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Models to Study the Pace of Play in Golf

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Successive groups of golfers playing on an 18-hole golf course can be represented as a network of 18 queues in series, but the model needs to account for the fact that, on most holes, more than one group can be playing at the same time, but with precedence constraints. We show how to approximate the model of group play on each hole by a conventional $G/GI/1$ single-server queue, without precedence constraints. To approximate the distribution of the sojourn time for each group on a full 18-hole golf course, we consider an idealized model consisting of 18 i.i.d. holes in series. We combine these approximations to obtain a series of 18 i.i.d. conventional single-server queues. We then apply heavy-traffic approximations to develop relatively simple analytical formulas to show how the mean and variance of the sojourn time depends on key parameters characterizing group play on each stage of a hole. Simulation experiments confirm that the approximations are effective. Thus, we provide useful tools for the analysis, design and management of golf courses. The techniques also should be useful more broadly, because many service systems combine the four complicating features of the system studied here: (i) network structure, (ii) heavy-traffic conditions, (iii) transient performance and (iv) precedence constraints.

Key words: pace of play on golf courses; queues with precedence constraints; transient behavior of queues; heavy traffic; queueing networks; queues in series

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1. Introduction

In this paper we develop new stochastic queueing models, and approximate performance formulas for those models, that can be used to help design and manage golf courses in order to control the pace of play. As emphasized by Riccio (2012, 2013), there is a need to control the pace of play, because the time to play a full round of golf has often become excessive. Kimes and Schruben (2002), Tiger and Salzer (2004) and Riccio (2012, 2013) already developed queueing models of groups playing on a golf course and applied computer simulation to study the pace of play. These models can be used to balance the competing objectives of having as many groups as possible play each day and keeping the expected times for those groups to play a full round as short as possible. They also can be used to examine the consequence of different tee-time schedules and the exceptional delays caused by lost balls and unusually slow groups. However, these models have not previously led to analytical performance formulas.

An important feature of golf that needs to be captured by these models is the fact that different groups of golfers interact on many of the holes. Typically, the groups move forward in a first-come first-served (FCFS) order, determined by their scheduled tee times on the first hole. However, typically, two groups can be playing on a par-4 hole at the same time, while three groups can be playing on a par-5 hole at the same time. This simultaneous play and the conventions for managing it introduces precedence constraints, as studied in queueing theory based on the max-plus algebra in Baccelli et al. (1992, 1989) and Heidergott et al. (2006). A conventional par-3 hole is more elementary because only one group can play on it at the same time, but there also is the modified par-3 hole "with wave-up," which allows two groups to play at the same time there too; see Riccio (2012, 2013).

These complicated precedence constraints tend to rule out conventional queueing theory and make simulation the only viable analysis tool. However, Whitt (2013) recently developed stochastic queueing models of group play on each of the holes (par-4, par-5, par-3, and par-3 with wave-up), with appropriate precedence constraints, and showed how to analytically determine the maximum possible throughput as a function of the times required for the groups to complete each stage of play on each hole. The maximum throughput was identified as the departure rate on a "fully-loaded" hole, i.e., when there always is a group ready to start playing whenever an opportunity arises.

An important contribution of that work is exposing how the random times required for groups to play the stages on each hole translate to the distribution of the time between successive groups clearing the green (completing play) on that hole when the hole is fully loaded. For example, on a par-4 hole three stages of play can be identified, which require random times S_i, $1 \le i \le 3$. It was shown for the model introduced there that the times between successive groups clearing the green on a fully-loaded par-4 hole should be distributed as

$$Y \equiv (S_1 \vee S_3) + S_2, \tag{1}$$

where $a \vee b \equiv \max\{a,b\}$; see §3 and §4. Formula (1) is useful because it shows how changes in the random stage playing times S_i will affect Y, which itself characterizes the possible pace of play on the hole.

In this paper, we go further by proposing a way to approximate the initial queueing model for group play on a par-4 hole, where two groups can be playing simultaneously, by a conventional $G/GI/1$ single-server queue with unlimited capacity and the FCFS service discipline, having i.i.d. service times, *without* any precedence constraints. The main idea is to let the approximating i.i.d. service times in the conventional queueing model be distributed as Y in (1) above. We show that this approach is effective to approximate the group waiting times before starting to play on each hole when the queue tends to be heavily loaded, which is the common case for golf courses. However, further adjustments are needed to approximate departure times and sojourn times; e.g., see §5.2.

We also develop parametric approximations for the distributions of the stage playing times S_i, which lead to parametric approximations for the distribution of Y in (1). The most promising of these is a triangular distribution to capture the usual relatively low variability, as in Tiger and Salzer (2004), plus a modification to allow for occasional lost balls; see §6.4 and §6.5. This produces tractable models of the triple (S_1, S_2, S_3) and Y depending on a 5-tuple of parameter values (m, r, a, p, L), where each parameter captures a separate property of the model. In particular, the three random variables S_i are assumed to be independent and are given symmetric triangular distributions on the intervals $[m_i - a, m_i + a]$, where $m_1 = m_3 = m_2/r = m$. Thus S_1 is distributed the same as S_3, both having mean m, while $E[S_2] = rm$, so that the ratio $E[S_2]/E[S_1] = r$ can be controlled separately. The variability of the three random variables S_i is specified by the single parameter a. We then assume that lost balls occur only on the first stage (including the tee shots) of each hole, with that happening on any hole with probability p and leading to a fixed large time L for stage 1 (corresponding to a maximum allowed delay). Thus the parameter pair (p, L) captures rare longer delays.

We also create an associated, idealized, more-highly-structured model of group play on the *entire golf course* that enable us to apply more of established queueing theory. In particular, we think that many insights can be gained by considering the simplified model in which all holes are identical. That is done partly because golf courses are often *designed* to require approximately equal time for groups to play each hole. When that is not nearly the case, then experience indicates that it should suffice to focus attention only on the *bottleneck* holes. The idealized model is intended for the case in which the course is approximately balanced. Since the par-4 hole is average in length and most common, we focus on analyzing a par-4 hole and a series of independent and identically distributed (i.i.d.) par-4 holes in series. If the course is not balanced, then the approximation may still be useful if the par-4 holes are the bottleneck holes, as evidently occurs when wave-up is used on the par-3 holes. Simulation can be used to verify that the insights gained from analyzing this idealized repetitive identical par-4 course apply to any other specific configuration of holes of the four types, using the models in Whitt (2013). Simulation can also be used to determine if even more detail provides valuable insight.

For the idealized model of multiple groups playing on an 18-hole golf course containing identical par-4 holes, we develop analytical approximation formulas. The approximation formula for the expected time required for group n to play the idealized 18-hole golf course, assuming constant intervals between scheduled tee times on the first hole, developed in §8 is

$$E[V_{18,n}] \approx E[Y]\left(n + 17 - \frac{n-1}{\rho} + 7.2\sqrt{nc_Y^2}\right) + 18E[S_3], \qquad (2)$$

where ρ is the traffic intensity (assumed to satisfy $\rho \geq 1$), S_3 is the random stage-3 playing time, while $E[Y]$ and c_Y^2 are the mean and squared coefficient of variation (scv, variance divided by the square of the mean) of the random variable Y in (1).

To develop the approximation in (2), we draw on heavy-traffic limits and approximations for the transient performance of a heavily loaded series of 18 $G/GI/1$ queues in series from Iglehart and Whitt (1970a,b) and Harrison and Reiman (1981). We especially draw on a previous simulation study by Greenberg et al. (1993) related to the heavy-traffic limits in Glynn and Whitt (1991).

There the focus was on analyzing the performance of a very large network of identical single-server queues in series. An efficient simulation algorithm was developed to simulate the performance of very large numbers of customers flowing through very large numbers of queues (e.g., millions) using techniques of distributed-event parallel simulation. Here we exploit results of those simulations to derive approximations for the mean and standard deviation of the heavy-traffic limit, which is a complex functional of 18-dimensional Brownian motion, exhibited in Theorem 3 of Glynn and Whitt (1991).

While we think that the issues addressed and the techniques developed in this paper can be useful to design and manage golf courses, we think that the value can be far greater, because the golf course illustrates four complicating features that are common in many service systems: (i) network structure, (ii) heavy-traffic conditions, (iii) transient performance and (iv) precedence constraints. There is a quite substantial literature on queueing systems with one or two of these features, but many service systems actually have all four. For example, that is substantiated by careful analysis of healthcare systems, as in Armony et al. (2011) and references therein.

Here is how the rest of this paper is organized: First, in §2 we illustrate how the approximation in (2) can be used to help design and manage golf courses. Then, in §3 we review the model of successive groups playing a par-4 hole from Whitt (2013) and in §4 we review results from Whitt (2013) for that model under the condition that the hole is fully loaded. The fully-loaded model was used to determine the maximum possible throughput. In §5 we develop the new approximating $G/GI/1$ model. In §6 we consider alternative models of the stage playing times that enable us to calculate the mean $E[Y]$ and scv c_Y^2 as needed in approximation (2) and other approximations. In particular, we consider exponential, uniform and triangular distributions for the stage playing times. (Triangular distributions were considered in Tiger and Salzer (2004).) We also analyze the consequence of rare long delays, as caused by lost balls. In §7 we report results of simulation experiments showing that the approximating $G/GI/1$ model can provide a useful approximation for the transient performance. In §8 we develop and evaluate the approximations for the sojourn time on the full golf course, consisting of 18 identical independent par-4 holes in series. Finally, in

§9 we draw conclusions. In the e-companion we show that the model approximation is also effective for steady-state performance, provided that the traffic intensity is not too low.

2. Application to Manage the Pace of Play

Before we review the model from Whitt (2013) and develop the new standard queueing model approximation, we start by showing how the approximate performance formula in (2) above can be applied in the design of a golf course. In particular, we show how to apply formula (2) to determine the number of groups that should be allowed to play each day, and thus the (assumed constant) interval between tee times Δ, as a function of the key model parameters $(E[Y], c_Y^2, E[S_3])$ and specified performance constraints. Thus, if there is a target number, ν, of groups to play each day, we can see how the other parameters should be set; i.e., which can guide the design and management of the golf course.

To do so, we formulate an optimization problem, aiming to maximize the number n of groups the play each day for specified model parameters $(E[Y], c_Y^2, E[S_3])$ subject to constraints. Let $V(\rho, n) \equiv E[V_{18,n}(\rho)]$ be the expected sojourn time on the course for group n (from tee time to clearing the green) as a function of the traffic intensity $\rho \equiv E[Y]/\Delta$, where Δ is the fixed interval between tee times. We approximate $V(\rho, n)$ by (2) above, assuming that $\rho \geq 1$. Let $G(\rho, n) \equiv E[G_{18,n}(\rho)]$ be the time for group n to clear the green on the last (18^{th}) hole, which is just $V(\rho, n)$ plus the tee time for group n, which is $(n-1)\Delta = (n-1)E[Y]/\rho$. It is natural to consider the following optimization problem:

$$\begin{aligned} &\text{maximize} \quad n \\ &\text{such that} \quad V(\rho, n) \leq \gamma \quad \text{and} \quad G(\rho, n) \leq \tau \quad \text{for} \quad \rho \geq 1. \end{aligned} \tag{3}$$

For example, if we were to aim for 4-hour rounds over a 14-hour day, then we would have $\gamma = 240$ minutes and $\tau = 840$ minutes. The tee times would then be restricted to the interval $[0, \tau - \gamma] = [0, 600]$ minutes.

From (2), we have the following functions of the model parameters and n:

$$V(\rho, n) = V(1, n) + (n-1)\left(1 - \frac{1}{\rho}\right) \quad \text{and}$$

$$G(\rho,n) = V(\rho,n) + \frac{(n-1)E[Y]}{\rho} = V(1,n) + (n-1)E[Y], \tag{4}$$

where

$$V(1,n) = A + B\sqrt{n}, \quad \text{with} \quad A \equiv 18([E[Y] + E[S_3]) \quad \text{and} \quad B = 7.2E[Y]\sqrt{c_Y^2}. \tag{5}$$

Since feasible numbers of groups must be integer, we round down to the nearest integer; let $\lfloor x \rfloor$ be the floor function, the greatest integer less than or equal to x.

