Online Companion to
Using Simulation to Help Manage the Pace of Play in Golf

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Abstract

(from the main paper) To help design and manage golf courses, Whitt (2015) developed a new stochastic queueing network model of a conventional 18-hole golf course and characterized the capacity of each hole. The primitives in that model are the random times required for each group of golfers to play each stage on each hole, where a stage is an appropriate set of steps. The model is relatively high level, but the model still captures the important property that multiple groups can play at the same time on the longer holes, subject to precedence constraints. In this paper, we develop a flexible and efficient simulation algorithm of that model and apply it to study important operational issues in golf. We substantiate the characterization of hole capacities and study: (i) the costs and benefits of a wave-up rule designed to increase the pace of play and (ii) the costs and benefits of alternative tee schedules, i.e., the intervals between successive groups starting play on the first hole. We estimate that the wave-up rule can increase the daily number of groups that can play each day by about 13%, while a two-level tee schedule rule can increase the daily number of groups that can play each day by about 3%. We show that the costs of making the tee interval too short tend to be far greater than the costs of making it too long.

Keywords: pace of play in golf, queueing models of golf, using simulation to manage golf courses, waving up in golf, alternative tee schedules on golf courses

1 Overview

This is an online companion to the main paper, Choi et al. (2017), which we will refer to that way. We present additional in which we present material supporting and elaborating upon our findings in the main paper.

The online companion is organized into four sections beyond this introduction section. Section 2 elaborates on our simulation algorithm. In particular, we discuss (i) the overall structure of our simulation algorithm, (ii) methodology for calculating statistical estimates, and (iii) efficiency of our algorithm design. Section 3 presents more data visualization figures like Figure 4 of the main paper. These provide a holistic perspective of how various parameters–such as tee times, the presence of the wave-up rule, and different course designs–affect the throughput levels of a golf course, and how they effect the sojourn and departure times of each playing group.

Section 4 provides additional waiting time statistics to portray a clearer context of the impact of making the course under-loaded. Section 5 shows the various throughput levels attained in golf courses that are balanced but have different hole sequences, thereby showing that hole sequence does not have a significant impact on the throughput as long as the courses are balanced.

Again, we use the notation — $P3$, $P3WU$, $SP3$, $P4$, and $P5$ — respectively, to denote the five hole types considered, namely, par-3 hole (unscaled par-3 without the wave-up rule), par-3 hole with wave-up, scaled par-3 hole, par-4 hole, and par-5 hole.

2 Simulation Algorithm: Design and Methodology

This section describes the simulation algorithm for the stochastic group play model reviewed in §2 of the main paper. We expand on the brief overview given in §1.2 of the main paper.

The basic goal was to develop a simulation algorithm that is flexible and scalable in implementing variety of golf course models, as well as variety of parameters associated with the golf courses. Our simulation algorithm provides flexibility by allowing users to choose various parameters, including but not limited to (i) hole sequence, (ii) number of playing groups per day, (iii) number of simulation replications, (iv) playing time parameters and distributions, (v) arrival time distributions, and (vi) hole types. Our simulation algorithm also is scalabile, because the algorithm can handle large numbers of playing groups and simulation replications of a complex golf course model. In order to make our simulation efficient, our simulation model makes use of matrices, as opposed to loops, to provider a faster means of tracking and calculating golf performance statistics of multiple playing groups.

In addition, our simulation algorithm allows the simulation results to be statistically analyzed in multiple ways. The model breaks down playing times and waiting times by each stage in each hole for all playing groups; such breakdown structure allows us to analyze each playing group's performance on a hole-by-hole, stage-by-stage basis. Additionally, the algorithm provides a platform for the simulation optimization algorithm presented in Section 4 of the main paper, which the tee interval that maximizes the throughput in the golf course subject to constraints.

2.1 Algorithm Design and Methodology

We now provide an overview of the simulation algorithm. We first describe the data structure of the algorithm, and then we specify the algorithm steps. A graphic overview of the simulation process is shown in Figure 1.

2.1.1 Data Structure

We now describe the key variables and their data structure.

Here are the user-specified variables (with our common choice are in parentheses):

- `n_grp`: number of groups playing in the course (102 groups)
  - We normally set `n_g` to be slightly larger than the actual number of groups to be analyzed. The reason is because each playing group's stage playing times depend on those of the precedent and the subsequent groups. For example, the playing times of the 100th group at the P3WU holes depend on those of the 99th and the 101st group. Therefore, even though we target to analyze the performance of only the first 100 groups, we set `n_g` to be larger, e.g. 102.
- `n_rep`: number of simulation replications (2000 replications)
- Playing time distribution parameters: $\vec{m}$, a
  - For our simulations, we used the triangular distribution to generate random playing times.

- $\vec{m}$ is the vector of mean stage playing time parameters and is different for each hole type. As outlined in §2.1 of the main paper, Par-3 and Par-4 holes have 3 stage, whereas P5 holes have 5 stages.
    - $\vec{m}^{(3)}$ =(mu1_P3,mu2_P3,mu3_P3): mean stage playing times for Par-3 stages
    - $\vec{m}^{(4)}$ =(mu1_P4,mu2_P4,mu3_P4): mean stage playing times for Par-4 stages
    - $\vec{m}^{(5)}$ =(mu1_P5,mu2_P5,mu3_P5,mu4_P5,mu5_P5): mean stage playing times for Par-5 stages
- a is the variability parameter that indicates the spread in the triangular distribution.
- For each of the hole types above, we set the mean parameters to be (3.50, 2.00, 2.67), (4.00, 2.00, 4.00) and (4.00, 2.00, 2.00, 1.33, 4.00), respectively. However, afterwards we made minor adjustments to achieve a balanced course, as described in §2.2 of the main paper.
    - More detailed description on generating random playing times via triangular distribution is available in the Section 2.2 of the main paper.

