations of the CMT, as in §3.4 of Whitt [2002a].

Theorem 1.2 (*the continuous mapping theorem, Theorem 3.4.1 of* Whitt [2002a]) *If*

$$\mathbf{Z}_n \Rightarrow \mathbf{Z} \quad in \quad \mathcal{S}_1, \tag{1.4}$$

where $\mathcal{S}_1$ is an appropriate topological space, such as $\mathcal{D}$, and if $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a continuous function from $\mathcal{S}_1$ to another such space $\mathcal{S}_2$, then

$$f(\mathbf{Z}_n) \Rightarrow f(\mathbf{Z}) \quad in \quad \mathcal{S}_2. \tag{1.5}$$

For example, given (1.1), we might ask if there is an associated limit for $M_n \equiv \max\{S_k : 0 \leq k \leq n\}$. There is, and it follows easily from Theorem 1.1.

Corollary 1.1 (*the CLT for the maximum partial sum*) *If, in addition to the independence and moment conditions of Theorem 1.1, the mean is $m = 0$, then*

$$n^{-1/2} M_n = \sup_{\{0 \leq t \leq 1\}} \{\mathbf{S}_n(t)\} \Rightarrow \sigma \sup_{\{0 \leq t \leq 1\}} \{\mathbf{B}(t)\} \in \mathbb{R} \quad as \quad n \to \infty, \tag{1.6}$$

so that

$$P(M_n \geq x\sqrt{n}/\sigma) \rightarrow 2P(N(0,1) > x) \quad as \quad n \rightarrow \infty \tag{1.7}$$

for any $x \geq 0$.

Proof We apply CMT with Donsker's theorem, using the function $f : \mathcal{D} \rightarrow \mathbb{R}$, where $f(x) \equiv \sup_{\{0 \leq t \leq 1\}} \{x(t)\}$. We can calculate the distribution of the limit, because it is well known that

$$P(\sup_{\{0 \leq t \leq 1\}} \{\mathbf{B}(t)\} > c) = 2P(N(0,1) > c); \tag{1.8}$$

by the reflection principle; see §10.2 of Ross [2014]. ∎

We make a few remarks:

(i) Corollary 1.1 was first obtained by Erdös and Kac (1946) for that one function, but Donsker's theorem, established in 1951, applies to any continuous function by combining Theorems 1.1 and 1.2 above.

(ii) By applying the CMT with the projection map at time $t = 1$, we immediately obtain the ordinary CLT from the FCLT. Hence, the FCLT is truly a generalization of the CLT.

(iii) The CMT has important applications to heavy-traffic limits for queues and networks of queues, as we indicate briefly in §4.

In the next section we discuss generalizations of the CMT in order to treat inverse processes. In particular, what we do can be regarded as an application of

Theorem 1.3 (*generalized continuous mapping theorem, Theorem 3.4.4 of* Whitt [2002a]) *Suppose that*

$$\mathbf{Z}_n \Rightarrow \mathbf{Z} \quad in \quad \mathcal{S}_1, \tag{1.9}$$

where $\mathcal{S}_1$ *is an appropriate topological space, such as* $\mathcal{D}$, *and* f *and* $f_n : \mathcal{S}_1 \rightarrow \mathcal{S}_2$, $n \geq 1$, *are functions from* $\mathcal{S}_1$ *to* $\mathcal{S}_2$. *Let* E *be the set of* x *in* $\mathcal{S}_1$ *such that* $f_n(x_n)$ *fails to converge to* $f(x)$ *for some sequence* $\{x_n\}$ *with* $x_n \rightarrow x$ *in in* $\mathcal{S}_1$. *If* $P(\mathbf{Z} \in E) = 0$, *then*

$$f_n(\mathbf{Z}_n) \Rightarrow f(\mathbf{Z}) \quad in \quad \mathcal{S}_2. \tag{1.10}$$

2 FCLT's for Renewal, Counting and Inverse Processes

In this section we review the FCLT for renewal processes and its generalization to more general counting and inverse processes. We start with §6.3.1 of Whitt [2002a] and then mention §§13.6-13.8.

