IEOR 6711: Stochastic Models I, Fall 2003, Professor Whitt

Solutions to Final Exam: Thursday, December 18.

Below are six questions with several parts. Do as much as you can. Show your work.

1. **Two-Pump Gas Station** (12 points)

Potential customers arrive at a full-service, two-pump gas station according to a Poisson process at a rate of 40 cars per hour. There are two service attendants to help customers, one for each pump. If the two pumps are busy, then arriving customers wait in a single queue, to be served in the order of arrival by the first available pump. However, customers will not enter the station to wait if there are already two customers waiting, in addition to the two in service. Suppose that the amount of time required to service a car is exponentially distributed with a mean of three minutes.

This problem is a minor variant of Problem 15 in Chapter 6, on page 393, of the easier Ross book, *Introduction to Probability Models*. To answer the questions posed here, we need to find the steady-state distribution of the CTMC. In general, the steady-state distribution can be found by solving the equation $\alpha Q = 0$, where α is the steady-state probability vector. This CTMC is a birth-and-death process, so we can find the steady-state solution directly, by solving the local balance equations

$$\alpha_k \lambda_k = \alpha_{k+1}\mu_{k+1} \quad \text{for all} \quad k \ .$$

But, first, we need to read carefully to make sure we get the model right. We need to recognize that there are five states: k for $0 \le k \le 4$. We thus can write down the steady-state probabilities in terms of an unknown x, and then normalize to find x:

$$\alpha_0 = 1 * x, \quad \alpha_1 = \frac{\lambda_0}{\mu_1} * x, \quad \alpha_2 = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2} * x \ ,$$

$$\alpha_3 = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3} * x \ , \quad \alpha_4 = \frac{\lambda_0}{\mu_1}\frac{\lambda_1}{\mu_2}\frac{\lambda_2}{\mu_3}\frac{\lambda_3}{\mu_4} * x \ .$$

A main issue is to be careful and get the rates right, using the same units. We could either use cars per hour or cars per minute, but we should be consistent. We should realize that a mean service time of 3 minutes means one service per 3 minutes or 20 services per hour.

So, here, inserting the birth rates and death rates, we get

$$\alpha_0 = 1 * x, \quad \alpha_1 = \frac{40}{20} * x, \quad \alpha_2 = \frac{40}{20}\frac{40}{40} * x \ ,$$

$$\alpha_3 = \frac{40}{20}\frac{40}{40}\frac{40}{40} * x \ , \quad \alpha_4 = \frac{40}{20}\frac{40}{40}\frac{40}{40}\frac{40}{40} * x$$

or

$$\alpha_0 = 1 * x, \quad \alpha_1 = 2 * x, \quad \alpha_2 = 2 * x \ , \quad \alpha_3 = 2 * x \ , \quad \alpha_4 = 2 * x \ ,$$

so that

$$\alpha_0 = 1/9, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 2/9 \ .$$

(a) In the long-run, what rate are cars served?

Cars are served at rate

$$20 * \alpha_1 + 40 * (\alpha_2 + \alpha_3 + \alpha_4) = 280/9 = 31.111\ldots \text{ per hour} \quad .$$

Alternatively, we could multiply the arrival rate times $1 - \alpha_4$ getting $40 \times (7/9) = 280/9$, as above.

(b) In the long-run, what proportion of potential customers fail to be served?

The probability an arrival cannot be served is $\alpha_4 = 2/9$. As a check on parts (a) and (b), we see that cars are served at the arrival rate times the probability that an arrival is eventually served:

$$\text{rate cars served } = \lambda(1 - \text{loss prob.}) = 40 \times (7/9) = 30.111$$

(c) What is the long-run proportion of time that the station is empty?

The long-run proportion of time that the station is empty is $\alpha_0 = 1/9$.

(d) In steady state, what is the probability that both pumps are busy?

In steady state, the probability that both pumps are busy is $\alpha_2 + \alpha_3 + \alpha_4 = 6/9 = 2/3$.

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2. **System Failure in a Call Center** (14 points)

Consider a call center with 100 agents answering telephone calls, modelled as an Erlang delay system, i.e., as an $M/M/100/\infty$ queue with Poisson arrival process having arrival rate λ, IID exponential service times each with mean $1/\mu$, 100 homogenous servers (agents) and unlimited waiting space.

Suppose that at some instant there is a system failure, which shuts down the call arrival process, but otherwise does not affect the system. Hence after that failure instant, there are no more arrivals, but the customers being served continue to receive their service. Suppose that there are 97 customers in service at the failure instant (and thus none waiting in queue). Let $X(t)$ be the number of customers still in the system receiving service t time units after the failure event.