- Lost ball parameters: `p` and `L`
    - `p`: Probability of a playing group losing a ball during the first shot for each hole
    - `L`: The alternative longer playing time required for a group to play on the first stage of a hole due to the lost ball

- `intv_array`: an array that establishes tee interval schedule settings for the arrival times. This array may contain multiple values; for example, with `intv_array`= [5.0, 5.5, 6.0, 6.5, 7.0], the simulation program will run 5 rounds of golf play simulation, each with different tee schedule policy (and with `n_rep` number of replications).

- Hole Parameters: `seq`, `n_h` and `n_s`
    - `seq`: an 1 by `n_h` array that records the sequence of holes in the golf course (this array can be scaled to contain fewer than 18 holes in the course).
    - `n_h`: Number of holes (18 holes)

- n_s: Number of stages (60 stages). n_s is calculated by adding the number of stages in each of the n_h holes. In particular,

  n_s= 3·(Number of Par-3 and Par-4 holes) + 5·(Number of Par-5 holes).

  Please refer to Section 2.1 of the main paper for stage diagrams for Par-3, Par-4 and Par-5 holes (3-stage diagram for Par-3 and Par-4 holes, and 5-stage diagram for Par-5 holes).

Here is the list of *output variables* in each replication of the simulation:

- A-matrix: an (n_g) by 1 array that records each playing group's arrival time at the golf course.
- P-matrix: an (n_g) by (n_s+2·n_h) records each playing group's stage clearing time after finishing each stage.

  As mentioned in the description for n_s, there are 3 stages in Par-3 and Par-4 holes and 5 stages in Par-5 holes. We add $2 \cdot 18$ columns in the P-matrix to record (i) hole arrival times of each group n (denoted by A_n) and (ii) beginning times of each group n (i.e. tee-off times, denoted by B_n). Please refer to §2.1 in the main paper for more details on performance measures. Since there are n_h number of holes in the course (typically 18), we multiply these two times by n_h, or 18.

- $w_{h,1}$ and $w_{h,2}$: n_g by 18 arrays that respectively record the first and second moments of waiting times of each group before beginning the play at each hole. The waiting times are calculated from subtracting A_n, the arrival time of group n at the hole, from B_n, the time that group n begins playing in the hole.
- x^1 and x^2: n_g by $\sum_{i=1}^{18} n_i$ arrays that respectively record the first and second moments of playing times of each group. The playing times are calculated from subtracting B_n, the time that group n begins playing in the hole, from G_n, the time that group n clears the green and departs from the hole.
- w^1_{total} and w^2_{total}: n_g by 1 arrays that records the first and second moments of total waiting times of each group in the entire golf course, averaged over n_rep number of replications.

- x^1_{total} and x^2_{total}: `n_g` by 1 arrays that records the first and second moments of total playing times of each group in the entire golf course, averaged over `n_rep` number of replications.

- `TSO, TSO_2, ETSO:` these arrays, each with a size of `n_g` by 1, respectively denote the first moment of the sojourn times of each playing group, the second moments, and the average sojourn times for `n_g` groups. Sojourn times are calculated by subtracting the first column of the P-matrix (the arrival times of the groups at the first hole) from the last column of the P-matrix (the clearing times of the groups at the last hole). `TSO_2` is the square of the first-moment array, `ETSO` is the average of the `TSO` arrays generated in `n_rep` number of replications.

  – Similarly as above, sample variance of the sojourn times can be generated from subtracting $(\texttt{TSO})^2$ from $(\texttt{TSO\_2})$.

2.1.2 Calculating Statistical Estimates

In this section, we indicate how the algorithm calculates the statistical estimates of sample means and variances for 2000 samples of waiting and playing times of each playing group. As mentioned previously, 2000 is our common parameter choice for `n_rep`, or the number of simulation replications, because we found that it is large enough to provide good statistical precision. The samples of waiting and playing times for each group come from running 2000 simulation replications.

Furthermore, we provide two-sided confidence intervals with 95% confidence over 2000 replications for waiting and playing times. We remark that, since the sample size is sufficiently large, there is no need to calculate Student t-Statistic, even when the variance is unknown.

- Estimating $w_{h,1}$ and $w_{h,2}$:

  – For variables $w_{h,1}$ and $w_{h,2}$ at hole h ($\forall h = 1, ...,\texttt{n\_h}$) for group j ($\forall j = 1, ...,\texttt{n\_g}$) over `n_r` i.i.d. replications, we provide the estimates as

$$\widehat{w^j_{h,1}} \equiv \sum_{k=1}^{\texttt{n\_r}} \left[w^j_{h,1}\right]_k/(\texttt{n\_r}), \text{ and}$$

$$\widehat{w^j_{h,2}} \equiv \sum_{k=1}^{\texttt{n\_r}} \left[w^j_{h,2}\right]_k/(\texttt{n\_r})$$

With the above estimates, the simulation calculates the difference $\widehat{w^j_{h,2}} - (\widehat{w^j_{h,1}})^2$ to yield variance s^2_w and subsequently calculates the square root of the value to yield standard deviation s_w, again for each group i at hole j.

– To compute half-width of the 95% confidence interval for the waiting times, we calculate $1.96 s_w/\sqrt{\texttt{n\_r}}$. Because our default setting for `n_rep` is 2000, the half-width would then be $1.96 s_w/\sqrt{2000}$.