THEOREM 1. (*optimal solution*) *The functions $V(\rho,n)$ in* (4) *is increasing in n and ρ, while the function $G(\rho,n)$ is increasing in n and independent of ρ, provided that $n \geq 0$ and $\rho \geq 1$. Hence, if there is an optimal solution, then one of the two constraints must be satisfied as an equality. If the first constraint on V is binding, then the optimal decision variables are $\rho_\gamma^* = 1$ and*

$$n_\gamma^* = \lfloor [(\gamma - A)/B]^2 \rfloor \tag{6}$$

for A and B in (5)*. If the second constraint on G is binding, then ρ_τ is unconstrained* (*but should be $\rho_\tau = 1$ to minimize $V(\rho, n_\gamma^*)$*) *and the optimal n is*

$$n_\tau^* = \lfloor (\sqrt{b^2 + 4ac} - b]/2a)^2 \rfloor \tag{7}$$

where $a = E[Y]$, $b = B = 7.2E[Y]\sqrt{c_Y^2}$ and $c = \tau - A + E[Y] = \tau - 17E[Y] - 18E[S_3]$.

Proof. First, suppose that the first constraint involving V is binding. Since $V(\rho,n)$ is increasing in both ρ and n, in order to achieve the largest value of n, it suffices to restrict attention to $\rho = \rho_\gamma^* = 1$. We then find n_γ^* by solving the equation $V(1,n) = \gamma$ using (5), which yields (6). Next, suppose that the second constraint is binding. First observe that ρ does not appear, so that it suffices to solve the equation $G(1,n) = \tau$, which yields a quadratic equation in $x \equiv \sqrt{n}$, whose solution is given in (7). ■

Theorem 1 implies that it suffices to focus on $\rho = 1$ in the optimization problem. This should be consistent with intuition, because it is impossible to achieve throughput faster than the bottleneck rate achieved at $\rho = 1$. Since we achieve $\rho = 1$ by setting $\Delta = E[Y]$, we see the importance of determining $E[Y]$, which is done in Whitt (2013).

EXAMPLE 1. To illustrate Theorem 1, suppose that $\tau = 840$, $\gamma = 240$, $E[Y] = 6$, $E[S_3] = 4$ and $c_Y^2 = 0.025$. Then $A = (18)(6+4) = 180$, $B = 7.47$ so that $n_\gamma^* = [(240-180)/7.47]^2 = \lfloor 64.5 \rfloor = 64$, while $a = 6$, $b = B = 7.47$ and $c = 666$, so that $n_\tau^* = \lfloor 97.9 \rfloor = 97$, Hence, we see that the first constraint on V is binding. The maximum value of n satisfying both constraints is $n_\gamma^* = 64$. Since the design is inefficient, we see that management has a strong incentive to increase n above $n_\gamma^* = 64$ in order to gain more revenue, but it can only do so by causing the expected times for playing a full round to exceed the target. Thus, this analysis evidently explains what is commonly occurring on golf courses today.

We say that a golf course design is *efficient* if the two constraints in (3) are *both* binding at the optimal solution. An efficient design has the advantage that it should not be necessary to increase throughput at the expense of golfer experience (excessive times to play a round). At the same time, it should not be necessary to restrict the throughput in order to achieve a target bound on the time to play a round. Efficiency depends on the constraint limits γ and τ as well as the model parameters. The following elementary result characterizes an efficient design.

PROPOSITION 1. (*efficient design*) *An efficient design for* (γ, τ) *occurs if and only if there is an* n_{ef} *such that*

$$V(1, n_{ef}) = \gamma \quad \textit{and} \quad V(1, n_{ef}) + (n_{ef} - 1)E[Y] = \tau \tag{8}$$

That in turn is achieved by $n_{ef} = n_\gamma^*$ *in* (6) *if and only if*

$$\tau = \gamma + (n_{ef} - 1)E[Y]. \tag{9}$$

Thus, for any specified $(\gamma, E[Y])$, *there is a unique* τ *that yields efficiency.*

To illustrate, in Example 1, since $n_\tau^* = 97 > 64 = n_\gamma^*$, that design is not efficient. Finally, suppose that we want to have ν groups play the course each day of length τ with $V(\rho, n) \le \gamma$ for all $n \le \nu$, where $0 < \gamma < \tau$. Thus, we let $\nu = n_\gamma^*$ in (6), so that

$$\nu = \frac{\gamma - 18(E[Y] + E[S_3])}{7.2E[Y]\sqrt{c_Y^2}} \tag{10}$$

We can then see what parameter triples $(E[Y], c_Y^2, E[S_3])$ satisfy target (10). We can aim for an efficient design by having $\nu = n_\tau^*$ as well.

3. Stochastic Model of Groups Playing a Par-4 Hole

In this section we review the stochastic model of successive groups of golfers playing a par-4 hole developed in Whitt (2013). The group typically contains a few individual players, but the model describes the progress of the group as a whole. The model incorporates the key property that two successive groups can be playing on a par-4 hole simultaneously.

There are *five steps* in the group play on a par-4 hole, each of which must be completed in order before the group moves on to the next step. The first step is the tee shot (one for each member of the group), denoted by T; the second step is walking up to the balls on the fairway, denoted by W_1; the third step is the fairway shot (one for each member of the group), denoted by F; the fourth step is walking up to the balls on or near the green, denoted by W_2; the fifth and final step is clearing the green (by all members of the group), denoted by G, which may involve one or more approach shots and one or more putts on the green for each player in the group.

The model next aggregates the five steps into *three stages*, which are important to capture the way successive groups interact while playing the hole. In particular, the three stages are:

$$(T, W_1) \to F \to (W_2, G) \tag{11}$$

Stage 1 is $(T, W1)$, stage 2 is F and stage 3 is (W_2, G).

Assuming an empty system initially, the first group can do all the stages, one after another without constraint. However, for $n \geq 1$, group $n+1$ cannot start stage 1 until *both* group $n+1$ arrives at the tee and group n has completed stage 2, i.e., has cleared the fairway. Similarly, for $n \geq 1$, group $n+1$ cannot start on stage 2 until *both* group $n+1$ is ready to begin there and group n has completed stage 3, i.e., cleared the green. These rules allow two groups to be playing on a par-4 hole simultaneously, but under those specified constraints. We may have groups n and $n+1$ on the course simultaneously for all n. That is, group n may first be on the course at the same time as group $n-1$ (who is ahead), but then later be on the course at the same time as group $n+1$ (who is behind). The groups remain in their original order, but successive groups interact on the hole. The group in front can cause extra delay for the one behind.

Let A_n be the arrival time of the n^{th} group at the tee of this hole on the golf course. Let $S_{j,n}$ be the time required for group n to complete stage j, $1 \leq j \leq 3$. Following queueing terminology, these are also called the *stage playing times*. These stage playing times are made up of the activities of the individual group members in each component of the stage. Thus, the stage playing times themselves should result from careful modeling and data analysis, and should depend on the size of the group and its characteristics, but these are the primitives in this model.

We now specify the performance measures, showing the result of the groups playing on the hole. Let B_n be the time that group n starts playing on this hole, i.e., the instant when one of the group goes into the tee box. Let T_n be the time that group n completes stage 1, including the tee and the following walk; let F_n be the time that group n completes stage 2, its shots on the fairway; and let G_n be the time that group n completes stage 3, and clears the green. Clearly, G_n is the group-n departure or completion time, while B_n is the group-n start time.

The mathematical model from Whitt (2013) relating the primitives to the performance random variables consists of the following four-part recursion:

$$\begin{aligned} B_n &\equiv A_n \bigvee F_{n-1}, \qquad T_n \equiv B_n + S_{1,n}, \\ F_n &\equiv (T_n \bigvee G_{n-1}) + S_{2,n} \qquad \text{and} \qquad G_n \equiv F_n + S_{3,n}, \end{aligned} \tag{12}$$

where $\equiv$ denotes "equality be definition" and $a \bigvee b \equiv \max\{a, b\}$. As initial conditions, assuming that the system starts empty, we set $F_0 \equiv G_0 \equiv 0$. The two maxima capture the two precedence constraints specified above.

The model in (12) extends directly to any number of par-4 models in series. We simply let the completion times G_n from one queue be the arrival times at the next queue (ignoring the times in between, which could be treated separately). The principal time-oriented performance measures are: the *waiting time* (of the n^{th} group before starting to play on the hole), $W_n = B_n - A_n$; the *cycle time* (the total time group n is actively playing this hole, possibly including some waiting there), $C_n \equiv G_n - B_n$; and the *sojourn time* (the total time spent by group n at the hole, waiting plus playing), $U_n = G_n - A_n = W_n + C_n$. In particular, note that W_n includes only the waiting time before starting to play; it does not include any waiting time after play has started, which can occur.

4. Model of a Fully Loaded Par-4 Hole

In order to determine the *capacity* of a par-4 hole, i.e., the maximum possible throughput, Whitt Whitt (2013) focused on a *fully-loaded hole*, i.e., all groups are at the hole at time 0 ready to play, i.e., $A_n \equiv 0$ for all n. Given the recursion in (12), it actually suffices to have only the weaker condition $A_n \le F_{n-1}$ for all $n \ge 1$. Since we will also exploit the results for the fully-loaded par-4 hole, we review the main results for it from Whitt (2013), where the proofs can be found.

Under the fully-loaded condition, the recursion in (12) reduces to

$$
\begin{aligned}
&B_n \equiv F_{n-1}, \qquad T_n \equiv B_n + S_{1,n}, \\
&F_n \equiv (T_n \bigvee G_{n-1}) + S_{2,n} \qquad \text{and} \qquad G_n \equiv F_n + S_{3,n},
\end{aligned}
\tag{13}
$$

Again, as initial conditions, assuming that the system starts empty, let $F_0 \equiv G_0 \equiv 0$.

4.1. The Intervals Between Successive Green Clearing Times

The main conclusion from Whitt (2013) is that the intervals between the times that successive groups clear the green on a fully-loaded par-4 hole has relatively simple structure that facilitates further analysis. The first step in exposing that structure is the observation that the fairway completion times can be represented directly as partial sums.

THEOREM 2. (*representation for F_n as a partial sum*) *For the fully-loaded par-4 hole with recursion in* (13),

$$F_n = F_{n-1} + Y_n, \quad n \ge 2, \quad \textit{so that} \quad F_n = \sum_{k=1}^{n} Y_k, \quad n \ge 1, \tag{14}$$

where

$$Y_n \equiv (S_{1,n} \bigvee S_{3,n-1}) + S_{2,n}, \quad n \ge 2, \quad \textit{and} \quad Y_1 = S_1 + S_2. \tag{15}$$

The representation in Theorem 2 can be exploited to establish asymptotic results for the successive green-clearing (completion) times G_n of n groups playing a fully-loaded par-4 hole, under basic stochastic assumptions. The basic idea is that G_n differs very little from F_n, so that G_n has the same asymptotic behavior as F_n as $n \to \infty$. The results require assumptions on the stage playing times. The next result is a strong law of large numbers (SLLN) for the green clearing times.

THEOREM 3. (*SLLN for the green clearance times G_n for the fully-loaded model*) *Consider the fully-loaded par-4 model in which the sequence of stage playing-time random vectors $\{(S_{1,n}, S_{2,n}, S_{3,n}) : n \ge 1\}$ is i.i.d. each distributed as the positive random vector (S_1, S_2, S_3) with finite means. Then*

$$\bar{G}_n \equiv \frac{G_n}{n} \to E[Y] < \infty \quad w.p.1, \tag{16}$$

where

$$Y \equiv (S_1 \bigvee S_3) + S_2. \tag{17}$$

Additional results, including a central limit theorem (CLT), appear in Whitt (2013).

4.2. Cycle Times in a Fully-Loaded Model

By the analysis in the previous subsection, the long-run average time between successive groups completing play of a fully-loaded par-4 hole is $E[Y]$ for Y in (17), while the time each group spends on the par-4 hole is the cycle time $C_n \equiv G_n - B_n$. The cycle times in a fully-loaded par-4 hole are in steady state for all $n \ge 2$, with a mean that is greater than $E[Y]$. Moreover, the difference can be quantified.