(a) Find an expression for $P(X(t) = 20)$.

The key observation is that times until the customers in service complete their service are IID exponential random variables, by the lack of memory property. Thus the probability

$P(X(t) = 20)$ is the probability of 77 service completions and 20 non-completions. That is a binomial probability with $n = 97$ and $p = e^{-\mu t}$. Specifically,

$$P(X(t) = 20) = \frac{97!}{20!77!}(e^{-\mu t})^{20}(1 - e^{-\mu t})^{77}$$

(b) Find an expression for $E[X(t)]$.

$E[X(t)] = np = 97 * e^{-\mu t}$

(c) Find an expression for $Var(X(t))$.

$Var(X(t)) = np(1-p) = 97 * e^{-\mu t}(1 - e^{-\mu t})$

(d) Redo part (b) under the assumption that there are initially 102 customers in the system, 100 in service plus 2 waiting in queue.

Start by conditioning on what happens to the first two departures. Get the following cases: (1) no departure before time t, (2) one departure before time t, (3) at least two departures before time t, with second departure occurring at time s, $0 < s < t$. Then $EX(t)$ equals 102 times probability of first event plus 101 times probability of second event plus expected value $E[X(t-s)]$, starting at 100, times the probability of the third event. That is

$$\begin{aligned}
E[X(t)] &= 102e^{-100\mu t} + 101(\int_0^t (100\mu)e^{-100\mu x}e^{-100\mu(t-x)}\,dx + \int_0^t (100\mu)^2 xe^{-100\mu x}(100e^{-100\mu(t-x)})\,dx \\
&= 102e^{-100\mu t} + 101\mu e^{-100\mu t}(100)t + 100e^{-100\mu t}\frac{(100\mu)^2t^2}{2} \\
&= e^{-100\mu t}(102 + 10,100\mu t + 1,000,000\frac{\mu^2t^2}{2})\ .
\end{aligned}$$

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3. **Markov Mouse in the Big City** (15 points)

In the Big City, Markov Mouse finds himself in a skyscraper maze. The skyscraper maze has 100 floors. Each floor is laid out like the maze discussed in class: On each floor there are nine rooms, numbered one through nine. Each floor has rooms laid out like

1	2	3
4	5	6
7	8	9

Markov Mouse may move from any room to a neighboring room on the same floor. Markov Mouse also may move up or down one floor from the center room, Room 5. However, no vertical movement (up or down) is possible from any room except Room 5. Specifically, on each floor, there are doors:

between rooms	1	and	2
between rooms	1	and	4
between rooms	2	and	3
between rooms	2	and	5
between rooms	3	and	6
between rooms	4	and	5
between rooms	4	and	7
between rooms	5	and	6
between rooms	5	and	8
between rooms	6	and	9
between rooms	7	and	8
between rooms	8	and	9

Markov Mouse can go through a door in either direction. From Room 5 on floors 2-99, Markov Mouse can move next to Rooms 2, 4, 6 and 8 on the same floor, or Markov Mouse can move to Room 5 on the floor above or to Room 5 on the floor below. From Room 5 on floor 1, Markov Mouse can move next to Rooms 2, 4, 6 and 8 on the same floor, or Markov Mouse can move to Room 5 on the floor above. (There is no floor below floor 1.) From Room 5 on floor 100, Markov Mouse can move next to Rooms 2, 4, 6 and 8 on the same floor, or Markov Mouse can move to Room 5 on the floor below. (There is no floor above floor 100.)

Suppose that Markov Mouse moves randomly from room to room, with each successive move being independent of the prior history. Suppose that, in each move, Markov Mouse is equally likely to make each of the available moves. Suppose that Markov Mouse starts in Room 1 on Floor 1.

(a) What is the the probability that Markov Mouse is in Room 1 on Floor 1 after making two moves?

The event can happen if and only if Markov Mouse goes from Room 1 to one of Rooms 2 or 4 in the first move and then immediately returns.

$$P(\text{returns to initial room in 2 moves}) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{3} .$$

(b) What is the the probability that Markov Mouse is in Room 1 on Floor 1 after making three moves?

The probability is 0, because the Markov chain is periodic with period 2. It can be in the odd rooms on Floor 1 only after an even number of moves.

(c) What is the expected number of moves made by Markov Mouse before he first returns to the initial room, Room 1 on Floor 1?