- Estimating x^1 and x^2:

  – Similarly as the estimates of x^1 and x^2, for variables x^1 and x^2 at hole h ($\forall h = 1, ..., 18$) for group j ($\forall j = 1, ...,$`n_g`) over `n_r` i.i.d. replications, we provide the estimates as

$$\widehat{x^1} \equiv \sum_{k=1}^{\texttt{n\_r}} \left[x^1(j,h)\right]_k/(\texttt{n\_r}), \text{ and}$$

$$\widehat{x^2} \equiv \sum_{k=1}^{\texttt{n\_r}} \left[x^2(j,h)\right]_k/(\texttt{n\_r})$$

With the above estimates, the simulation calculates the difference $\widehat{x^2} - \left(\widehat{x^1}\right)^2$ to yield variance s^2_x and subsequently calculates the square root of the value to yield standard deviation s_x, again for each group j at hole h.

  – To compute half-width of the 95% confidence interval for the waiting times, we calculate $1.96 s_x/\sqrt{\texttt{n\_r}}$. Because our default setting for `n_r` is 2000, the half-width would then be $1.96 s_x/\sqrt{2000}$.

- Estimating w^1_{total} and w^2_{total}:

  – The total waiting time for group j ($\forall j = 1, ...,$`n_g`) across all the holes h ($\forall h = 1, ..., 18$) in the course is calculated as $\sum_{h=1}^{18} \widehat{w^j_{h,1}}$ for each replication k. Similarly, w^2_{total} is defined as the sum of the second moments (i.e. $\sum_{h=1}^{18} \widehat{w^j_{h,2}}$). Subsequently, the first and second moments of the total waiting time are estimated as

$$\widehat{w^1_{total}} = \left[\sum_{k=1}^{\texttt{n\_r}} \left(\sum_{h=1}^{18} \left[\widehat{w^j_{h,1}}\right]_k\right)\right]/(\texttt{n\_r}), \text{ and}$$

$$\widehat{w^2_{total}} = \left[\sum_{k=1}^{\texttt{n\_r}} \left(\sum_{h=1}^{18} \left[\widehat{w^j_{h,2}} \right]_k \right) \right] / (\texttt{n\_r})$$

- Estimating x^1_{total} and x^2_{total}:
  - The total playing time for group j ($\forall j = 1, ..., \texttt{n\_g}$) across all the holes h ($\forall h = 1, ..., 18$) in the course is calculated as $\sum_{j=1}^{18} \left[x^1(j,h) \right]_k$ for each replication k. Similarly, x^2_{total} is defined as the sum of the second moments (i.e. $\sum_{j=1}^{18} \left[x^2(j,h) \right]_k$). Subsequently, the first and second moments of the total playing time are estimated as:

$$\widehat{x^1_{total}} = \left[\sum_{k=1}^{\texttt{n\_r}} \left(\sum_{h=1}^{18} \left[x^1(j,h) \right]_k \right) \right] / (\texttt{n\_r}), \text{ and}$$

$$\widehat{x^2_{total}} = \left[\sum_{k=1}^{\texttt{n\_r}} \left(\sum_{j=1}^{18} \left[x^2(j,h) \right]_k \right) \right] / (\texttt{n\_r})$$

2.1.3 Algorithm Steps

Below is a list of steps in the simulation algorithm. An overview of the steps listed below is also presented in Figure 1.

(i) First, `seq`, or the hole sequence array, with a dimension of $1 \times \texttt{n\_h}$ ($\texttt{n\_h}$ is the number of holes in the course) is defined in the simulation algorithm. Again, our common parameter choice for `n_h` is 18.

(ii) Next, all relevant parameters—such as `n_g` (the number of playing group), tee intervals, playing/waiting time distributions—are declared and defined.

(iii) Once all the relevant parameters are defined, the algorithm initiates the first replication of the simulation by generating `A` (array of arrival times of all the playing groups) according to the tee schedule policy defined in the simulation model (with a dimension of $N \times 1$, in which `n_g` denotes the number of playing groups). For example, setting an even deterministic tee schedule at 5 minutes means that the groups would arrive at times 0, 5, 10, 15, etc.

(iv) Once the arrival time array is generated, the algorithm begins generating stage playing times for each playing group at each of the stages in the first hole. For generating stage playing times, the algorithm calls a different sub-function for each hole type ($P3/SP3$, $P3WU$, $P4$, and $P5$). Each sub-function entails the recursive mathematical representation of the stochastic models of group play, which are outlined in Section 2 of the main paper.

(v) Subsequently, the main function of the simulation algorithm will supply the type of playing time distribution and relevant parameter values to the appropriate sub-function based on the hole type of the first hole. The sub-function will recursively compute and record hole arrival, waiting, and playing times for each stage in the first hole, for all `n_g` groups.

(vi) The playing time matrix, which has a dimension of `n_g` by n_1 (the number of playing stages at hole 1), is returned to the main function of the algorithm.Then, the hole departure times from the matrix (i.e. the last column) are copied to the arrival time array to begin simulations for the second hole.

(vii) Subsequently, Step 4 to 6 is repeated for the next 17 holes until the last playing group departs from the last hole in the sequence. The stage playing time arrays for 18 holes are augmented to create "P-matrix": a `n_g` by $\sum_{i=1}^{18} n_i$ matrix that records the stage playing times of each playing group.

(viii) The entire simulation, from steps 1 to 7, is then run for multiple independent and identically distributed replications. The simulation model is currently set to run `n_sim` (e.g. 2000) replications, which we found is sufficient to yield a reasonably high level of statistical precision.

(ix) Finally, the golf simulator concludes with computation of various statistics, including means, second moments, and variances of sojourn times, waiting times, and playing times. Please refer to Sec 2.1.2 for more details on our methodology for calculating statistical estimates.