We express the main result without any stochastic assumptions. Let $S_k \equiv X_1 + \cdots + X_k$, $k \geq 1$, be the partial sums of k nonnegative real-valued random variables X_k and let $S_0 \equiv 0$, where $\equiv$ denotes equality by definition. Let $N(t)$ count the number of points less than or equal to t, i.e.,

$$N(t) \equiv \max\{k \geq 0 : S_k \leq t\}, \quad t \geq 0. \tag{2.1}$$

The results in this section follow from the observation that these processes are essentially inverses of each other. In particular, without any stochastic assumptions,

$$S_k \leq t \quad \text{if and only if} \quad N(t) \geq k. \tag{2.2}$$

Let $\mathbf{S}_n$ and $\mathbf{N}_n$ be the associated random functions in $\mathcal{D} \equiv \mathcal{D}([0,\infty))$ defined for $n \geq 1$, by

$$\begin{aligned} \mathbf{S}_n &\equiv c_n^{-1}(S_{\lfloor nt \rfloor} - mnt), \quad t \geq 0 \\ \mathbf{N}_n &\equiv c_n^{-1}(N(nt) - nt/m), \quad t \geq 0. \end{aligned} \tag{2.3}$$

We now establish an equivalence of convergence for the normalized processes $\mathbf{S}_n$ and $\mathbf{N}_n$ in (2.3). We actually obtain joint convergence from either one. For that purpose, let $x \circ y$ be the composition, i.e., $(x \circ y)(t) \equiv x(y(t))$, $t \geq 0$.

Theorem 2.1 (*Thm 6.3.1 on p. 202 of* Whitt [2002a]) *Suppose that $m > 0$, $c_n \to \infty$, $n/c_n \to \infty$ and $\mathbf{S}(0) = 0$ for $\mathbf{S}$ defined below. If*

$$\textit{either} \quad \mathbf{S}_n \Rightarrow \mathbf{S} \quad \textit{or} \quad \mathbf{N}_n \Rightarrow \mathbf{N} \quad \textit{in} \quad (\mathcal{D}, M_1), \tag{2.4}$$

then both hold, separately as well as jointly, i.e.,

$$(\mathbf{S}_n, \mathbf{N}_n) \Rightarrow (\mathbf{S}, \mathbf{N}) \quad \textit{in} \quad (\mathcal{D}^2, M_1) \tag{2.5}$$

where

$$\mathbf{N} = -m^{-1}\mathbf{S} \circ m^{-1}\mathbf{e}, \quad \textit{i.e.,} \quad \mathbf{N}(t) = -m^{-1}\mathbf{S}(t/m), \quad t \geq 0. \tag{2.6}$$

or, equivalently,

$$\mathbf{S} = -m\mathbf{N} \circ m\mathbf{e}, \quad \textit{i.e.,} \quad \mathbf{S}(t) = -m^{-1}\mathbf{N}(mt), \quad t \geq 0. \tag{2.7}$$

We make the following remarks:

(i) There is a typo in equations (3.7) and (3.8) in Theorem 6.3.1 of Whitt [2002a]; the minus signs in (2.6) and (2.7) above are missing in Whitt [2002a]. It is clear that something there from the first paragraph of §6.3.2 of Whitt [2002a].

(ii) An early paper establishing much of Theorem 2.1 is Iglehart and Whitt [1971]. It plays a key role in the heavy-traffic limit theorems in Iglehart and Whitt [1970a,b]. An updated version very close to Whitt [2002a] appears in §7 of Whitt [1980].

(iii) The proof of Theorem 2.1 is a consequence of Corollary 13.8.1 to Theorem 13.8.2 in Whitt [2002a], which in turn is a consequence of Theorem 13.7.1 of Whitt [2002a], which we discuss below. The topic is "preservation of convergence for the inverse function with centering."

(iv) The main case for the spatial normalization constants in (2.3) is $c_n = \sqrt{n}$, which arises in Donsker's theorem for $\mathbf{S}_n$, Theorem 4.3.2 of Whitt [2002a], yielding a Brownian motion (BM) limit. Other spatial scaling arises with heavy tails and strong dependence, as discussed in Chapters 4 and 6 of Whitt [2002a].

(v) The M_1 appears in the convergence in (2.5) because, in general, the convergence in (2.5) is understood to be in the nonstandard M_1 topology, but that distinction does not arise if the limit process has continuous sample paths, as is the case for BM. The Skorohod topologies all agree for continuous limits. The M_1 topology is important for non-BM limits such as stable processes. For quick background on the Skorohod topologies, see §3.3 of Whitt [2002a]. For more on the applied significance, see Ch. 6 of Whitt [2002a]; for more on the theory, see Ch. 12 of Whitt [2002a] and of course the source, Skorohod [1956]. For a recent paper involving the M_1 topology, see Pang and Whitt [2010].