THEOREM 4. (*the cycle times C_n for the fully-loaded model*) *In the fully-loaded par-4 model, for $n \ge 2$, the cycle times simplify to*

$$C_n \equiv G_n - B_n = Y_n + S_{3,n}, \tag{18}$$

so that, if the stage playing times come from three independent sequences of i.i.d. random variables $\{S_{i,n} : n \ge 1\}$, $i = 1, 2, 3$, then the distribution of C_n is independent of n for all $n \ge 2$, with $E[C_n] = E[Y] + E[S_3]$ for all $n \ge 2$ and, trivially, $C_n \Rightarrow C \equiv Y + S_3^*$, where S_3^* is a random variable independent of Y and itself distributed as S_3.

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4.3. The Moments of Y

Since the random variable Y in (17) plays a critical role, it is important to have expressions for its moments. Let G and G_i be the cdf's of Y and S_i, respectively, e.g., $G_i(x) \equiv P(S_i \le x)$ and let $\bar{G}_i(x) \equiv 1 - G_i(x)$ be the complementary cdf (ccdf).

THEOREM 5. (*moments of* Y) *Consider the fully-loaded par-4 model.* (*a*) *If* S_1 *and* S_3 *are independent, then*

$$E[Y] = \int_0^\infty G^*(x)\,dx, \quad \textit{where} \quad G^*(x) \equiv \bar{G}_1(x) + \bar{G}_2(x) + \bar{G}_3(x) - \bar{G}_1(x)\bar{G}_3(x). \tag{19}$$

(*b*) *If, in addition, the random variables* S_1, S_2 *and* S_3 *are mutually independent and the cdf's* G_1 *and* G_3 *have densities* g_1 *and* g_3, *then*

$$Var(Y) = Var(S_2) + Var(S_1 \bigvee S_3), \tag{20}$$

where,

$$E[(S_1 \bigvee S_3)^2] = \int_0^\infty E[S_1^2 | S_1 > x] P(S_1 > x) g_3(x)\,dx + \int_0^\infty E[S_3^2 | S_3 > x] P(S_3 > x) g_1(x)\,dx \tag{21}$$

and

$$E[(S_1 \bigvee S_3)] = \int_0^\infty [\bar{G}_1(x) + \bar{G}_3(x) - \bar{G}_1(x)\bar{G}_3(x)]\,dx. \tag{22}$$

We develop specific models of the stage playing times in §6 below, which we apply to obtain even more tractable expressions for the moments of Y. We now introduce our model approximation.

5. The Approximating Conventional $G/GI/1$ Single-Server Queue

We now propose to approximate the given par-4 model specified in §3 by a conventional $G/GI/1$ single-server model with unlimited waiting space, the FCFS discipline, the given arrival process of groups to the hole and i.i.d. (aggregate) service times, with an independent exceptional first service time. *In this approximating model, only one group is being served at a time, ignoring all other groups.* For that purpose, we assume that the stage playing times come from three independent sequences of i.i.d random variables $\{S_{i,n} : n \geq 1\}$ for $1 \leq i \leq 3$, where $S_{i,n}$ is distributed as S_i with cdf G_i.

The underlying idea is to let the new aggregate service times be approximately equal to the successive intervals between groups clearing the green, $\Delta_n \equiv G_n - G_{n-1}$, $n \geq 1$, in the fully loaded model analyzed in §4. We anticipate that this approximation strategy should be effective when the exact model is heavily loaded, which is the principal case of interest case for golf courses.

By Theorem 3.4 of Whitt (2013), these random variables Δ_n for $n \geq 2$ come from a somewhat complicated 2-dependent sequence. However, from Theorem 3, we see that we can choose a closely related sequence of i.i.d service times that produces the same SLLN behavior as Δ_n by letting these approximate i.i.d. service times be distributed as $Y \equiv (S_1 \bigvee S_3) + S_2$ for $n \geq 2$, as in (17). (By Theorem 3.3 of Whitt (2013), they also have the same CLT behavior.) However, we let the first service time be $Y_1^* \equiv S_1 + S_2 + S_3$ as experienced by the first group to play the hole. Hence that is our main approximation. (Thus we use the same notation Y_n.)

5.1. The Classical Recursion

Thus, the approximating $G/GI/1$ model is specified by the sequence of arrival times $\{A_n : n \geq 1\}$ and the sequence of mutually independent service times $\{Y_n : n \geq 1\}$, where $Y_n \stackrel{\mathrm{d}}{=} Y$, $n \geq 2$, for Y in (17) and $Y_1 \stackrel{\mathrm{d}}{=} Y_1^* \equiv S_1 + S_2 + S_3$. Then we can apply the classical single-server queue recursion:

$$B_n \equiv A_n \bigvee D_{n-1} \qquad \text{and} \qquad D_n \equiv B_n + Y_n, \tag{23}$$

where $D_0 \equiv 0$. The variables D_n are the departure times in the conventional single-server model. However, the actual approximate departure times we use are defined differently in §5.2. (We do *not* use D_n as an approximation for G_n in §3.)

The exceptional behavior of F_n for $n = 1$ in Theorem 2 explains the exceptional first service time. We remark that there is a literature on queues with exceptional first service, which can be traced from citations to the early paper Welch (1964), but that literature focuses on queues in which the first service time of *every* busy period is exceptional. In contrast, here only the very first group has a different service time. This exceptional first service time will affect the transient behavior, but not the limiting behavior, the steady-state limits for $\rho < 1$ and, after scaling, the heavy-traffic limits for $\rho \geq 1$.

We regard the main approximations following directly from (23) as being for the group waiting times at the tee (before teeing off), defined by $W_n \equiv B_n - A_n$. We next develop the remaining approximations.

5.2. Approximations for the Group Sojourn and Departure Times

The appropriate approximation for the group sojourn times requires care, because we should not just add the waiting time W_n to the approximating aggregate service time Y_n, as we would in the conventional $G/GI/1$ single-server model in which customers are served one at a time. Instead, we let the the sojourn time approximation for the model in §3 be independent sum of the waiting time and the *cycle time* in §4.2, i.e.,

$$U_n \approx W_n + C_n, \tag{24}$$

where C_n is independent of W_n. That is appropriate because the cycle time is the time spent by the group on the hole (in the fully loaded model). By Theorem 4, the cycle time is the independent sum of Y and S_3, so that both the means and variances add:

$$E[U_n] \approx E[W_n] + E[Y] + E[S_3] \quad \text{and} \quad Var(U_n) \approx Var(W_n) + Var(Y) + Var(S_3), \quad n \geq 2. \tag{25}$$

where the mean and variance of the waiting time W_n are obtained from the $G/GI/1$ model above. For the exceptional first, group,

$$E[U_1] \approx E[S_1] + E[S_2] + E[S_3] \quad \text{and} \quad Var(U_1) \approx Var(S_1) + Var(S_2) + Var(S_3). \tag{26}$$

The departure times from group n from the hole (and thus the arrival times at a subsequent hole) are necessarily $G_n = A_n + U_n$. Thus we have the approximations

$$E[G_n] \approx E[A_n] + E[U_n] \quad \text{and} \quad Var(G_n) \approx Var(A_n) + Var(U_n), \tag{27}$$

using (25) and (26) above.

We will primarily consider the case of a a fixed deterministic schedule of start times (tee times) with constant intervals in between on the first hole. For deterministic arrivals, the arrival time mean and variance used in (27) are $E[A_n] = A_n = (n-1)EY/\rho$ and $Var(A_n) = 0$ above.

For a following queue, the arrival times A_n are the departure times G_n, using (27), and not using D_n in the recursion (23). For any queue after the first, the arrival time mean and variance, will be obtained from (27) for the previous queue. We observe that successive waiting times are typically not nearly independent. Hence, successive sojourn times U_n and interdeparture times $G_n - G_{n-1}$ are typically not independent either.

Given the derived green clearing times G_n, we have associated approximations for the number of of groups at the hole, either waiting or playing, as seen be group n upon arrival, N_n^a, and at an arbitrary time, $N(t)$. As in formulas (2.3) and (2.4) of Whitt (2013), these are defined as

$$N_n^a \equiv n - 1 - \max\{k \geq 0 : G_k \leq A_n\}, \quad n \geq 1. \tag{28}$$

and

$$N(t) \equiv N_n^a + 1, \quad A_{n-1} \leq t < A_n. \tag{29}$$

6. Alternative Stage Playing Time Models

In this section we develop alternative models for the stage playing times S_i that lead to alternate overall approximating aggregate service-time random variables Y to use in the approximating $G/GI/1$ model. We first consider exponential stage stage playing times, because of the appealing tractability of the exponential distribution. However, the exponential distribution evidently is too highly variable for the golf course application. To consider more appealing less-variable alternatives, we also consider uniform and triangular stage playing times; triangular distributions were used in Tiger and Salzer (2004). Finally, we develop a model to account for the extra variability caused by an exceptional long delay, as occurs with a lost ball. Rare long stage service times can have a dramatic impact, as shown by Riccio (2012, 2013).

6.1. Independent Exponential Stage Playing Times

We can obtain explicit expressions for the moments of Y when S_1, S_2 and S_3 are mutually independent exponential random variables. The following is an easy consequence of Theorem 5.

COROLLARY 1. (*moments of Y for independent exponential stage playing times*) *For the fully-loaded par-4 hole in which S_i are mutually independent exponential random variables with means μ_i^{-1}, $i=1,2,3$, then*

$$E[Y]=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}-\frac{1}{\mu_1+\mu_3} \tag{30}$$

and

$$Var(Y)=\frac{1}{(\mu_1+\mu_2)^2}+Var(Z), \tag{31}$$

where

$$E[Z^2]=P(S_1<S_3)E[S_3^2]+P(S_3<S_1)E[S_1^2]=\left(\frac{\mu_1}{\mu_1+\mu_3}\right)\frac{2}{\mu_3^2}+\left(\frac{\mu_3}{\mu_1+\mu_3}\right)\frac{2}{\mu_1^2} \tag{32}$$

and

$$E[Z]=P(S_1<S_3)E[S_3]+P(S_3<S_1)E[S_1]=\left(\frac{\mu_1}{\mu_1+\mu_3}\right)\frac{1}{\mu_1}+\left(\frac{\mu_3}{\mu_1+\mu_3}\right)\frac{1}{\mu_3}. \tag{33}$$

If, in addition, $\mu_1=\mu_3$, then $Z\stackrel{d}{=}S_1\stackrel{d}{=}S_3$, so that

$$Var(Y)=Var(S_2)+Var(S_1\bigvee S_3)=E[S_2]^2+\left(\frac{E[S_1]}{2}\right)^2+Var(S_1). \tag{34}$$

Proof. First, we obtain (30) directly from the relation

$$Y\stackrel{d}{=}(S_1\bigvee S_3)+S_2=S_1+S_2+S_3-S_1\bigwedge S_3, \tag{35}$$

and the property that the minimum of exponential variables is exponential with a rate equal to the sum of the rates. For (31), we use (20) and

$$(S_1\bigvee S_3)=(S_1\bigwedge S_3)+Z, \tag{36}$$

where, by the lack of memory property, these are independent with (32) and (33) holding. For the special case in which $S_1\stackrel{d}{=}S_3$, those formulas simplify. ■

6.2. Parametric Models with Special Structure

We now introduce additional structure in the stage playing time distributions in order to obtain models with only a few parameters. In particular, in the spirit of p. 94 of Riccio (2012), we assume that S_1, S_2 and S_3 are mutually independent random variables with

$$S_1 \stackrel{\mathrm{d}}{=} S_3, \tag{37}$$

so that one parameter can be the means $E[S_1] = E[S_3]$; i.e., we have

$$m = E[S_1] = E[S_3] = \frac{E[S_2]}{r}. \tag{38}$$

Hence there are only the two parameters m and r beyond the distributions of S_1 and S_2. Moreover, we let S_2 have a distribution of the same type as S_1.