In regards to the steps above, we make one concluding remark. In the P-matrix, we calculate and populate the playing times group-by-group—before doing so hole-by-hole. The reason we cannot populate the matrix the other way (i.e. populating playing times for group 1 hole-by-hole for 18 holes—before doing so for group 2 and subsequent groups) is because of the wave-up rule for $P3WU$ holes. The wave-up rule stipulates that each group's playing time depends not only on that of the precedent group, but also on that of the subsequent group.

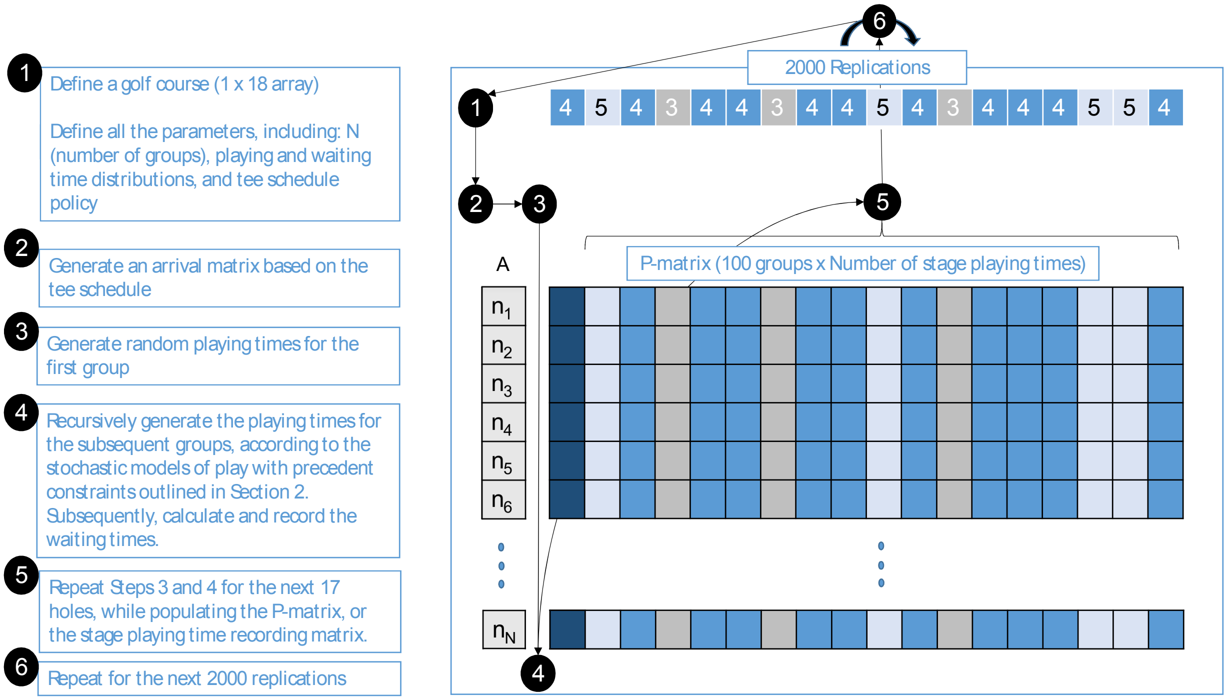

Figure 1: an overview of the data structure and steps of the simulation algorithm.

2.2 Advantages of Our Simulation Algorithm Design

Because of the complexity of golf course simulations–and because of the large number of replications (i.e. 2000) run during each round of simulation–we have made our simulation design as efficient as possible. There are several aspects of our simulation design that help us achieve the goal.

2.2.1 Computation Scale of the Simulation Experiments

We now elaborate upon §1.2 of the main paper and provide a more detailed context of the scale of running our simulation algorithm. Overall, the scale of our simulation experiment is fairly large in terms of run length, and the large scale can be attributed to several factors. Primarily, the run length of our simulation experiment can be determined by three factors: (i) the number of i.i.d. replications, (ii) stage playing times generated per day of golf play, and (iii) various design factors considered.

First, the number of i.i.d. replications (or, as denoted above, `n_r`) plays a key role in determining the run length of the simulation. Our common choice for the parameter `n_r`) is 2000, which, we believe, is sufficient to provide robust statistical precision for various variable estimates. Indeed, the statistical precision is evidenced by the small values of half-widths displayed in the bottom row of Table 3 in §3.1 of the main paper.

Second, we consider the number of stage playing times generated per day of golf play. As stipulated above in the list of input variables for our simulation experiments, our common parameter choice for `n_s`, which denotes the number of stage playing times, is 60. For all course designs, we have three stage playing times for 3 Par-3 and 12 Par-4 holes, and five stage playing times for three Par-5 holes; thus, the total number of stage playing times sum up to 60. Since the common parameter choice for the number of groups is set to be 102, each simulation replication, then, will generate 6120 stage playing times.

Third, we also consider a number of various design factors. There are at least three additional design factors when we run the simulation experiment: (i) the number of course designs, (ii) the number of alternative tee intervals and (iii) the number of Par-3 hole types. As mentioned in the introduction to §3 of the main paper, golf courses commonly have 12 identical Par-4 holes, 3 identical Par-3 holes and 3 identical Par-5 holes. However, each golf course has a different combination of hole sequence; using the combinatoric formula, there is a total of $\frac{18!}{3!\cdot 12!\cdot 3!} = 371,280$ different combinations available. Practically speaking, we do recognize that the possible number of hole combinations in the real world is roughly 20, but the number can be easily larger.

We also consider the number of alternative tee intervals (also denoted as `n_t`). For each course design and for each Par-3 hole type, we obtain the simulation results for 30 different tee time intervals. Lastly, as mentioned in the previous paragraph, we also consider the three different types of Par-3 hole types. Considering these three design factors, we conclude that total number of design factors to be considered, at the minimum, equals to $20 \cdot 30 \cdot 3 = 1800$.