The expected number of moves between successive visits to Room 1 on Floor 1 is the reciprocal of the stationary probability. The stationary probability is the number of doors out of that room divided by the sum of the number of doors out of all the rooms. There are two doors out of Room 1 on each floor. So the numerator is 2. There are 4 corner rooms (Rooms 1, 3, 7 and 9) on each of the 100 floors, from which there are 2 doors; there are 4 side rooms (Rooms 2, 4, 6 and 8) on each of the 100 floors, from which there are 3 doors; there is 1 center room (Room 5) on each for 98 floors, from which there are 6 moves; there is 1 center room (Room 5) on 2 floors (Floors 1 and 100), from which there are 5 moves. Thus the denominator is

$$4 * 100 * 2 + 4 * 100 * 3 + 1 * 98 * 6 + 1 * 2 * 5 = 800 + 1200 + 588 + 10 = 2598 \ .$$

Thus the stationary probability that Markov Mouse is in Room 1 on Floor 1 is $2/2598 = 1/1299$. Hence the mean time until returning to that room is 1299 moves.

(d) Consider the probability that Markov Mouse is in Room 1 on Floor 1 after n moves. Give an approximation for this probability for large values of n.

Most of the hard work has been done in part (c) above, but we have to be careful because the Markov chain is periodic. Let us refer to Room 1 on Floor 1 as simply Room 1. We then want to find the n-step transition probability $P^n_{1,1}$. We have

$$P^{2n+1}_{1,1} = 0 \quad \text{for all} \quad n \geq 0$$

and

$$\lim_{n \to \infty} P^{2n}_{1,1} = 2\pi_1 = \frac{2}{1299} \ .$$

(e) Justify your answers in part (c) and (d).

The skyscraper maze, just like the single-floor maze, produces a periodic Markov chain, with period 2. Thus the stationary distribution is not really the limiting steady-state probability. Indeed, there is no limit for $P^n_{i,j}$ as $n \to \infty$, because of the periodicity. The main structural result for the periodic finite Markov chain is Theorem 4.3.1 on page 173 and Remark 3 on page 177.

The main justification desired is for the simple solution for the stationary distribution. The special structure occurs because the random movement in the skyscraper maze can be identified with a random walk on an edge-weighted graph, as in Proposition 4.7.1 on page 206. The special form for the stationary distribution that holds there is due to the time reversibility that holds for this model. The supporting details are given in Section 4.7 and in Proposition 4.7.1. A strong answer would include the proof of Proposition 4.7.1.

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4. **Ordered Server Selection** (32 points)

Consider an $M/M/\infty$ queue in which customers arrive according to a Poisson process having rate λ. Each customer starts to receive service immediately upon arrival from one of an unlimited number of servers. Suppose that the service times are IID exponential random variables with mean $1/\mu$.

Suppose that, on arrival, each customer will choose the lowest numbered server that is free. Thus we can think of all arrivals occurring at server 1. Those customers who find server 1 free begin service there. Those customers finding server 1 busy immediately overflow and become arrivals at server 2. Those customers finding both servers 1 and 2 busy immediately overflow and become arrivals at server 3, and so forth.

(a) What is the long-run proportion of time that server 1 is busy?

First, we observe that this problem is a minor variant of Problem 5.30 on page 291 of Ross. Second, we observe that when we focus on the first k servers, the system behaves like the Erlang loss (B) model, i.e., the $M/M/k/0$ model. The steady-state probability that the first k servers are all busy is the same as it is in the $M/M/k/0$ model, which in turn is the steady-state blocking probability in the Erlang loss model. The Erlang loss model is analyzed in detail in Section 5.7.2 on pages 275-278 of Ross. The number of busy servers in the Erlang loss model is a birth-and-death process. It is also possible to analyze the ordered-server-selection model as a CTMC, but it is easier to exploit the connection to the Erlang loss model, because birth-and-death processes are much easier to analyze than general CTMC's.

To answer this question, we can thus think of the Erlang loss model with one server, i.e., the $M/M/1/0$ model. That is a two-state birth-and-death process. The answer is a special case of the Erlang loss formula, given in Theorem 5.7.4 of Ross. But the answer is easy to derive directly. The long-run proportion of time that server 1 is busy is

$$P(\text{server 1 busy }) = \frac{\lambda}{\lambda + \mu} = \frac{\alpha}{1 + \alpha} ,$$

where

$$\alpha \equiv \frac{\lambda}{\mu} .$$

(b) What is the long-run overflow rate from server 1 to server 2?