Example 2. (a concrete exponential example) We now consider a concrete example of the exponential stage playing times satisfying (37) and (38) in (38). For this example, we have

$$E[Y] = \frac{(3+2r)m}{2}, \quad Var(Y) = \frac{(5+4r^2)m^2}{4} \quad \text{and} \quad c_Y^2 \equiv \frac{Var(Y)}{E[Y]^2} = \frac{5+4r^2}{(3+2r)^2}. \tag{39}$$

Following p. 94 of Riccio (2012) again, we now let $m = 6$ and $r = 1/2$. First, for $r = 1/2$, $E[Y] = 3m/2$, $var(Y) = 3m^2/2$ and $c_Y^2 = 3/8 = 0.375$. From (39), we see that, for $m = 6$, $E[Y] = 12$ and $Var(Y) = 54$. We look at $m = 6$ throughput the paper, but the value $m = 6$ gives reasonable sojourn times on the golf course only in the case of deterministic stage playing times. For realistic random stage service times, a more realistic value evidently would be $m = 4$.

6.3. Independent Uniform Stage Playing Times

As a first step toward considering more realistic models of the stage playing times, we now assume that all three random stage playing times are independent uniform random variables satisfying (37) and (38). In particular, we assume that

$$S_i \stackrel{\mathrm{d}}{=} m_i - a_i + 2a_i U, \tag{40}$$

where $U \equiv U[0,1]$ is uniformly distributed on the interval $[0,1]$ and $0 \le a_i \le 1$. That is tantamount to assuming that S_i is uniformly distributed on the interval $[m_i - a_i, m_i + a_i]$. Since $E[U] = 1/2$ and $Var(U) = 1/12$, we see that $E[S_i] = m_i$ and $Var(S_i) = a_i^2/3$.

We further simplify by assuming that the parameters a_i in (40) coincide, i.e.,

$$a_1 = a_2 = a_3 = a, \tag{41}$$

so that only two parameters remain: m and a.

With those simplifying assumptions, we have

$$S_1 \vee S_3 = m - a + 2a(U_1 \vee U_2)), \tag{42}$$

where U_1 and U_2 are two i.i.d. uniform random variables on $[0,1]$. Since

$$F_{U_1 \vee U_2}(t) \equiv P(U_1 \vee U_2 \le t) = P(U_1 \le t)^2 = t^2, \quad 0 \le t \le 1,$$

we have $E[U_1 \vee U_2] = 2/3$, $E[(U_1 \vee U_2)^2] = 1/2$ and $Var(U_1 \vee U_2) = 1/18$. Hence,

$$E[S_1 \vee S_3] = m + \frac{a}{3} \quad \text{and} \quad Var(S_1 \vee S_3) = \frac{2a^2}{9}, \tag{43}$$

so that

$$E[Y] = (1+r)m + \frac{a}{3} \quad \text{and} \quad Var(Y) = \frac{5a^2}{9}. \tag{44}$$

Example 3. (the uniform analog of Example 2)

Paralleling Example 2, suppose that $m = 6$ and $r = 1/2$. If $a = 3$ (which is as large as possible), then $E[Y] = 10.0$, $Var(Y) = 5.0$ and $c_Y^2 = 0.05$; if $a = 1$, then $E[Y] = 9.333$, $Var(Y) = 0.5555$ and $c_Y^2 = 006377$. Notice that the mean is substantially smaller than in the corresponding exponential example, but c_Y^2 is much smaller. From (2), we see that it is important to look at $\sqrt{c_Y^2}$. In the exponential case, $\sqrt{c_Y^2} = 0.612$, whereas for $a = 1$, $\sqrt{c_Y^2} = 0.07985$, which is 7.7 times smaller.

6.4. The Case of Independent Triangular Stage Playing Times

Aiming for an even more realistic model, as in Tiger and Salzer (2004), we now assume that S_i has a symmetric triangular distribution. In particular, paralleling (40), let

$$S_i \stackrel{\mathrm{d}}{=} m_i - a_i + 2a_i T, \tag{45}$$

where $T \equiv T[0,1]$ is a (symmetric) triangular distribution on the interval $[0,1]$ with density

$$f_T(x) = 4t, \quad 0 \le t \le 0.5, \quad \text{and} \quad 4 - 4t, \quad 0.5 \le t \le 1. \tag{46}$$

so that $E[T] = 1/2$ and $Var(T) = 1/24$.

That is tantamount to assuming that S_i has a triangular distribution on the interval $[m_i - a_i, m_i + a_i]$, so that $E[S_i] = m_i$ and $Var(S_i) = a_i^2/6$. We further simplify by assuming that (41) and (38) hold, so that there are only the three parameters m, r and a. With that simplifying assumption, we have

$$S_1 \vee S_3 = m - a + 2a(T_1 \vee T_2), \tag{47}$$

where T_1 and T_2 are two i.i.d. triangular random variables on $[0,1]$.

Since

$$P(T \le t) = 2t^2, \quad 0 \le t \le 1/2, \quad \text{and} \quad P(T \le t) = 1 - 2(1-t)^2, \quad 1/2 \le t \le 1,$$

we have

$$\begin{aligned} P(T_1 \vee T_3 \le t) &= P(T \le t)^2 = 4t^4 \quad \text{and} \quad f_{T_1 \vee T_3}(t) = 16t^3, \quad 0 \le t \le 1/2, \\ P(T_1 \vee T_3 \le t) &= (1 - 2(1-t)^2)^2 \quad \text{and} \quad f_{T_1 \vee T_3}(t) = 8[-1 + 5t - 6t^2 + 2t^3], \quad 1/2 \le t \le 1, \end{aligned} \tag{48}$$

so that $E[T_1 \vee T_3] = 37/60$ and $Var(T_1 \vee T_3) = 101/3600$. Hence,

$$E[S_1 \vee S_3] = m + \frac{7a}{30} \quad \text{and} \quad Var(S_1 \vee S_3) = \frac{101a^2}{900} \tag{49}$$

and

$$E[Y] = (1 + r)m + \frac{7a}{30} \quad \text{and} \quad Var(Y) = \frac{251a^2}{900}. \tag{50}$$

EXAMPLE 4. (triangular analog of Example 2) Just as in Examples 2 and 3, suppose that $m = 6$ and $r = 1/2$. If $a = 3$ (which is as large as possible), then $E[Y] = 9.7$, $Var(Y) = 2.51$ and $c_Y^2 = 0.02667$; if $a = 1$, then $E[Y] = 9.233$, $Var(Y) = 0.2789$ and $c_Y^2 = 0.003271$. Notice that the mean and variance are both slightly smaller than in corresponding the uniform case. We will find that it is important to look at $\sqrt{c_Y^2}$. In the exponential case, $\sqrt{c_Y^2} = 0.612$, whereas for $a = 1$, $\sqrt{c_Y^2} = 0.0572$, which is 10.7 times smaller.

6.5. Modification for Occasional Lost Balls

The uniform and triangular distributions capture the lower variability we expect to have in group playing times under normal circumstances. However, there can be unexpected delays, such as are caused by a lost ball, which makes the time much longer than it would be otherwise. To avoid excessive delays, golf courses often impose an upper limit on the playing time of each group on each hole, such as 20 minutes.

To model these rare events in a relatively simple way, we consider random extra delays at an upper limit. For simplicity, we assume that a lost ball can only occur on the tee shot, so we only modify the distribution of S_1. We first let this upper limit be the constant value L minutes. We then assume that such unexpected events occur for each group on the first stage of each hole with probability p. So we introduce the two extra parameters p and L.

Thus given any of the models for S_i discussed above, this modification leads to a new distribution for S_1. Let the new random time for group play on stage 1 be $\bar{S}_1$, and let $\bar{Y}$, be the new random time between successive times to clear the green, which is still defined by (17), but with $\bar{S}_1$ replacing S_1. Now we have

$$P(\bar{S}_1 = S_1) = 1 - p \quad \text{and} \quad P(\bar{S}_1 = L) = p. \tag{51}$$

Then the first two moments of $\bar{S}_1$ are

$$E[\bar{S}_1] = (1-p)E[S_1] + pL \quad \text{and} \quad E[\bar{S}_1^2] = (1-p)E[S_1^2] + pL^2, \tag{52}$$

so that

$$Var(\bar{S}_1) = (1-p)[\sigma_{S_1}^2 + p(L - E[S_1])^2]. \tag{53}$$

Then

$$\bar{Y} = (\bar{S}_1 \vee S_3) + S_2.$$

However, for bounded stage playing times such as occur with the uniform and triangular distributions, we can go further. If, in addition to (37) and (38), we have

$$P(\bar{S}_1 > S_3) = 1, \tag{54}$$

then we have

$$\begin{aligned} \bar{Y} &= \bar{S}_1 + S_2, \quad E[\bar{Y}] = p(L + rm) + (1-p)E[Y] \quad \text{and} \\ E[\bar{Y}^2] &= p(L + rm)^2 + (1-p)E[Y^2]. \end{aligned} \tag{55}$$

Hence we can combine the lost-ball feature with one of the models in §6.3 and §6.4 to obtain a tractable model. If we add condition (41), then we obtain tractable models depending on the parameter 5-tuple (m, r, a, p, L). We can thus incorporate the rare lost ball with the usual low variability of the uniform or triangular distribution to obtain a final estimate of the pace of play. In particular, we can combine (55) and (50) to obtain the first two moments of $\bar{Y}$ for the $tri + LB$ model with parameter 5-tuple (m, r, a, p, L):

$$\begin{aligned} E[\bar{Y}] &= p(L + rm) + (1-p)\left((1+r)m + \frac{7a}{30}\right) \\ E[\bar{Y}^2] &= p(L + rm)^2 + (1-p)\left(\frac{251a^2}{900} + E[\bar{Y}]^2\right). \end{aligned} \tag{56}$$

Example 5. (Example 4 revisited with the triangular distribution and lost balls) For example, Let S_i have triangular distributions as in §6.4 with parameters m, r and a. Since $E[S_1] = m$ and $Var(S_1) = a^2/6$.

$$E[\bar{S}_1] = (1-p)m + pL \quad \text{and} \quad E[\bar{S}_1^2] = (1-p)\left(m^2 + \frac{a^2}{6}\right) + pL^2, \tag{57}$$

Suppose that we use the parameters $m = 6$ and $r = 1/2$ for the triangular distribution, as in Example 4, and let $p = 0.05$ and $L = 12$ for the lost balls. Then

$$E[\bar{S}_1] = 0.95(6) + (0.05)(12) = 6.3 \quad \text{and} \quad E[\bar{S}_1^2] = (0.95)\left(36 + \frac{9}{6}\right) + (0.05)(144) = 42.825 \tag{58}$$

so that

$$Var(\bar{S}_1) = 42.825 - (6.3)^2 = 42.825 - 39.69 = 3.135 \tag{59}$$

Since $Var(S_1) = 1.50$, the variance of S_1 increased by more than a factor of 2.

The next step is to determine the distribution of $\bar{Y}$. Notice that

$$P(\bar{Y} = 12 + S_2) = 0.05 = 1 - P(\bar{Y} = Y). \tag{60}$$

Hence,

$$\begin{aligned} E[\bar{Y}] &= (0.05)(15) + (0.95)(9.7) = 0.75 + 9.215 = 9.965 \quad \text{and} \\ E[\bar{Y}^2] &= (0.05)(226.5) + (0.95)(96.6) = 11.325 + 91.77 = 103.095 \end{aligned} \tag{61}$$

so that

$$\begin{aligned} Var(\bar{Y}) &= 103.095 - (9.965)^2 = 103.095 - 99.30 = 3.795 \quad \text{and} \\ c_{\bar{Y}}^2 &= 3.795/(9.965)^2 = 0.03822 \end{aligned} \tag{62}$$

As expected, $c_{\bar{Y}}^2$ is greater than $c_Y^2 = 0.02667$, by a factor of about 1.5, but $c_{\bar{Y}}^2$ is still 10 times smaller than for the exponential distribution. It is thus natural to regard the exponential distribution as a crude upper bound.