Furthermore, we note that the scale of our simulation experiments can become larger for two-level tee intervals (see §5 of the main paper for more details), for which we consider various combinations of multiple tee parameters. We considered 3 cut-offs (i.e. the group number at which the tee interval increases), 3 types of Par-3 holes for 93 combinations of two-level tee intervals. We also have run our simulation experiment for 5 different course designs. This gives us additional 4185 scenarios to consider in addition to the 1800 cases mentioned above.

Considering all of the factors mentioned above, we can calculate the total number of stage playing times generated, which is $6120 \cdot 5985 \cdot 2000 \approx 7.3 \cdot 10^{10}$ (or approximately 73 billion stage playing times). To portray a clearer picture of the required computational capacity to run our simulation experiment, we note that the entire simulation experiment is estimated to take approximately 1.8 million seconds or approximately 500 full hours, based on our observations of time performance measurements of the MATLAB software.

An important remark is that the above 500-hour requirement meets the bare minimum amount of time required to conduct our research. The time requirement does not include other efforts in designing our simulation experiment; we have run additional simulation runs to carefully choose and test our parameter choices, evaluate robustness of our model, etc. We plan on extending our simulation study to carefully study the impact of slow groups in the golf course.

2.2.2 Other Advantages of the Simulation

There also are other advantages of our simulation design. First, our simulation program is modularized into individual functions. In our simulation algorithm, the main function is designed as the main underlying framework for running different types of golf play scenarios. The random playing times for each hole are generated not in the main function, but in one of the following sub-functions: (i) $P3$ hole (also includes $SP3$ holes), (ii) $P3WU$ hole, (iii) $P4$ hole, and (iv) $P5$ hole. Allocating various parts of the simulation algorithm to different functions helps us to quickly identify certain parts that need improvements.

Second, we have minimized the usage of loops by extensively using arrays. The clear advantage of running simulations with arrays is that doing so helps the simulation program to run faster than running loops does since dealing with arrays often involve fewer lines of codes.

Furthermore, we have divided the columns of time-stamp array P more granularly by different playing stages rather than holes. Such array construction indeed provides a more detailed analysis of playing and waiting times for each stage; but more importantly, it helps us to easily identify bottleneck stages.

3 Data Visualization

A significant component of the simulation algorithm is creating data-visualization figures that help reveal important insights about system performance.

One such data visualization figure was illustrated in Figure 3 of the main paper. We present others now. These figures show how sojourn times and departure times vary with different tee interval times and playing groups.

In §4.1 of the main paper, we formulated an optimization framework that aims to maximize the throughput (i.e. the number of groups that can play per day) subject to (i) sojourn time constraint and (ii) departure time constraint. The visualization figures in this section provide a comprehensive view of sojourn time and departure time trends across a wide range of tee times. We created the visualization figures to better understand how the throughput levels change as tee interval changes, and to understand the behaviors of sojourn time convergence at each tee interval: at which group sojourn times converge, and what value they converge to. Through depicting this dynamic interaction behavior, the visualization figures in this section seek to visually depict the tradeoff between sojourn times and departure times in the optimization framework in §4.1 of the main paper.