(vi) It is naturally to ask if it is not possible to exploit the inverse relation in (2.2) to obtain results in $\mathbb{R}$ instead of in $\mathcal{D}$. Indeed, it is, but the proofs are actually much harder; see Glynn and Whitt [1988] and §3.5 of Whitt [2002b].

(vii) The condition $\mathbf{S}(0) = 0$ exposes a subtle point. First, in applications, the condition is usually satisfied. It appears in the theoretical basis in Theorem 13.7.1 of Whitt [2002a]; see p. 82 of Whitt [1980] for more on the history.

We now state the corollary for renewal processes. We use the squared coefficient of variation (scv, variance divided by the square of the mean).

Corollary 2.1 (*the FCLT for a renewal process*) *If in the setting above X_k are i.i.d. nonnegative random variables with finite mean $E[X] = m > 0$ and variance $Var(X) = \sigma^2 > 0$, then, in addition to Donsker's theorem concluding that $\mathbf{S}_n \Rightarrow \mathbf{S} \quad in \quad (\mathcal{D}, M_1)$, where $c_n = \sqrt{n}$ and $\mathbf{S} = \sigma\mathbf{B}$ with $\mathbf{B}$ being a standard BM, we have*

$$\mathbf{N}_n \Rightarrow \mathbf{N} \quad in \quad (\mathcal{D}, M_1), \tag{2.8}$$

where

$$\mathbf{N} = -m^{-1}\mathbf{S} \circ m^{-1}e \stackrel{\mathrm{d}}{=} \sqrt{\sigma^2/m^3}\mathbf{B} = \sqrt{\lambda c_a^2}\mathbf{B}, \tag{2.9}$$

where $\lambda \equiv 1/m$ and $c_a^2 \equiv \sigma^2/m^2$ is the scv of the time between renewals.

Proof The limit (2.8) with the first expression in (2.9) follows directly from Theorem 2.1. The second expression follows from the basic scaling property of BM, i.e., for $c > 0$,

$$\mathbf{B} \circ ce \stackrel{\mathrm{d}}{=} \sqrt{c}\mathbf{B}. \quad \blacksquare \tag{2.10}$$

By applying the continuous mapping theorem with the projection map at time $t = 1$, we immediately obtain the ordinary CLT for a renewal process, which can be found in many textbooks. The importance of the line of reasoning above is that the results extend beyong the familiar i.i.d. setting of a renewal process: We get a FCLT and a CLT for the counting process whenever we have a FCLT for the partial sums. One specific example is the superposition of independent renewal processes, which is not itself a renewal proocess unless all processes are Poisson.

Corollary 2.2 (*the ordinary CLT for a renewal process*) *Under the conditions of Theorem* 2.1,

$$t^{-1/2}(N(t) - t/m) \Rightarrow N(0, \lambda c_a^2) \quad in \quad \mathbb{R} \quad as \quad t \to \infty, \tag{2.11}$$

where $\lambda \equiv 1/m$ and $c_a^2 \equiv \sigma^2/m^2$ is the scv of the time between renewals.

We now elaborate on the technical foundations, which is expressed by Theorem 13.7.1 of Whitt [2002a]. This theorem expresses an equivalence of convergence for sequences of deterministic elements of $\mathcal{D}$. It applies immediately to yield the corresponding statement for the stochastic processes. The main idea in this representation to formulate the relation in a way that exposes the inverse relation in its essential form. For that purpose, let the inverse of a nonnegative real-valued function x in $\mathcal{D}$ that is unbounded above be defined by

$$x^{-1} \equiv \inf\{s \geq 0 : x(s) > t\}, \quad t \geq 0; \tag{2.12}$$

see §13.6 of Whitt [2002a].

We make a few remarks:

(i) A proper treatment requires that we look at various subsets of $\mathcal{D}$; see pp. 428 and 441 of Whitt [2002a].

(ii) Note that the definition (2.12) yields a right-continuous function, and so is in $\mathcal{D}$. The corresponding left-continuous in verse has $\geq$ instead of $>$ in (2.12). For further discussion seee p. 442 of Whitt [2002a].

(iii) The inverse relations in (2.2) and (2.12) are not exactly consistent. The gap is bridged in §13.8 of Whitt [2002a].

We now state a result that is appealing for its simplicity. Before worrying about the details of the proof, it is good to see that this result makes sense by looking at a picture.