We show histograms of Y when the stage playing times have the triangular distribution with $(m, r, a) = (6, 0.5, 3)$ in Figure 1 and that triangular distribution modified to account for lost balls with $(p, L) = (.05, 12)$ in Figure 2. The lost balls clearly produce a heavier upper tail, but within a reasonable range, because Y remains bounded above by $(\bar{S}_1 \vee S_3) + S_2 \le 12 + (3 + 3) = 18$ (compared to $((S_1 \vee S_3) + S_2 \le ([6 + 3] \vee [6 + 3]) + (3 + 3) = 15$ for the triangular distribution).

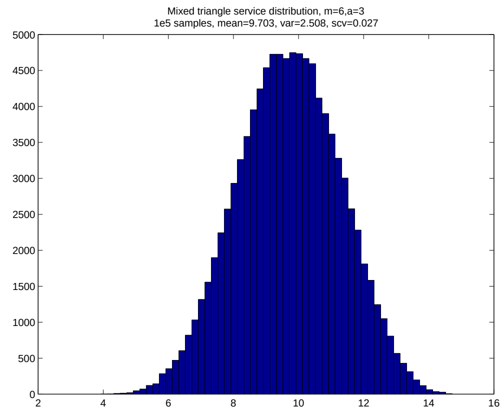

Figure 1 Histogram of the distribution of Y when the stages have a triangular distribution with $(m,a) = (6,3)$

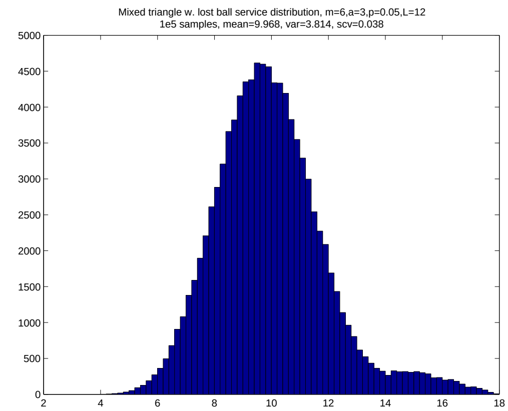

Figure 2 Histogram of the distribution of Y when the stages have a triangular distribution modified to account for lost balls with $(p,L) = (.05, 12)$

7. Simulation Comparisons for the Transient Performance

In this section we show that the approximating $G/GI/1$ model in §5 with exceptional first service in §5, with the sojourn time adjustment in §5.2, provides an effective approximation for the transient performance of the exact model in §3. We consider the case of identical par-4 holes in series. To do so, we report simulation results estimating the mean and standard deviation of the sojourn time of group n on each hole and on the first h holes for the cases $(h,n) = (10,20)$ and $(18,100)$.

7.1. Sojourn Times

We now label the variables with two subscripts, the first indexing the hole and the second indexing the group. With this notation, we now aim to approximate the sojourn time of group n on hole h, $U_{h,n} \equiv G_{h,n} - A_{h,n}$, for various holes h and the total sojourn time of group n on the first h holes,

$$V_{h,n} \equiv U_{1,n} + \cdots + U_{h,n}. \tag{63}$$

We illustrate by displaying simulation results for the cases $(h,n) = (10,20)$ and $(18,100)$.

For our first experiment, we consider the all-exponential model with independent all-exponential stage playing times with parameters $(m,r) = (6,0.5)$, having means $E[S_1] = E[S_3] = 6$, $E[S_2] = 3$, as in Example 2, and constant intervals between tee times on the first hole. The interval between tee times is used to adjust the traffic intensity. We perform the transient simulations for three values of the traffic intensity ρ, defined by $\rho \equiv \lambda E[Y]$: 0.9, 1.0, and 1.1. The simulations are based on 2000 independent replications. This consistently makes the half-width of 95% confidence intervals less than 5% of the estimated means and 10% of the estimated standard deviations.

We find that the sojourn times over several holes tend to be approximately normally distributed, so that the mean and standard deviation serve to describe the entire distribution. Figure 3 illustrates by showing the histogram of the sojourn times $V_{10,20}$ for $\rho = 0.9$ (on the left) and $\rho = 1.1$ (on the right) estimated for the exact model in §3. (The approximation produces very similar plots.)

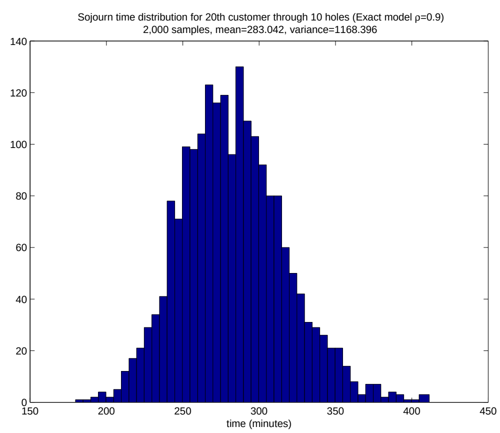

Figure 3 Histogram of the sojourn times $V_{10,20}$ in the all exponential model with for $\rho = 0.9$

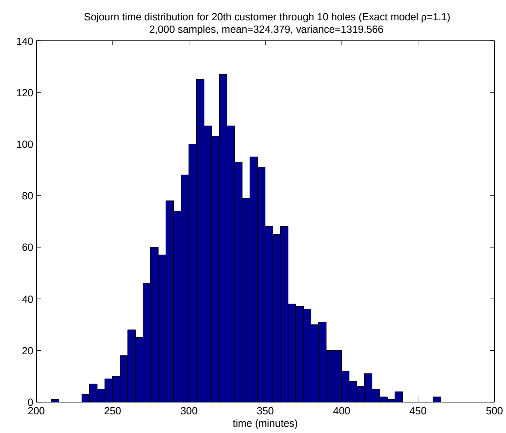

Figure 4 Histogram of the sojourn times $V_{10,20}$ in the all exponential model with for $\rho = 1.1$

We give simulation estimates of the mean and standard deviation of these sojourn times $U_{h,20}$ for the exact and approximate models in Table 1 for holes $h = 1, 2, 3, 6$ and 10 and the total sojourn time $V_{10,20}$. Overall, we see that the mean sojourn time may increase from $h = 1$ to $h = 2$ but then gradually declines thereafter; the standard deviations evidently decline only after $h = 3$. We see that the approximate model consistently overestimates the mean and standard deviation but not by too much. It overestimates the mean and standard deviation of $V_{10,20}$ for $\rho = 0.9$ by 7% and 12%, respectively.

Table 1 Simulation comparison of the transient performance predicted by the exact model in §3 and the approximate model in §5: estimates of the mean and standard deviation of the sojourn times of group 20 on several holes, $(U_{h,20})$, and over the first 10 holes, $(V_{10,20})$, for a series of i.i.d. par-4 holes with exponential stage playing times, for three traffic intensities $\rho = 0.9$, 1.0 and 1.1

traffic intensity	$\rho = 0.9$		$\rho = 1.0$		$\rho = 1.1$	
perf. measure	mean	std dev	mean	std dev	mean	std dev
hole 1, exact model	28.0	18.1	36.6	22.3	48.4	26.5
approx model	30.6	20.3	38.9	23.1	52.6	27.9
hole 2, exact model	32.7	20.0	37.8	24.1	42.2	25.5
approx model	33.5	21.1	40.1	23.7	43.2	25.2
hole 3, exact model	31.1	20.4	34.8	21.7	35.6	22.5
approx model	33.9	21.7	36.9	22.6	38.1	23.7
hole 6, exact model	28.5	18.2	29.0	18.0	29.7	22.8
approx model	29.4	19.5	30.7	18.3	31.4	19.5
hole 10, exact model	25.1	16.1	26.0	16.8	25.8	16.8
approx model	27.1	17.5	27.7	16.6	27.9	16.6
first 10 holes, exact model	283.8	34.2	305.9	35.1	326.6	36.6
approx model	303.8	38.5	328.2	38.0	346.7	40.2

We now focus on the different stage playing time distributions. In particular, as in Examples 2, 4 and 5, we now consider the exponential distribution with $(m,r) = (6, 0.5)$, the triangular distribution $(tri(m,a))$ for $(m,r,a) = (6, 0.5, 3)$, that same triangular distribution with the lost ball parameters $(p,L) = (0.05, 12)$. In all cases we let $E[S_1] = E[S_3] = 2E[S_2] = m = 6$. In Examples 2, 4 and 5 we have calculated the means and variances of Y for each of these three cases. They are, respectively,

$$(E[Y], Var(Y)) = (12.00, 54.00), \quad (9.70, 2.51) \quad (9.97, 3.80).$$

Even though the stage playing times have identical means in all three models, they have different variability, which affects both the mean and variance of Y. The mean and variance of Y in turn strongly affect the mean sojourn times $E[U_{h,n}]$ and $E[V_{h,n}]$.

Table 2 gives simulation estimates of the mean and standard deviation of these sojourn times $U_{h,100}$ for the exact and approximate models for holes $h = 1, 2, 3, 6, 10, 18$ and the total sojourn time $V_{18,100}$. (Now we consider $(h,n) = (18, 100$ instead of $(10, 20)$.) These results are again based on 2000 independent replications. For the less variable triangular stage playing time distribution, the half-width of 95% confidence intervals is consistently less than 1% of the mean estimate and

5% of the standard deviation estimate.

Again, we see that the mean sojourn time may increase from $h=1$ to $h=2$ but then gradually declines thereafter; the standard deviations evidently decline only after $h=3$. We see that the approximate model consistently overestimates the mean and standard deviation but not by too much.

Table 2 Simulation comparison of the transient performance predicted by the exact model in §3 and the approximate model in §5: estimates of the mean and standard deviation of the sojourn times of group 100 on several holes, $(U_{h,100})$, and over the full course of 18 holes, $(V_{18,100})$, for a series of i.i.d. par-4 holes with traffic intensity $\rho=1.1$ and three stage playing time distributions, as in Examples 2, 4 and 5.

distribution	tri.	$(m,a)=(6,3)$	tri. + LB	$(p,L)=(.05,12)$	expon.	$m=6$
perf. measure	mean	std dev	mean	std dev	mean	std dev
hole 1, exact model	103.6	15.8	106.4	19.0	142.7	66.1
approx model	108.4	15.9	111.1	19.3	144.8	65.9
hole 2, exact model	31.6	13.6	35.3	16.2	87.3	59.7
approx model	33.1	14.2	36.9	17.2	87.3	57.0
hole 3, exact model	26.4	10.3	29.6	12.7	65.7	46.3
approx model	27.7	10.6	30.4	13.2	68.3	47.1
hole 6, exact model	21.9	6.7	24.3	8.9	47.7	35.4
approx model	23.10	7.7	25.0	9.7	50.6	34.8
hole 10, exact model	20.4	5.8	21.8	7.0	40.7	30.1
approx model	21.7	6.8	23.0	8.16	41.3	27.9
hole 18, exact model	18.7	4.3	19.9	16.8	33.3	23.8
approx model	20.7	6.0	21.4	5.7	35.3	23.4
first 18 holes, exact	468.8	10.1	503.7	14.1	908.5	58.6
approx model	498.3	12.4	526.7	15.3	938.5	61.8

We also simulated the standard $D/GI/1$ model with service times distributed as Y, without any exceptional first service times. We found that the exceptional first service times did not significantly alter the results, but it is important to include the sojourn time adjustment in §5.2.

7.2. Negative Dependence

When we consider the sojourn time over several holes, we might think that the variance should be approximately the sum of the variances on the individual holes, but the simulation experiments show that is not nearly the case. The estimated sum of the variances of the sojourn times on the first 10 holes in Table 1 and over the first 18 holes in Table 2 is much larger than the variance of the sojourn time over all the holes. We make an explicit comparison in Table 3. Fortunately, we see that this effect is well captured by the approximation. Table 3 shows that the ratio is greatest for the least variable triangular distribution.