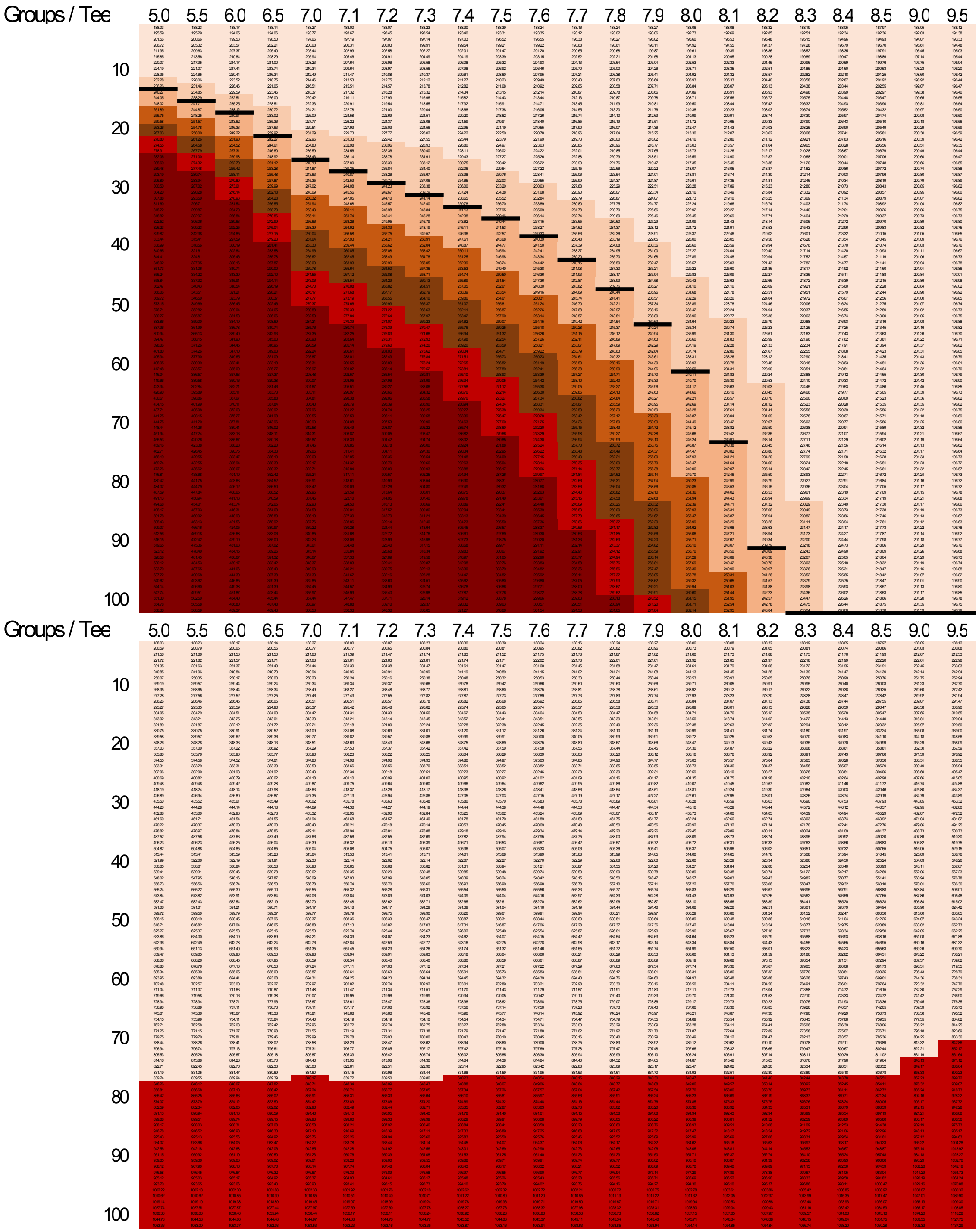

Figure 2: Sojourn times (above) and departure times (below) as a function of the group and the tee interval for a golf course with P3 holes (without the wave-up rule) in the base course.

3.1 Interpreting the Color Visualization

3.1.1 Case 1: $P3$ holes Without the Wave-up Rule

The first pair of figures in Figure 2 show average sojourn times (top) and departure times (bottom) of each group (averaged over `n_r` replications) as a function of the group number (y-axis) and the fixed tee interval (x-axis) for the base case with unscaled $P3$ holes. Here, the base case refers to the hole sequence 454—434—454—434—454—434, as discussed in §3 of the main paper. When enlarged, these figures will show the actual numeric sojourn and departure times.

The top and the bottom figures are color-coded figures; the colors associated with the sojourn and departure times for each group at each fixed tee interval will vary based on how large these times are. For both top and bottom figures, darker color indicates a larger numerical value.

For the top figure, starting from the top (i.e. group 1) to the bottom (i.e. group 100), we see 7 different colors. These colors, from the lightest to the darkest, respectively indicate the sojourn time ranges of $[0, 230]$, $(230, 240]$, $(240, 250]$, $(250, 260]$, $(260, 270]$, $(270, 280]$ and $(280, \infty)$ minutes. In the top figure, the thick black lines indicate whether the sojourn-time constraint is violated; sojourn times that appear below the black lines indicate that the constraint is violated.

For the bottom figure, we see 2 different colors. The lighter color indicates that a departure time for a particular group and tee occurs before 840 minutes (14 hours) since the golf course begins its operations, while the red color indicates that the departure time occurs after 840 minutes (thus violates the departure-time constraint). In the bottom figure in (Figure 2), the number of groups is bounded by the red region according to the departure-time constraint.

From Figure 2, we draw couple of insights. The top figure of Figure 2 shows that more groups will satisfy the sojourn-time constraint as tee interval increases; when groups arrive less frequently, they experience shorter sojourn times. Consistent with intuition, the average sojourn time for the group increases as the group number increases, but decreases as the tee interval gets longer (and the arrival rate decreases). The average departure time for the group also increases as the group number increases, but increases as the tee interval gets longer (i.e. as the arrival rate decreases).

Furthermore, Figure 2 visually shows that the sojourn-time constraint tends to be much more binding than the departure-time constraint. There is a rapid change in the constraint region for the sojourn times at the top, but only a gradual change in the departure times below.