The long-run overflow rate is the probability that server 1 is busy times the arrival rate, so the answer is

$$\text{overflow rate from server 1 to server 2 } = \frac{\alpha}{1 + \alpha} \times \lambda = \frac{\lambda^2}{\lambda + \mu} .$$

(c) Consider the overflow process from server 1 to server 2, with the system initially empty; i.e., let $N(t)$ be the number of overflows in the time interval $[0, t]$. What kind of stochastic process is $\{N(t) : t \geq 0\}$? Is it a Markov process? Is it a counting process? Is it a (possibly nonhomogeneous) Poisson process? Is it a (possibly delayed) renewal process? Explain.

The overflow process $\{N(t) : t \geq 0\}$ is a counting process. It is not a Markov process, because the future beyond some time t (which is not the epoch of an overflow) depends upon whether server 1 is busy or not at time t. That implies that the overflow process is not a Poisson process. However, the overflow process *is* a delayed renewal process, because the the process starts over (renews) at each overflow epoch. The times between successive overflows are IID. However, the time until the first overflow has a different distribution, so the overflow process is a delayed renewal process instead of just a renewal process; see Section 3.5 in Ross.

(d) What is the long-run proportion of time that both servers 1 and 2 are busy?

As indicated in the first paragraph, that is just the steady-state blocking probability in the $M/M/2/0$ model, which is computed using basic birth-and-death process properties. In particular,

$$P(\text{servers 1 and 2 both busy}) = \frac{\alpha^2/2}{1+\alpha+\alpha^2/2},$$

where, as before,

$$\alpha \equiv \frac{\lambda}{\mu} .$$

(e) Starting with an empty system, what is the expected time until server 2 first becomes busy?

Let $T_{i,2}$ be the time to until server 2 first becomes busy, starting with the system empty ($i = 0$) or with server 1 busy ($i = 1$). We develop an equation for the mean $ET_{0,2}$, by considering what happens at the successive transitions:

$$\begin{aligned} ET_{0,2} &= \frac{1}{\lambda} + ET_{1,2} \\ &= \frac{1}{\lambda} + \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} ET_{0,2} , \end{aligned}$$

so that

$$ET_{0,2} \frac{\lambda}{\lambda+\mu} = \frac{1}{\lambda} + \frac{1}{\lambda+\mu} ,$$

so that

$$ET_{0,2} = \frac{\frac{1}{\lambda} + \frac{1}{\lambda+\mu}}{\frac{\lambda}{\lambda+\mu}} = \frac{2\lambda+\mu}{\lambda^2} .$$

(f) What is the expected number of busy servers among the first two servers in steady state?

It suffices to use the steady-state distribution in the Erlang loss model $M/M/2/0$. Let N be the steady-state number of busy servers in the $M/M/2/0$ model. Then what we want is

$$EN = 1 \times P(N=1) + 2 \times P(N=2)$$

$$
\begin{aligned}
&= \frac{1 \times \alpha}{1+\alpha+\alpha^2/2} + \frac{2 \times \alpha^2/2}{1+\alpha+\alpha^2/2} \\
&= \frac{\alpha+\alpha^2}{1+\alpha+\alpha^2/2} \\
&= \frac{2\alpha+2\alpha^2}{2+2\alpha+\alpha^2}\ ,
\end{aligned}
$$

where, as before,

$$\alpha \equiv \frac{\lambda}{\mu}\ .$$

(g) What proportion of time is server 2 busy? (Hint: Combine (a) and (f).)

You can analyze this directly by constructing a four-state CTMC with states: $\{0\}$, $\{1\}$, $\{2\}$ and $\{1,2\}$, describing the sets of busy servers. Alternatively, we can get the answer by subtracting (a) from (f). Let $I_i = 1$ if server i is busy in steady state. Then (a) gives $E[I_1]$ and (f) gives $E[I_1 + I_2]$, so we get what we want, $E[I_2]$, by subtracting:

$$
\begin{aligned}
\text{proportion of time server 2 busy} &= E[I_2] \\
&= E[I_1 + I_2] - E[I_1] \\
&= \frac{2\alpha+2\alpha^2}{2+2\alpha+\alpha^2} - \frac{\alpha}{1+\alpha} \\
&= \frac{2\alpha^2+\alpha^3}{(2+2\alpha+\alpha^2)(1+\alpha)}\ .
\end{aligned}
$$

(h) Find an expression for the proportion of time that server k is busy, for general k.