Theorem 2.2 (*Thm 13.7.1 on p. 448 of* Whitt [2002a]) *let x_n be elements of $\mathcal{D}$ that are nonnegative, nondecreasing and unbounded above; let e be the identity function, i.e., $e(t) \equiv t$, $t \geq 0$, and let c_n be constants satisfying $c_n \to \infty$. If*

$$c_n(x_n - e) \to y \in (\mathcal{D}, M_1) \quad \textit{with} \quad y(0) = 0, \tag{2.13}$$

then

$$c_n(x_n^{-1} - e) \to -y \in (\mathcal{D}, M_1). \tag{2.14}$$

If the limit function y has no positive jumps, as when it is continuous, then the limit holds in the standard J_1 topology. That reduces to the topology of uniform convergence on bounded intervals if y is continuous.

For interesting and useful generalizations of Theorem 2.2 above, see Theorems 13.7.2-13.7.4 of Whitt [2002a].

3 Renewal-Reward Processes and Other Random Sums

We now apply the results of §2 on the preservation of convergence for the inverse function with centering together with the preservation of convergence for the composition function with centering in order to obtain a FCLT for renewal-reward processes and generalizations. In this section we present the results with the usual independence assumptions. These results may be hard to extract from the more abstract presentation in Whitt [2002a], but they are in fact a direct consequence.

For the renewal reward processes, suppose we again have the renewal process N in §2 with the i.i.d. times between renewals X_k having finite means and variances, using the notation $E[X] = m_a = \lambda^{-1}$ and $Var(X) = \sigma_a^2$, so that λ is the arrival rate and a denotes arrival process. In addition, suppose that we have the i.i.d. rewards Y_k, also having a finite mean and variance, with $EY = m_r$ and $Var(Y) = \sigma_r^2$, using the subscript r to denote the rewards. We will be establish a FCLT for the renewal-reward process

$$Z(t) \equiv \sum_{k=1}^{N(t)} Y_k, \quad t \geq 0. \tag{3.1}$$

Note that $Z(t)$ represents the total reward at time t under the assumption that a reward Y_k is earned at the time of the k^{th} point in the renewal process N.

Let $\mathbf{S}_{a,n}$, $\mathbf{N}_n$, $\bar{N}_n$, $\mathbf{S}_{r,n}$ and $\mathbf{Z}_n$ be associated random functions in $\mathcal{D} \equiv \mathcal{D}([0,\infty))$ defined for $n \geq 1$, by

$$\begin{aligned}
\mathbf{S}_{a,n} &\equiv n^{-1/2}(S_{a,\lfloor nt \rfloor} - m_a nt), \quad t \geq 0, \\
\mathbf{N}_n &\equiv n^{-1/2}(N(nt) - \lambda nt), \quad t \geq 0, \\
\bar{N}_n &\equiv n^{-1}N(nt), \quad t \geq 0, \\
\mathbf{S}_{r,n} &\equiv n^{-1/2}(S_{r,\lfloor nt \rfloor} - m_r nt), \quad t \geq 0, \\
\mathbf{Z}_n &\equiv n^{-1/2}(Z(nt) - \lambda m_r nt), \quad t \geq 0.
\end{aligned} \tag{3.2}$$

The following is an analog of Theorem 2.1, which corresponds to Theorem 7.4.1 of Whitt [2002a].

Theorem 3.1 (*Thm 7.4.1 on p. 239 of* Whitt [2002a]) *Suppose that* $\{X_k\}$ *and* $\{Y_k\}$ *are independent sequences of i.i.d. random variables with finite means* $m_a > 0$ *and* $m_r > 0$ *and variances* σ_a^2 *and* σ_r^2*. Then*

$$(\mathbf{S}_{a,n}, \mathbf{S}_{r,n}, \mathbf{N}_n, \bar{N}_n, \mathbf{Z}_n) \Rightarrow (\sigma_a \mathbf{B}_a, \sigma_r \mathbf{B}_r, \mathbf{N}, \lambda \mathbf{e}, \mathbf{Z}) \tag{3.3}$$

for the processes defined in (3.2)*, where*

$$\mathbf{N} = -\lambda \sigma_a \mathbf{B}_a \circ \lambda \mathbf{e} \stackrel{\text{d}}{=} \sqrt{\lambda^3 \sigma_a^2} \mathbf{B}_a \quad \textit{and} \quad \mathbf{Z} \stackrel{\text{d}}{=} \sigma_z \mathbf{B}, \tag{3.4}$$

with $\mathbf{B}$ *being a standard BM and*

$$\sigma_z^2 = \lambda\sigma_r^2 + m_r^2\lambda^3\sigma_a^2 = \lambda m_r^2(c_r^2 + c_a^2), \tag{3.5}$$

with $c_r^2 \equiv \sigma_r^2/m_r^2$ *and* $c_a^2 \equiv \sigma_a^2/m_a^2$ *being the scv's.*