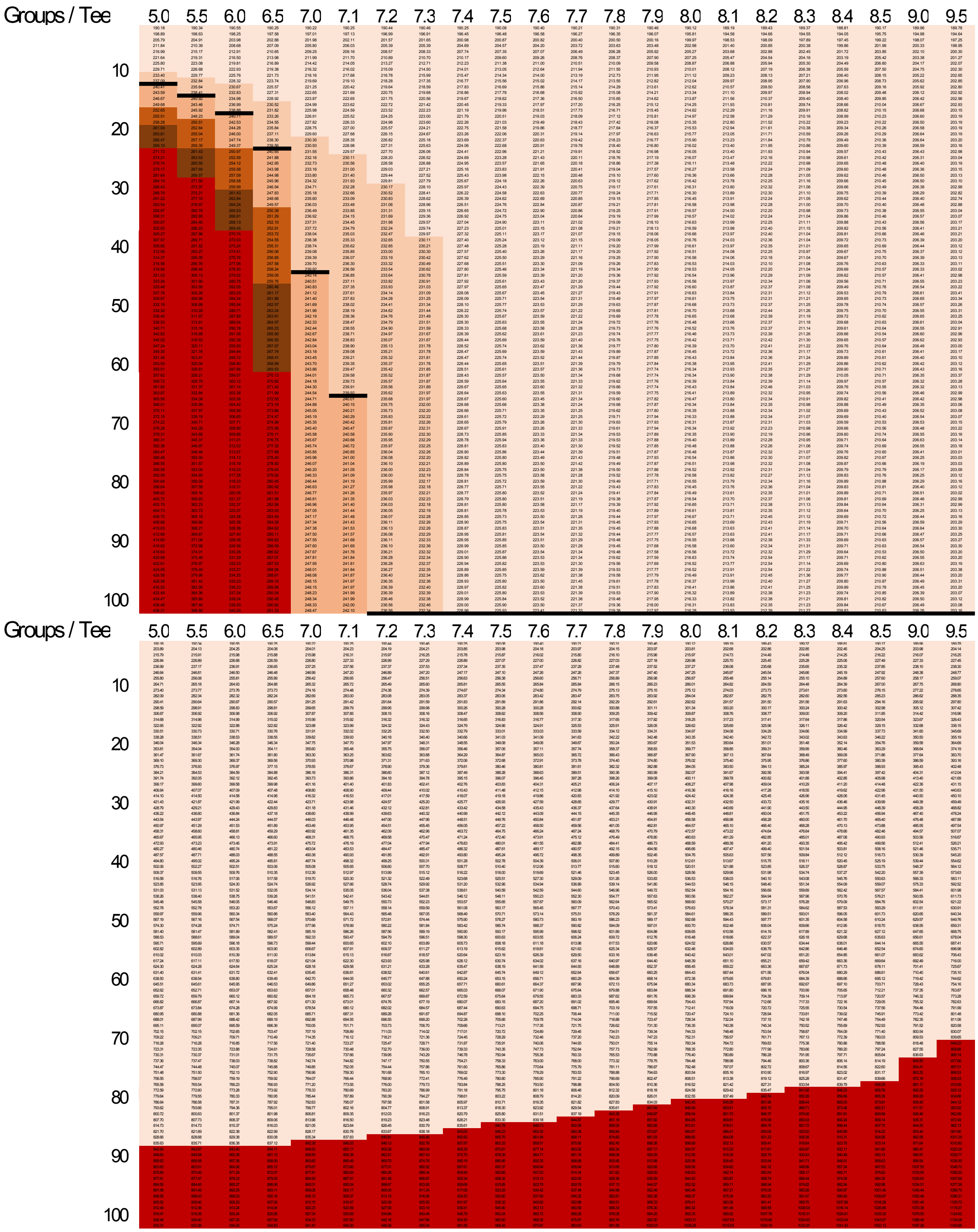

Figure 3: Sojourn times (above) and departure times (below) as a function of the group and the tee interval for a golf course, but now with P3WU holes in the base course. Note that the colored-regions (i.e. constraint regions) are much smaller in the sojourn time and departure time figures when P3WU holes are used.

3.1.2 Case 2: $P3WU$ holes

The second pair of figures in Figure 3 show average sojourn times (top) and departure times (bottom) of each group (averaged over `n_r` replications) as a function of the group number (y-axis) and the fixed tee interval (x-axis) for the base case, but with $P3WU$ holes. Again, the base case refers to the hole sequence 454—434—454—434—454—434, as discussed in §3 of the main paper.

Similarly to Figure 2, the top and the bottom figures in Figure 3 are color-coded. The top figure shows 7 different colors to indicate different ranges of sojourn times; these colors, from the lightest to the darkest, respectively indicate the sojourn time ranges of $[0, 230]$, $(230, 240]$, $(240, 250]$, $(250, 260]$, $(260, 270]$, $(270, 280]$ and $(280, \infty)$ minutes. Sojourn times that appear below the black lines indicate that the constraint is violated. In the bottom figure, the ivory color indicates that the departure time occurs before 840 minutes, and the red color indicates that the departure time occurs after 840 minutes. The red color indicates that the departure-time constraint is violated.

When compared to Figure 2, Figure 3 shows the advantage of the wave-up rule. First, the top figure shows that the sojourn times of the groups across various tee intervals are much lower than those in Figure 2; the rate of change in the constraint region of Figure 3 is much more rapid than the rate in Figure 2. For instance, when the tee interval is 7.2, while only 27 groups in Figure 2 satisfy the sojourn-time constraint, all of the 100 groups in Figure 3 satisfy the sojourn-time constraint.

For the departure time figure in the bottom, Figure 3 shows that the departure times decrease as the tee interval gets longer. The decreasing trend in departure times across tee intervals is more noticeable in Figure 3 than in Figure 2.

Let us further compare the two figures. In the top figure of Figure 3, we see that, when the tee interval is equal to or greater than 7.2 minutes, the sojourn-time constraint is no longer binding, but the departure-time constraint is. When the departure time figure (bottom) is juxtaposed with the sojourn time figure (top), we see that the maximum throughput of 84 occurs when the tee interval is 7.2. On the other hand, the maximum throughput in Figure 2 is attained at a larger tee interval around 8.2 minutes, where the maximum throughput is 74. Indeed, we see the advantage of the $P3WU$ holes over $P3$ holes. Please see Table 5 in §4.1 of the main paper for more details on the throughput. Further notice that, in Figure 3, when the tee interval is greater than 7.2, sojourn times converge as group numbers increase, whereas the sojourn times converge in Figure 2 when the tee is 8.4 or greater. Convergence in sojourn times, indeed, are strong indicators that the golf course is substantially under-loaded.

On the other hand, when examined closely, Figure 3 also shows the disadvantage of the wave-up rule. When the tee interval is 9.0 minutes or higher, sojourn and departure times of the playing groups in Figure 3 are higher than those in Figure 2. This trend may be attributed to the complexity of the $P3WU$ hole, as mentioned in §4.1 of the main paper.