This is a direct generalization of part (g). What we want is $E[I_k]$. We get it by

$$
\begin{aligned}
\text{prob(server k busy)} &= E[I_k] \\
&= E[I_1 + \cdots + I_k] - E[I_1 + \cdots + I_{k-1}] \\
&= \sum_{i=1}^{k} \frac{i \times (\alpha^i/i!)}{\sum_{i=0}^{k}(\alpha^i/i!)} - \sum_{i=1}^{k-1} \frac{i \times (\alpha^i/i!)}{\sum_{i=0}^{k-1}(\alpha^i/i!)} \\
&= \frac{\sum_{i=1}^{k} i \times (\alpha^i/i!)}{\sum_{i=0}^{k}(\alpha^i/i!)} - \frac{\sum_{i=1}^{k-1} i \times (\alpha^i/i!)}{\sum_{i=0}^{k-1}(\alpha^i/i!)}\ .
\end{aligned}
$$

(i) If the service-time distribution were general instead of exponential (with the service times still being IID, having the same mean $1/\mu$), how would the answers in parts (a)-(h) change? Explain.

As discussed in class and in Section 5.7.2 of Ross, the Erlang loss model $M/G/s/0$ has the insensitivity property: The steady-state distribution of the number of customers present depends on the service-time distribution only through its mean. Hence all answers except to (c) and (e) are unchanged. The exponential service-time distribution is used in parts (c) and (e). For (c), the overflow process ceases to be a delayed renewal process. The epochs of overflows are not renewal events for the system.

For (e), the answer changes, but we can still do a similar analysis. We still can write

$$E[T_{0,2}] = \frac{1}{\lambda} + ET_{1,2} \ ,$$

but $ET_{1,2}$ changes. We can proceed much as before to get an expression, but it is more complicated. Let G be the cdf, g the pdf and $G^c(t) \equiv 1 - G(t)$ the ccdf of a service time. Then we can write

$$E[T_{1,2}] = \int_0^\infty tG^c(t)\lambda e^{-\lambda t}\, dt + \int_0^\infty (t + E[T_{0,2}])g(t)e^{-\lambda t}\, dt \ ,$$

so that

$$E[T_{0,2}](1 - \int_0^\infty g(t)e^{-\lambda t}\, dt) = \int_0^\infty tG^c(t)\lambda e^{-\lambda t}\, dt + \int_0^\infty tg(t)e^{-\lambda t}\, dt \ ,$$

and

$$E[T_{0,2}] = \frac{\int_0^\infty tG^c(t)\lambda e^{-\lambda t}\, dt + \int_0^\infty tg(t)e^{-\lambda t}\, dt}{1 - \int_0^\infty g(t)e^{-\lambda t}\, dt} \ .$$

We can further express the integrals as Laplace transforms of the pdf g (the denominator) and other distributions associated with the cdf G (the two integrals in the numerator), all evaluated at the argument λ.

5. **Independent Markov Chains** (12 points)

Let $\{X_k(t) : t \geq 0\}$ for $1 \leq k \leq 5$ be mutually independent irreducible continuous-time Markov chains (CTMC's) with transition-rate matrices

$$Q_{i,j}^{(k)} \equiv \lim_{h \downarrow 0} P(X_k(t+h) = j | X_k(t) = i)$$

and limiting steady-state probability vectors $\alpha^{(k)}$, i.e.,

$$\alpha_j^{(k)} \equiv \lim_{t \to \infty} P(X_k(t) = j | X_k(0) = i) \ .$$

(a) Consider the vector-valued stochastic process $\{(X_1(t), X_2(t), \cdots, X_5(t)) : t \geq 0\}$. Is it a CTMC? If so, what is its transition-rate matrix?

The whole problem is a generalization of Problem 5.23 on page 290 of Ross. It suffices to consider only the product of two processes, because higher numbers can be treated by induction. That is, the product of five is equal to the product of the fifth one with the product of the first four.

We will henceforth consider the product of only two processes. Yes, it is a CTMC; it satisfies the definition of a Markov process. It is important to note that transitions can only occur in one coordinate at a time. The transition-rate matrix Q can thus be written as

$$Q_{(i,j),(i',j)} = Q_{i,i'}^{(1)} \quad \text{and} \quad Q_{(i,j),(i,j')} = Q_{j,j'}^{(2)}$$

with all other rates being zero.

(b) Does the vector-valued stochastic process $\{(X_1(t), X_2(t), \cdots, X_5(t)) : t \ge 0\}$ have a limiting steady-state distribution. If so, what is it?