Table 3 A comparison of the variance of the sojourn time of n groups over h holes with the sum of the variances of the sojourn times in the six experiments in Tables 1 and 2

case	var. of sum	sum of vars.	ratio
exponential with $\rho = 0.9$	$Var(V_{10,20})$	$\sum_{i=1}^{10} Var(U_{i,20})$	
exact	1197	3418	2.85
approx.	1412	3413	2.41
exponential with $\rho = 1.0$	$Var(V_{10,20})$	$\sum_{i=1}^{10} Var(U_{i,20})$	
exact	1234	3835	3.10
approx.	1440	3936	2.73
exponential with $\rho = 1.1$	$Var(V_{10,20})$	$\sum_{i=1}^{10} Var(U_{i,20})$	
exact	1342	4353	3.24
approx.	1617	4393	2.72
triangular with $\rho = 1.1$	$Var(V_{18,100})$	$\sum_{i=1}^{18} Var(U_{i,100})$	
exact	102.7	1071	10.4
approx.	154.1	1297	8.4
tri. and LB with $\rho = 1.1$	$Var(V_{18,100})$	$\sum_{i=1}^{18} Var(U_{i,100})$	
exact	200.3	1651	8.2
approx.	236.2	1881	8.0
exponential with $\rho = 1.1$	$Var(V_{18,100})$	$\sum_{i=1}^{18} Var(U_{i,100})$	
exact	3823	23291	6.09
approx.	3434	22893	6.66

We also estimated the correlations $c(j,k)$ of the sojourn times on holes j and $j+k$ for all j and k. We find that these correlations are consistently negative with absolute values that decrease in j and k, with $c(j,1) \approx -0.14$ for the triangular distribution and $c(j,1) \approx -0.10$ for the tri and $tri + LB$ distributions, respectively, for $j \geq 4$.

8. The Time Required for a Group to Play a Full Round of Golf

In this section we develop and evaluate heavy-traffic (HT) approximations for the time required for each successive group to play a full round of golf on the idealized course containing a series of 18 i.i.d. par-4 holes.

Let $G_{h,n}$ be the time that group n clears the green on hole h (completes play on the first h holes). Let $A_{1,n}$ be the time that group n is scheduled to start (tee off) at the first hole. (The group may in fact have to wait for other groups in front of them.) Let $V_{h,n} = U_{1,n} + \cdots + U_{h,n}$ be the sojourn time of group n on the first h holes. These three performance descriptors are related by

$$V_{h,n} = G_{h,n} - A_{1,n}. \tag{64}$$

Here we assume that the tee times are evenly spaced deterministic times so that

$$A_{1,n} = \frac{n-1}{\lambda} = \frac{(n-1)E[Y]}{\rho}, \tag{65}$$

where λ is the arrival rate and $1/\lambda$ is the fixed interval between successive scheduled group tee times on the first hole, with λ chosen so that the traffic intensity $\rho = \lambda E[Y]$ is some specified value, with Y as in (17). Here we consider cases in which $\rho \geq 1$, so that groups are usually ready to tee off at the first green whenever the opportunity arises.

8.1. The Standard Series-Queue Model

We first consider the standard series-queue model, with i.i.d $\cdot/GI/1$ queues in series, each having service times distributed as Y in (17). We evaluate HT approximations from Iglehart and Whitt (1970a,b), elaborated upon in Glynn and Whitt (1991), Greenberg et al. (1993), Harrison and Reiman (1981), Reiman (1984) based on stochastic-process limits where we let $n \to \infty$ for fixed $\rho \geq 1$. We compare to the simulations of both this same model and the exact golf-course model, focusing on the sojourn time of group $n = 100$ on the first $h = 18$ holes.

A reference case is the series of deterministic $\cdot/D/1$ queues with the deterministic arrival process and deterministic service times with mean $E[Y]$, Then

$$G_{h,n}^{D,std} = (n+h-1)E[Y], \quad A_{1,n} = \frac{(n-1)E[Y]}{\rho} \quad \text{and} \quad V_{h,n}^{D,std} = G_{h,n}^{D,std} - A_{1,n}. \tag{66}$$

For example, with $n = 100$, $h = 18$, $\rho = 1.1$ and $S_1 = S_6 = 2S_2 = 6$ and $Y = 9.0$, so that

$$G_{18,100}^{D,std} = (117)9 = 1053, \quad A_{1,100} = \frac{(99)9}{1.1} = 810 \quad \text{and} \quad V_{18,100}^{D,std} = 1053 - 810 = 243. \tag{67}$$

From this example, we see that these deterministic stage service times would make the pace of play far too slow. Before we introduce the important variability, which will slow the pace of play even more, we see that the required time for 100 groups to play on the course is $1053/60 \approx 17.5$ hours, which could just be realized in mid-summer, from 5:00 am until 10:30 pm, while the sojourn time of the last group is just over $243/60 \approx 4$ hours. That sojourn time is composed of $18 \times 9 = 162$ minutes of service time and 81 minutes waiting to tee off at the first hole. Thus, we probably want to have $\rho < 1.1$. More importantly, the extra variability raises these times to untenable levels.

To treat the case of stochastic service times, we apply the heavy-traffic (HT) limit in which $n \to \infty$ for a series of standard $\cdot/GI/1$ queues, where the service times at the queues come from independent sequences of i.i.d. random variables distributed as Y in (17), with mean $E[Y]$ and scv $c_Y^2 \equiv Var(Y)/E[Y]^2$. Given that $\rho \geq 1$, there will almost always be groups ready to start play on the first hole whenever the opportunity arises. Thus,we can apply Theorem 3.2 of Glynn and Whitt (1991), which reduces to previous HT limit in Iglehart and Whitt (1970a,b), as noted in Remark 3.1 there. That HT limit is in turn a special case of more general HT limits for open networks of queues in Harrison and Reiman (1981), Reiman (1984) and Ch. 14 of Whitt (2002). The connection to this earlier work is seen by observing that, for $h \geq 2$, $G_{h,n}^{D,std}$ is approximately the departure time in a series network of $h-1$ queues with traffic intensity $\rho = 1$ and interarrival-time distribution equal to the service time distribution on Y. (We consider the service time on the first hole as the interarrival time to the rest of the model.)

For $h \geq 2$, The HT limit takes the form

$$\frac{G_{h,n}^{D,std} - (n+h-1)E[Y]}{\sqrt{n}} \Rightarrow \sigma_Y \hat{D}_h(1) \quad \text{as} \quad n \to \infty, \tag{68}$$

where $\hat{D}_h(1)$ is a complex function of (h)-dimensional standard Brownian motion (BM), as arises in the case $E[Y] = Var(Y) = 1$. (The term $h-1$ on the left could be omitted because it is asymptotically negligible as $n \to \infty$ when divided by $\sqrt{n}$, but we include it to be consistent with (66) above.) The key approximation stemming from the HT limit is

$$G_{h,n}^{D,std} \approx E[Y]\left(n + h - 1 + \sqrt{nc_Y^2}\hat{D}_h(1)\right). \tag{69}$$

It remains to evaluate the distribution of the random variable $\hat{D}_h(1)$ appearing in (68) and (69). To evaluate its mean and standard deviation, we now exploit simulation results from Greenberg et al. (1993). In particular, we apply Table 5 of Greenberg et al. (1993) to produce the approximation

$$E[\hat{D}_h(1)] \approx b_h\sqrt{h} \quad \text{and} \quad SD[\hat{D}_h(1)] \approx c_h \tag{70}$$

where b_h and c_h are constants that in general should depend on h with $b_h \uparrow 2$ as $h \uparrow \infty$, while c_h decreases. For $h = 10$, $b_h \approx 1.62$ and $c_h \approx 0.65$; for $h = 100$, $b_h \approx 1.95$ and $c_h \approx 0.45$. Hence, we use the approximations $b_{18} \approx 1.7$ and $c_{18} \approx 0.6$, yielding

$$E[\hat{D}_{18}(1)] \approx 1.7\sqrt{h} \approx 7.2 \quad \text{and} \quad SD[\hat{D}_{18}(1)] \approx 0.6 \tag{71}$$

Since $V_{h,n}^{std} = G_{h,n}^{D,std} - A_{1,n}$, we can combine the results to approximate the distribution of the sojourn time $V_{h,n}^{std}$. Since $A_{1,n}$ is deterministic,

$$SD[V_{h,n}^{std}] = SD[G_{h,n}^{D,std}]. \tag{72}$$

Combining (69), (71) and (72), we obtain the approximations

$$\begin{aligned} E[V_{18,n}^{std}] &\approx E[Y]\left(n + 17 - \frac{n-1}{\rho} + 7.2\sqrt{nc_Y^2}\right) \quad \text{and} \quad , \\ SD[V_{18,n}^{std}] &\approx 0.6E[Y]\sqrt{nc_Y^2}. \end{aligned} \tag{73}$$

For the case $n = 100$, we obtain

$$E[V_{18,100}^{std}] \approx E[Y]\left(27 + 7.2\sqrt{100c_Y^2}\right) \quad \text{and} \quad SD[V_{18,100}^{std}] \approx 0.6E[Y]\sqrt{100c_Y^2} \tag{74}$$

To evaluate the quality of the approximation in (74), we simulated the standard model with i.i.d. service times distributed as Y in (17) with traffic intensity $\rho = 1.1$ (arrival rate $E[Y]/1.1$) and $\rho = 1.0$ for the stage playing time distributions in Table 2. The results are shown in Table 4. For $\rho = 1.0$, the approximations for the mean sojourn time of group 100 over 18 holes with the tri, $tri + LB$ and exp stage service time distributons are, respectively, 3.9% high, 10.6% high and 3.7% high.

For the cases with $\rho \geq 1$, Table 4 shows that (73) and (74) provide useful approximations for the mean $E[V_{18,n}]$ and standard deviation $SD[V_{18,n}^{std}]$, showing the dependence upon the five key

Table 4 Comparison with simulation estimates of the heavy-traffic approximation in (74) for the mean and standard deviation of $V^{std}_{18,100}$, the sojourn time in the standard model, for service times distributed as Y in (17) with four different stage playing times satisfying (38) with $m = 6$, based on 2000 replications, $\rho = 0.9$, 1.0 and 1.1

stage playing time dist.	exact	(sim)	HT approx	(75)	(76)
$\rho = 1.1$	mean	SD	mean	SD	$\sqrt{c_V^2}$
deterministic, $m = 6$	243	0.0	$9.00(27 + 0.00) = 243$	0.0	0.0000
triangular, $(m, a) = (6, 3)$	363	9.5	$9.70(27 + 11.75) = 376$	9.5	0.0253
tri+LB, $(p, L) = (.05, 12)$	398	14.0	$9.97(27 + 16.42) = 433$	11.6	0.0268
exponential, $m = 6$	826	56.4	$12.00(27 + 44.1) = 853$	44.1	0.0517
$\rho = 1.0$					
deterministic, $m = 6$	162	0.0	$9.00(18 + 0.00) = 162$	0.0	0.0000
triangular, $(m, a) = (6, 3)$	278	9.5	$9.70(18 + 11.75) = 289$	7.3	0.0253
tri+LB, $(p, L) = (.05, 12)$	310	13.8	$9.97(18 + 16.42) = 343$	9.2	0.0268
exponential, $m = 6$	722	56.7	$12.00(18 + 44.1) = 749$	38.7	0.0517
$\rho = 0.9$					
deterministic, $m = 6$	162	0.0	$9.00(18 + 0.00) = 162$	0.0	
triangular, $(m, a) = (6, 3)$	201	5.6	$162 + 9.7(18)(9)(0.0267) = 204$	7.3	
tri+LB, $(p, L) = (.05, 12)$	224	9.8	$162 + 9.97(18(9)(0.0382) = 224$	9.2	
exponential, $m = 6$	597	52.9	$162 + 12(18(9)(0.375)(0.727) = 691$	38.7	

variables $E[Y]$, ρ, n, h and c_Y^2. For example, from (73), we see that the mean $E[V_{18,n}]$ is directly proportional to $E[Y]$.

Our main focus is on cases with $\rho \geq 1$, but Table 4 also includes results for $\rho = 0.9$ to show what happens. Table 4 shows that there are no dramatic changes; we can obtain reasonable rough estimates for the mean values at $\rho = 0.9$ by subtracting the difference of the values at $\rho = 1.1$ and $\rho = 1.0$ from the value for $\rho = 1.0$. However, the approximations for $\rho = 0.9$ have a very different basis. For $\rho < 1$, we use a variation of the approximation for the steady-state mean from Whitt (1983), which is discussed in the appendix. For the standard deviation, we draw on (73), assuming that $SD(V^{std}_{18,n}(\rho))$ for $\rho < 1$ has the same value as in the approximation for $\rho = 1.0$.