Proof The proof can be simple and direct, given §2. In particular, we first obtain the joint limit for the first four components of (3.3). The first two limits in (3.3) separately follow from Donsker's theorem. The third limit, joint with the second, is Theorem 2.1 above. The fourth limit follows immediately from the third by the continuous mapping theorem, because we have introduced greater spatial scaling. These four limits hold jointly by Theorems 11.4.4 and 11.4.5 of Whitt [2002a]. Finally we apply the continuous mapping theorem with composition plus simple addition to get all five limits by writing

$$\mathbf{Z}_n = \mathbf{S}_{r,n} \circ \bar{N}_n + m_r\mathbf{N}_n \tag{3.6}$$

i.e.,

$$\mathbf{Z}_n(t) \equiv n^{-1/2}\left(\sum_{k=1}^{N(nt)} -\lambda m_r nt\right), \quad t \geq 0, \tag{3.7}$$

while

$$(\mathbf{S}_{r,n} \circ \bar{N}_n)(t) = n^{-1/2}\left(\sum_{k=1}^{N(nt)} -m_r N_n(nt)\right), \quad t \geq 0, \tag{3.8}$$

and

$$m_r\mathbf{N}_n(t) = m_r n^{-1/2}\left(N(nt) - \lambda m_r nt\right), \quad t \geq 0. \tag{3.9}$$

We then add (3.8) and (3.9) to get (3.7), observing the the second term in (3.8) cancels the first term in (3.9).

We now derive alternative expressions for the limit process $\mathbf{Z}$. First, directly from (3.6) we obtain

$$\begin{aligned}
\mathbf{Z} &= \mathbf{S}_r \circ \lambda\mathbf{e} + m_r\mathbf{N} \\
&= \sigma_r\mathbf{B}_r \circ \lambda\mathbf{e} - m_r\lambda\sigma_x\mathbf{B}_x \circ \lambda\mathbf{e} \\
&= (\sigma_r\mathbf{B}_r - m_r\lambda\sigma_x\mathbf{B}_x) \circ \lambda\mathbf{e} \\
&\stackrel{\mathrm{d}}{=} \sqrt{\sigma_r^2 + m_r^2\lambda^2\sigma_x^2}\,\mathbf{B} \circ \lambda\mathbf{e} \\
&\stackrel{\mathrm{d}}{=} \sqrt{\lambda\sigma_r^2 + m_r^2\lambda^3\sigma_x^2}\,\mathbf{B} = \sqrt{\lambda m_r^2(c_r^2 + c_x^2)}\,\mathbf{B},
\end{aligned} \tag{3.10}$$

which justifies the expression for σ_z^2 in (3.5). ∎

We make the following remarks:

(i) We have not stated Theorem 3.1 above in the same level of generality as Theorem 2.1. To do that, we could have started by assuming the joint limit $(\mathbf{S}_{r,n}, \mathbf{N}_n) \Rightarrow (\mathbf{S}_r, \mathbf{N})$ and omitted all the independence assumptions. By the reasoning above, we would get $\mathbf{Z}_n \Rightarrow \mathbf{Z}$, where $\mathbf{Z}$ is given in the first line of (3.10) above.

(ii) The last step in the proof of Theorem 3.1 above to get the last component can be regarded as a consequence of Corollary 13.3.1 of Whitt [2002a], which we express below in the same form as Theorem 2.2 above.

Here is a variant of Corollary 13.3.1 of Whitt [2002a].

Theorem 3.2 (*Cor 13.3.1 on p. 432 of* Whitt [2002a]) *let x_n, x and z be elements of $\mathcal{D}$; let y_n be elements of $\mathcal{D}$ that are nonnegative and nondecreasing; let e be the identity function, i.e., $e(t) \equiv t$, $t \geq 0$; let c_n be constants satisfying $c_n \to \infty$; and let b_n be constant satisfying $b_n \to b$. If*

$$(x_n - c_n e, c_n(y_n - b_n e)) \to (x, y) \in (\mathcal{D}^2, M_1), \tag{3.11}$$

where the functions $x \circ be$ and y have no common discontinuities, then $y_n \to be$ and

$$(x_n \circ y_n - c_n b_e) \to x \circ be + y \in (\mathcal{D}, M_1). \tag{3.12}$$

If the limit functions x and y are continuous, convergence holds in the topology of uniform convergence on bounded intervals.