The limiting probability is the product of the component limiting probabilities:

$$\alpha_{(i,j)} = \alpha_i^{(1)} \alpha_j^{(2)} \ .$$

That is immediate, because

$$P(X_1(t) = i, X_2(t) = j) = P(X_1(t) = i) P(X_2(t) = j) \quad \text{for all} \quad t \ .$$

It can also be shown that the Q and α defined above satisfy $\alpha Q = 0$.

(c) Suppose that the five individual CTMC's $\{X_k(t) : t \ge 0\}$ for $1 \le k \le 5$ are all time reversible. Prove or disprove: The vector-valued stochastic process $\{(X_1(t), X_2(t), \cdots, X_5(t)) : t \ge 0\}$ is then time reversible.

The result is true. Again it suffices to consider the two-dimensional case. It suffices to show that

$$\alpha_{(i,j)} Q_{(i,j),(i',j)} = \alpha_{(i',j)} Q_{(i',j),(i,j)}$$

and

$$\alpha_{(i,j)} Q_{(i,j),(i,j')} = \alpha_{(i,j')} Q_{(i,j'),(i,j)}$$

for all (i,j), (i,j') and (i',j). That covers all possible transitions, because there can be a transition in only one coordinate at a time. Substituting α and Q into these equations, we see that indeed the reversibility condition holds.

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6. **Stochastic Order Relations** (15 points)

This problem concerns the standard, commonly accepted notion of stochastic order, which we denote by $\le_{st}$. (Ross puts the *st* under the $\le$.)

(a) What is meant by $X \le_{st} Y$ for real-valued random variables X and Y?

The standard definition is

$$P(X > a) \le P(Y > a) \quad \text{for all} \quad a \ ;$$

see (9.1.1) on page 404. An alternative definition is

$$E[f(X)] \le E[f(Y)] \quad \text{for all} \quad \text{nondecreasing real-valued functions } f \ ;$$

see Proposition 9.1.2 on page 405 of Ross.

(b) Consider a real-valued stochastic process $\{X(t) : t \geq 0\}$. What does it mean to say that $X(t)$ is stochastically increasing in t?

This problem is an elaboration of Section 9.2.1 on pages 416-418 of Ross. To say that $X(t)$ is stochastically increasing in t means that

$$X(t_1) \leq_{st} X(t_2) \quad \text{for all} \quad 0 \leq t_1 < t_2 \ .$$

(c) Again consider a real-valued stochastic process $\{X(t) : t \geq 0\}$, but now suppose that it is a CTMC. What does it mean to say that the entire stochastic process $\{X(t) : t \geq 0\}$ is stochastically increasing in the initial state $X(0)$?

This is the content of Proposition 9.2.3 on page 417. The idea here is that we comparing entire stochastic processes instead of just random variables. The meaning is

$$\{X(t) : t \geq 0 | X(0 = i\} \leq_{st} \{X(t) : t \geq 0 | X(0 = j\} \quad \text{for all} \quad 0 \leq i < j \ ,$$

i.e.,

$$E[f(\{X(t) : t \geq 0\}) | X(0) = i] \leq E[f(\{X(t) : t \geq 0\}) | X(0) = j]$$

for all $0 \leq i < j$ and for all nondecreasing real-valued functions f defined on the space of sample paths of the CTMC for which the expectations are well defined. Or, alternatively (which can be shown to be equivalent), as on p. 417 of Ross,

$$E[f(X(t_1), X(t_2), \cdots, X(t_k)) | X(0) = i] \leq E[f(X(t_1), X(t_2), \cdots, X(t_k)) | X(0) = j]$$

for all $0 \leq i < j$, all k time points $0 < t_1 < t_2 < \cdots < t_k$, all nondecreasing real-valued functions f on $\mathbb{R}^k$ for which the expectations are well defined, and for all $k \geq 1$.

(d) Prove or disprove: If a real-valued stochastic process $\{X(t) : t \geq 0\}$ is a birth-and-death process, then it is stochastically increasing in the initial state $X(0)$.

It is true. This is Proposition 9.2.3 on page 417. The proof appears there. The idea is to do a simple almost-sure construction, i.e., a coupling, in which the smaller process always lies below the larger process.

(e) Prove or disprove: If a a real-valued stochastic process $\{X(t) : t \geq 0\}$ is a birth-and-death process with $X(0) = 0$, then $X(t)$ is stochastically increasing in t.

It is true. This is Proposition 9.2.4 on page 417. The proof appears on page 418. It uses part (d).