8.2. Approximation for the Golf Model

To obtain the corresponding approximation for the sojourn time in the golf model, we need to include the sojourn time adjustment in §5.2. To do so for the mean, we should add $18E[S_3]$ to the approximation for the mean $E[V^{std}_{18,n}]$ in (73) and (74) above. That yields the final approximation for the mean sojourn time for our golf model; i.e., for $h = 18$ and $30 \leq n \leq 300$, we suggest the approximation

$$E[V_{18,n}] \approx E[Y]\left(n + 17 - \frac{n-1}{\rho} + 7.2\sqrt{nc_Y^2}\right) + 18E[S_3]. \tag{75}$$

Unlike the approximation formula for the standard model in (73), the approximation for the mean $E[V_{18,n}]$ in (75) is *not* directly proportional to $E[Y]$. However, it is directly proportional to m if we assume that (38) holds.

For the standard deviation, there evidently is dependence among these stage playing times used in the adjustment. Hence, we advocate the simple approximation

$$\sqrt{c^2_{V_{18,n}}} \approx \sqrt{c^2_{V^{std}_{18,n}}} = \frac{SD[V^{std}_{18,n}]}{E[V^{std}_{18,n}]}, \tag{76}$$

where the terms on the right are given in (73). This method allows us to exploit the thorough study in Greenberg et al. (1993).

We compare the approximation to simulation estimates in Table 5.

Table 5 Comparison of the heavy-traffic approximation for the mean sojourn time of group 100 on the full 18-hole golf course, $E[V_{18,100}]$, in (75) with simulation estimates for four different stage playing times satisfying (38) with $m = 6$ based on 2000 independent replications, $\rho = 0.9$, 1.0 and 1.1

stage playing time dist.	exact	(sim)	model approx	(sim)	HT approx	(75)
$\rho = 1.1$	mean	SD	mean	SD	mean	SD
deterministic, $m = 6$	351	0.0	351	0.0	351	0.0
triangular, $(m,a) = (6,3)$	469	10.1	498	12.4	484	12.2
tri+LB, $(p,L) = (.05,12)$	503	14.2	526	15.3	531	14.2
exponential, $m = 6$	908	61.7	938	61.0	961	50.0
$\rho = 1.0$	mean	SD	mean	SD	mean	SD
deterministic, $m = 6$	270	0.000	270	0.000	270	0.000
triangular, $(m,a) = (6,3)$	382	9.9	411	12.7	396	10.0
tri+LB, $(p,L) = (.05,12)$	416	14.4	437	15.3	440	11.8
exponential, $m = 6$	807	59.0	832	60.5	852	44.1
$\rho = 0.9$	mean	SD	mean	SD	mean	SD
deterministic, $m = 6$	270	0.000	270	0.000	270	0.000
triangular, $(m,a) = (6,3)$	306	6.6	312	8.8	312	7.9
tri+LB, $(p,L) = (.05,12)$	330	11.9	335	11.9	332	8.9
exponential, $m = 6$	683	56.2	707	60.6	852	44.0

Table 5 show that the HT approximation gives a useful approximation for the mean $E[V_{18,n}]$ and standard deviation $SD[V_{18,n}]$. From Tables 4 and 5, we see that the errors in Table 5 are primarily due to the quality of the heavy-traffic approximation for the standard model in this setting.

The approximation formulas in (75) and (76) effectively reveal the important (and somewhat complex) dependence upon the six key variables $E[Y]$, ρ, n, k, c_Y^2 and $E[S_3]$. For the mean, first

$$E[V_{18,n}] \approx E[Y]A_n + 18E[S_3], \tag{77}$$

where

$$A_n \equiv A_n(\rho, c_Y^2) = (n+17) + \frac{n-1}{\rho} + 7.2\sqrt{nc_Y^2}. \tag{78}$$

For example, suppose that $n = 1.1$ and $\rho = 1.1$. For the exponential stage playing time model with $E[S_1 = E[S_3] = 2ES_2] = m = 6$, $E[Y] = 12$, $E[S_3] = 6$ and $A_n = 117 - 90 + 7.2\sqrt{37.5} = 27 + 44 = 71$, so that $7.2\sqrt{nc_Y^2}$ contributes 62% of A_n. In contrast, for the triangular distribution with $(m,a) = (6,3)$, $E[Y] = 9.7$, $E[S_3] = 6$ and $A_n = 117 - 90 + 7.2\sqrt{2.667} = 27 + 11.8 = 38.8$. For the triangular distribution, the mean $E[Y]$ is 81% of the mean in the exponential case, while $7.2\sqrt{nc_Y^2}$ contributes only 30% of A_n.

9. Conclusions

In this paper we developed a modeling framework for analyzing the pace of play on golf courses. We started in §2 by illustrating how the reslting approximation formulas can be used to analyze the pace of play. Then in §3 we reviewed the stochastic queueing model of successive groups of golfers playing on a generic par-4 hole from Whitt (2013). This model captures the random times required for groups to complete play on separate stages of the hole, but the model is complicated by the fact that more than one group can be playing on each hole at the same time. In §5 we proposed a new approach for approximating this model by a conventional $G/GI/1$ single-served model in which only one group is playing on the hole at any time. The main idea is to let the approximating "aggregate" service times be chosen to appropriately match the successive times between successive groups clearing the green (competing play) on that hole in the fully-loaded version of the original model. Hence in §4 we also reviewed the analysis of the fully-loaded par-4 model. We use this approximating $G/GI/1$ model to directly approximate the waiting times, but a modification is needed to approximate the sojourn and departure times, as shown in §5.2.

Both the original stochastic models from Whitt (2013) and the corresponding approximating conventional models introduced here extend directly to models of play on an 18-hole golf course with any configuration of these holes, but we suggest focusing on 18 identical holes to conveniently derive important insights, because then relatively simple analytical approximation formulas are readily available. In particular, we propose (75) and (76) as approximations for the mean and standard deviation of the sojourn time of group n on the entire 18-hole golf course (provided that n is not too small, which is appropriate if we focus on the last group to play).

To focus on key aspects of group play on each hole, in §6 we developed alternative models of the stage playing time distributions. We think that the model combining the triangular distribution with the possibility of lost balls is especially promising. It has the parameter 5-tuple (m, r, a, p, L), where each parameter captures a different aspect of the model. Extending the model to asymmetric triangular distributions, skewed to the right, would evidently be even more realistic.

We conducted simulation experiments to show that these approximating conventional $G/GI/1$ single-served models and the analytical approximation formulas provide good performance approximations for the more complex original model under heavy loading, which is the principal case for golf courses. The analytical performance formulas for the conventional model can be used to obtain important insights. The approximation formulas can be used to quickly perform back-of-the-envelope calculations to understand how the golf course performance depends on the interval between successive tee times and the stage playing times.

There are many remaining problems. One important next step is to examine data of group play on golf courses and estimate the distributions of (i) the waiting times and the times to play each hole for successive groups, and (ii) the stage playing times S_i on each hole. It also remains to examine alternative ways to control the pace of play.

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References

Abate, J., W. Whitt. 1987. Transient behavior of regulated Brownian motion, I: Starting at the origin. *Advances in Applied Probability* **195**(3) 560–598.

Armony, M., S. Israelit, A. Mandelbaum, Y. Marmor, Y. Tseytlin, G. Yom-Tov. 2011. Patient flow in hospitals: a data-based queueing-science perspective. New York University, http://www.stern.nyu.edu/om/faculty/armony/.

Baccelli, F., G. Cohen, G. J. Olsder, J.-P. Quadrat. 1992. *Synchronization and Linearity: An Algebra for Discrete Event Systems*. Wiley, New York.

Baccelli, F., A. Makowski, A. Shwartz. 1989. The fork-join queue and related systems with synchronization constraints: Stochastic ordering and computable bounds. *Advances in Applied Probability* **21**(3) 629–660.

Glynn, P. W., W. Whitt. 1991. Departures from many queues in series. *Annals of Applied Probability* **1**(4) 546–572.

Greenberg, A. G., O. Schlunk, W. Whitt. 1993. Using distributed-event parallel simulation to study many queues in series. *Probability in the Engineering and Informational Sciences* **7**(2) 159–186.

Harrison, J. M., M. I. Reiman. 1981. Reflected Brownian motion on an orthant. *Annals of Probability* **9**(2) 302–308.

Heidergott, B., G. J. Olsder, J. W. van der Woude. 2006. *Max Plus at work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and its Applications*. Princeton University Press, Princeton, New Jersey.

Iglehart, D. L., W. Whitt. 1970a. Multiple channel queues in heavy traffic, i. *Advances in Applied Probability* **2**(1) 150–177.

Iglehart, D. L., W. Whitt. 1970b. Multiple channel queues in heavy traffic, ii: Sequences, networks and batches. *Advances in Applied Probability* **2**(2) 355–369.

Kimes, S., L. Schruben. 2002. Golf revenue management: a study of tee time intervals. *Journal of Revnue and Pricing Management* **1**(2) 111–119.

Kraemer, W., M. Langenbach-Belz. 1976. Approximate formula for the delay in the queueing system $GI/G/1$. *Congessbook, Proceedings of the Eighth International Teletraffic Congress, Melbourne* **235** 1–8.

Reiman, M. I. 1984. Open queueing networks in heavy traffic. *Mathematics of Operations Research* **9** 441–458.

Riccio, L. 2012. Analyzing the pace of play in golf: the golf course as a factory. *International Jounral of Golf Science* **1** 90–112.

Riccio, L. 2013. *Golf's Pace of Play Bible*. xxx, New York.

Tiger, A., S. Salzer. 2004. Daily play at a golf course: using spreadsheet simulation to identify system constraints. *INFORMS Transactions on Education* **4**(2) 28–35.

Welch, P. D. 1964. On a generalized M/G/1 queuing process in which the first customer of each busy period receives exceptional service. *Operations Research* **12**(5) 736–752.

Whitt, W. 1983. The queueing network analyzer. *Bell System Technical Journal* **62**(9) 2779–2815.

Whitt, W. 1989. Planning queueing simulations. *Management Science* **35**(11) 1341–1366.

Whitt, W. 2002. *Stochastic-Process Limits*. Springer, New York.

Whitt, W. 2013. The maximum throughput rate for each hole on a golf course. Columbia University, http://www.columbia.edu/∼ww2040/allpapers.html.

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E-Companion

We now investigate how the model approximation performs for steady-state distributions. For the less variable tri and $tri+LB$ stage playing time distributions, the steady-state approximations are quite useful, but for the more variable exponential stage playing time distribution the steady state distribution is not so useful because steady-state is not reached during the day. Nevertheless, it is interesting to see how the $GI/GI/1$ model approximation performs in steady-state, which of course requires that the rate in be less than the maximum rate out. We first introduce analytical approximations that we can exploit in this case.

EC.1. QNA Approximations for the Steady-State Waiting Time

Unfortunately, the approximating $G/GI/1$ model itself is complicated, even in the $D/GI/1$ case of primary interest to us for the first queue. (Subsequent queues are substantially more complicated.) Thus, for the resulting $G/GI/1$ approximating model, we also propose steady-state performance approximations from the Queueing Network Analyzer (QNA) in Whitt (1983). Fortunately, the steady-state behavior does not depend on the exceptional first service times.

These QNA approximations depend on the mean service time $E[Y]$, the traffic intensity $\rho \equiv \lambda E[Y]$ and the squared coefficients of variation (scv, variance divided by the square of the mean) c_a^2 and c_s^2 of an interarrival time and a service time, i.e., Y. These in turn draw heavily upon Kraemer and Langenbach-Belz (1976). Here we only consider the case $c_a^2 \le 1$ and $c_s^2 \le 1$, which seems most relevant here, but the other cases are also given in Whitt (1983). (For the $D/GI/1$ model, $c_a^2 = 0$.)