4 Applications to Heavy-Traffic Limits for Time-Varying Queues

The results above have been applied extensively to establish heavy-traffic limits for queues and networks of queues as can be seen from Iglehart and Whitt [1970a,b], Reiman [1984] and Whitt [2002a]. The limits in §2 yield FCLT's for arrival processes in queueing models, given corresponding limits for partial sums. The limits in §3 yield FCLT's for the process representing the total input of work in $[0, t]$, which in turn plays a key role in FCLT's for the workload process. These arguments are applied in recent heavy-traffic limits for time-varying queues, in particular, in the proofs of Theorem 3.2 in Whitt [2014], Theorem 1 in Ma and Whitt [2016], Theorem 3 of Whitt and You [2016] and Theorems 3.1 and 6.1 in Whitt [2016]. These in turn are applied for the workload process in the time-varying robust queueing in Whitt and You [2016], for the cumulative idleness in an interval in §3.3 of Sun and Whitt [2016] and in FCLT versions of time-varying Little's law, extending Whitt and Zhang [2016], which is motivated by the data analysis in Whitt and Zhang [2015].

References

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P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, first edition, 1968.

P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, second edition, 1999.

D. Gamarnik and A. Zeevi. Validity of heavy traffic steady-state approximations in generalized Jackson networks. *Advances in Applied Probability*, 16(1):56–90, 2006.

P. W. Glynn and W. Whitt. Ordinary CLT and WLLN versions of $L = \lambda W$. *Mathematics of Operations Research*, 13:674–692, 1988.

D. L. Iglehart and W. Whitt. Multiple channel queues in heavy traffic, I. *Advances in Applied Probability*, 2(1): 150–177, 1970a.

D. L. Iglehart and W. Whitt. Multiple channel queues in heavy traffic, II: Sequences, networks and batches. *Advances in Applied Probability*, 2(2):355–369, 1970b.

D. L. Iglehart and W. Whitt. The equivalence of functional central limit theorems for counting processes and associated partial sums. *Ann. Math. Statist.*, 42(4):1372–1378, 1971.

N. Ma and W. Whitt. A performance algorithm for periodic queues. Columbia University, Available at: http://www.columbia.edu/∼ww2040/allpapers.html, 2016.

G. Pang and W. Whitt. Continuity of a queueing integral representation in the M_1 topology. *Annals of Applied Probability*, 10:214–237, 2010.

M. I. Reiman. Open queueing networks in heavy traffic. *Math. Oper. Res.*, 9(3):441–458, 1984.

S. M. Ross. *Introduction to Probability Models.* Academic Press, New York, eleventh edition, 2014.

A. A. Skorohod. Limit theorems for stochastic processes. *Theor. Probability Appl.*, 1:261–290, 1956.

X. Sun and W. Whitt. Creating work breaks from available idleness. Columbia University, Available at: http://www.columbia.edu/∼ww2040/allpapers.html, 2016.

W. Whitt. Some useful functions for functional limit theorems. *Mathematics of Operations Research*, 5:67–85, 1980.

W. Whitt. *Stochastic-Process Limits.* Springer, New York, 2002a. Available at: http://www.columbia.edu/∼ww2040/jumps.html.

W. Whitt. Internet Supplement to Stochastic-Process Limits. Columbia University, Available at: http://www.columbia.edu/∼ww2040/supplement.html, 2002b.

W. Whitt. Heavy-traffic limits for queues with periodic arrival processes. *Operations Research Letters*, 42: 458–461, 2014.

W. Whitt. Heavy-traffic limits for a single-server queue leading up to a critical point. Columbia University, Available at: http://www.columbia.edu/∼ww2040/allpapers.html, 2016.

W. Whitt and W. You. Time-varying robust queueing. Columbia University, Available at: http://www.columbia.edu/∼ww2040/allpapers.html, 2016.

W. Whitt and X. Zhang. A data-generated queueing model of an emergency department. Columbia University, http://www.columbia.edu/∼ww2040/allpapers.html, presented at INFORMS 2015, 2015.

W. Whitt and X. Zhang. A periodic Little's law. in preparation, Columbia University, http://www.columbia.edu/∼ww2040/allpapers.html, 2016.