From (44) and (45) of Whitt (1983), the approximate mean steady-state waiting time is

$$E[W] \approx \frac{E[Y]\rho(c_a^2 + c_s^2)g}{2(1-\rho)}, \tag{EC.1}$$

where

$$g(\rho, c_a^2, c_s^2) \equiv \exp\left[\frac{-2(1-\rho)(1-c_a^2)^2}{3\rho(c_a^2 + c_s^2)}\right]. \tag{EC.2}$$

From (48) and (49) of Whitt (1983), the approximate steady state delay probability is

$$P(W > 0) \approx \rho + (c_a^2 - 1)\rho(1-\rho)h, \quad \text{where} \quad h(\rho, c_a^2, c_s^2) \equiv \frac{1 + c_a^2 + \rho c_s^2}{1 + \rho(c_s^2 - 1) + \rho^2(4c_a^2 + c_s^2)}. \tag{EC.3}$$

From (50)-(54) of Whitt (1983), the approximate variance of the steady-state waiting time is

$$Var(W) = (EW)^2 c_W^2, \quad \text{where} \quad c_W^2 = \frac{c_D^2 + 1 - P(W>0)}{P(W>0)},$$
$$c_D^2 \approx 2\rho - 1 + \frac{4(1-\rho)d_s^3}{3(c_s^2+1)^2} \quad \text{and} \quad d_s^3 \equiv (2c_s^2+1)(c_s^2+1). \tag{EC.4}$$

The QNA approximation is intended for complex networks of queues, and takes a relatively simple form for the series of $G/GI/1$ queues we consider. In particular, given the parameters of one queue, the departure rate necessarily equals the arrival rate and, from (38) of Whitt (1983), the approximating scv of the departure process is

$$c_d^2 \approx \rho^2 c_s^2 + (1-\rho^2)c_a^2. \tag{EC.5}$$

For the k^{th} queue of i.i.d. queues in series, the resulting simple approximation is

$$c_{a,k+1}^2 = c_{d,k}^2 \approx (1-\rho^2)^k c_a^2 + [1-(1-\rho^2)^k]c_s^2. \tag{EC.6}$$

As k increases $c_{d,k}^2 \to c_s^2$. For suitably large k and ρ, $c_{a,k+1}^2 = c_{d,k}^2 \approx c_s^2$.

Finally, we remark that the overall mean waiting time and sojourn time on the full 18-hole course can be expressed as the sum of the means on the individual holes. The QNA approximation for the variance of the sum is the sum of the component variances, but we have seen that is inappropriate here. The variance of the sum is more complicated because of the dependence noted in §7.2. In particular, for the case of the exponential stage playing times and $\rho = 0.9$, the sum of the variances of the sojourn times at the 18 queues is more than 4 times greater than the estimated variance of the sum of the sojourn times.

EC.2. Simulation Comparisons for Steady-State Performance

We now report results of a simulation experiment comparing the steady-state performance of the exact par-4 model in §3 to the performance of the approximating $G/GI/1$ model with exceptional first service in §5 when $\rho = 0.9$. The exceptional first service does not influence the steady-state behavior.

For the steady-state performance, We compare three different approximations: The first is based on simulation estimates of the $G/GI/1$ model with i.i.d. service times Y as in (17). The second is based on simulation estimates of the same $G/GI/1$ model except the service-time distribution is fit to a gamma distribution. The third is based on the QNA analytical approximation formulas.

We first consider one queue and then we consider 18 queues in series. Since the golf course tends to be heavily loaded, our main steady-state case has $\rho = 0.9$, but we also examine lighter loads $\rho = 0.7$ and $\rho = 0.5$.

EC.2.1. One Queue with traffic Intensity $\rho = 0.9$

Consistent with the common use of deterministic evenly spaced tee times, we consider the case of a $D/GI/1$ model with deterministic arrival times and mutually independent exponential stage playing times as in §6.1.

Following p. 94 of Riccio (2012), we consider a concrete example with mean stage playing times $E[S_1] = E[S_3] = 6$ and $E[S_2] = 3$. As above, we assume that S_i is exponential for all i. Hence, $E[Y] = 12$ and $Var(Y) = 54$, so that the scv is $c_Y^2 = Var(Y)/E[Y]^2 = 0.375$. We find that this pdf is well approximated by a gamma pdf with shape parameter $k = 2.66$ and scale parameter $\theta = 4.505$, having mean $k\theta = 11.98 \approx 12$ and variance $k\theta^2 = 53.98 \approx 54$. (For comparison, note that if the stage playing times were deterministic, then $Y = (S_1 \vee S_3) + S_2 = 9$, so that the exponential distributions inflate the mean $E[Y]$ by 1/3 from its deterministic-case value.) Consistent with the heavy loading of golf courses, we let the traffic intensity be $\rho = \lambda E[Y] = 0.9$, so that $\lambda = 0.9/12 = 0.075$ and the fixed interval between group tee times is 13.3333.

Each simulation run has $10,000$ groups (arrivals) and $2,000$ runs were made. The simulation estimates of steady-state performance uses the data for groups $7,501$ to $9,500$; thus the total number of data points is 4×10^6 for these estimates. Table EC.1 shows simulation results for three models, the exact model and the two $D/GI/1$ approximate models, with service times distributed as Y and gamma$(2.66, 4.605)$, plus the QNA analytical approximations from §EC.1. The results show that: (i) the gamma distribution does not significantly alter the $G/GI/1$ approximation, (ii) the $G/GI/1$ model approximations are quite accurate and (iii) the relatively simple analytical QNA approximations for the performance measures are quite accurate for this case as well. Moreover, consistent with heavy-traffic limits, the conditional waiting time $D \equiv W|W > 0$ fits an exponential very well.

Table EC.1 Simulation comparisons for the steady-state performance on the first hole: exact model in §*3* versus the $D/GI/1$ approximate model in §*5* with aggregate service according to Y and gamma fit, and the QNA formulas from §*EC.1*. Waiting-time (W) and sojourn-time (U) performance measures are shown for deterministic arrival times, $\rho = 0.9$ and independent exponential stage playing times with $E[S_1] = E[S_3] = 6$ and $E[S_2] = 3$.

model	$E[W]$	$Var(W)$	$P(W > 0)$	$E[W\|W > 0]$	$c^2_{W\|W>0}$	$E[U]$	$Var(U)$
sim. (exact)	16.2	506	0.673	23.93	1.01	33.7	613
sim. (app,Y)	17.4	527	0.729	23.79	0.97	35.4	617
sim (app,gam)	17.4	517	0.732	23.66	0.98	35.4	607
QNA, §*EC.1*	16.6	472	0.727	22.83	0.97	34.6	562

EC.2.2. Eighteen i.i.d. par-4 Holes in Series

We now evaluate the approximation for 18 i.i.d. par-4 holes in series. As before, we consider a deterministic arrival process at the first hole, with traffic intensity $\rho = 0.9$ at all queues. The flow through the network makes the arrival process at each successive queue somewhat more variable, causing the congestion to steadily increase. The QNA approximating arrival process scv is now $c_a^2 \approx c_s^2 = 0.375$ instead of $c_a^2 = 0$ at queue 1. Table EC.2 shows the performance results just as in Table EC.1 for the *last* (18^{th}) queue, using $c_a^2 \approx 0.375$ in the QNA approximation. Table EC.2 shows that the performance approximations are very good for the $G/GI/1$ model, just as in Table EC.1, but the QNA approximations tend to be somewhat high. Nevertheless the y should bed useful for engineering applications.

EC.2.3. Lighter Loads

We also did simulation comparisons for lighter loads. As might be anticipated, we found that the quality of the performance approximation provided by the approximating $G/GI/1$ model degrades significantly as ρ decreases. For lower ρ, the approximation provides only a crude upper bound.

Table EC.2 Simulation comparisons for the steady-state performance on the 18^{th} hole: exact model in §*3* versus the $G/GI/1$ approximate model in §*5* with aggregate service according to Y and gamma fit, and the QNA formulas from §*EC.1*. Waiting-time (W) and sojourn-time (U) performance measures are shown for deterministic arrival times, $\rho = 0.9$ and independent exponential stage playing times with $E[S_1] = E[S_3] = 6$ and $E[S_2] = 3$.

model	$E[W]$	$Var(W)$	$E[U]$	$Var(U)$
exact (sim)	37.1	1860	54.8	1976
approx (Y, sim)	37.4	1743	55.4	1797
approx (gam, sim)	37.1	1754	55.1	1808
approx (QNA, §*EC.1*)	38.5	2134	56.5	2188

For example, the exact (approximate) estimates of the mean steady-state wait EW for $\rho = 0.7$ and 0.5 were 2.20 (2.97) and 0.333 (0.568), respectively. The performance is roughly consistent with the approximation

$$E[W, exact] \approx \sqrt{\rho} E[W; approx]. \tag{EC.7}$$

EC.2.4. Application to Fewer than 100 Groups

From the simulations above based on 10,000 groups, we see that both the model approximation and the QNA approximation from Whitt (1983) are effective for approximating the steady state behavior of the system. That is supported by the simulation results above for 10,000 groups. However, the actual golf course will have only $20 - 100$ groups playing each day. Clearly, the steady-state approximations will also apply to these transient settings if the system is nearly in steady state for these smaller numbers. We investigated this issue and have drawn a few tentative conclusions. Mainly, we conclude that this issue is worth further study.

Our first conclusion, as expected, because it is consistent with established results about the approach to steady state, especially for single-server queues and reflected Brownian motion, as in Abate and Whitt (1987) and §4.3 of Whitt (1989), is that the approach to steady state becomes much slower as the variability in the arrival and service processes increases. In particular, we find direct steady-state approximations work quite well for the tri and $tri + LB$ distributions, but *not* for the exponential distribution.

Our second conclusion is that the time required for the k^{th} queue to approach steady state is increasing in k. For the example with exponential stage playing times, the estimated expected waiting and sojourn times for group 100 on the first hole are approximately the same as for group 10,000 on the first hole, which we take to be approximately the steady-state value. However, the estimated expected waiting time for group 100 on the 18^{th} hole is only 72.7% of the estimated expected waiting time for group 10,000 on the 18^{th} hole. This phenomenon is evidently consistent with experience that different performance descriptions of the same model approach their steady-state limits at different rates, but it is worth further study.

Based on our simulation experiments, we propose some simple approximations for the performance of the models when $\rho < 1$. These are used in Table 4, which includes results for $\rho = 0.9$. First, for $n = 100$ groups (and other n not too small), we conclude that direct steady-state approximations from Whitt (1983) should provide reasonable rough approximations for the tri and $tri + LB$ distributions. In particular, for these cases we suggest the approximation formulas

$$E[V_{18,n}^{std}(\rho)] \approx 18E[Y] + \frac{E[Y]\rho c_Y^2}{1-\rho} \quad \text{and} \quad SD(V_{18,n}^{std}(\rho)) \approx SD[V_{18,n}^{std}(1)]. \tag{EC.8}$$

The approximation for the mean in (EC.8) is based on (EC.1) assuming, first, that and $c_{a,k}^2 \approx c_s^2$ at each queue, so that $c_{a_k}^2 + c_s^2)/2 \approx c_s^2$ for each k, as supported by (EC.6), and, second, that $g \approx 1$ in (EC.2), which tends to be nearly correct when $c_a^2 \approx c_s^2$.

In Whitt (1983) the variance of the sum of the sojourn times is approximated by the sum of the variances at the individual queues. We have consistently found that is in appropriate, with the variance of the sum being much less than the sum of the variances. That property is found to hold for $\rho = 0.9$ just as for $\rho \geq 1$, but with somewhat less force. We find that a much better approximation for $SD(V_{18,n}^{std}(\rho))$ for $\rho = 0.9$ is the well-studied approximation for $\rho = 1.0$. The simulation results in Table 4 confirm that the standard deviation is very nearly the same for $\rho = 0.9$ as it is for $\rho = 1.0$.

However, the same approximation performs very poorly for the mean $E[V_{18,n}^{std}(\rho)]$ when $\rho = 0.9$ and the stage playing times are exponential, yielding the value 891 compared to the simulation estimate of 597. However, since the simulation estimates for the mean waiting times for groups 100 and 10,000 on hole 18 were 28.0 and 38.5, respectively, we reduce the expected waiting times used in (EC.8) for the exponential stage playing times by $28.0/38.5 = 0.727$.