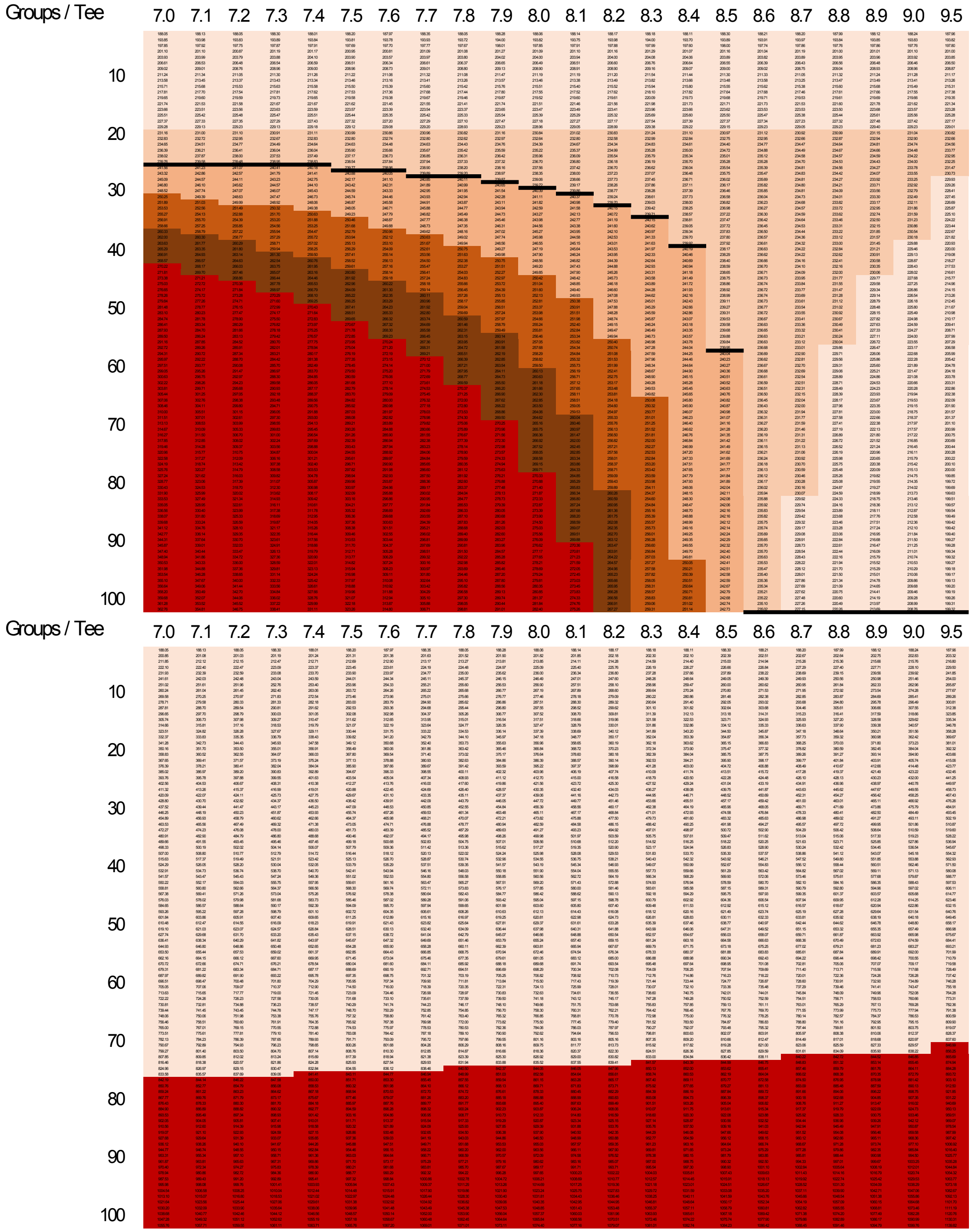

Figure 4: Sojourn times (above) and departure times (below) as a function of the group and the tee interval for a golf course with unscaled P3 holes, but now with uneven (two-level) tee times. More specifically, the tee times start off small at 7.00 for the first 20 groups, then increases after the 20th group; the tee times shown on the horizontal axis labels in the figure above denotes the tee times after the 20th group.

3.1.3 Case 3: $P3$ holes with alternative tee schedule

Finally, the third pair of figures in Figure 4 show average sojourn times (top) and departure times (bottom) of each group (averaged over `n_r` replications) as a function of the group number (y-axis) and the fixed tee interval (x-axis) for the base case with unscaled $P3$ holes – but now with alternative tee schedule, as discussed in §5 of the main paper. As discussed in the caption for Figure 4, the tee times are 7.00 for the first 20 groups, but increases for the later groups. Again, the base case refers to the hole sequence 454—434—454—434—454—434, as discussed in §3 of the main paper.

Similarly to Figures 2 and 3, the top and the bottom figures in Figure 4 are color-coded. The top figure shows 7 different colors to indicate different ranges of sojourn times, again indicating the same set of sojourn time ranges as indicated in the descriptions for two cases above. Sojourn times that appear below the black lines indicate that the constraint is violated. Similarly, in the bottom figure, the ivory and the red colors respectively indicate that the departure time occurs before and after 840 minutes, and the red color indicates that the departure-time constraint is violated.

Because the first 20 groups in Figure 4 play with tee interval of 7.00 (regardless of the tee interval for next 80 groups), we see that the sojourn times of the first 20 groups are the same across the x-axis. Then, the sojourn times for each y-axis fall into one of the two categories: (i) the sojourn times continue to increase as the golf course becomes over-loaded, or (ii) the sojourn times decrease as the course becomes under-loaded. In the top figure, when the tee interval (for the later 80 groups) is less than 8.0, no more than 27 groups can satisfy the sojourn time constraint, where as the top figure in Figure 2 (unscaled $P3$ holes without the alternative tee schedule) shows that 60 groups can satisfy the sojourn time constraint. While there appears to be a disadvantage in having the alternative tee schedule because the sojourn times appear to be higher in Figure 4 overall, the alternative schedule does realize higher throughput levels for $P3WU$ holes and $SP3$ holes. Please see §5 in the main paper for greater details on the impact of alternative tee schedule on throughput levels.

4 Extension of the Waiting Time Study

In this section, we provide additional results on comparing alternative course designs to supplement our findings in Section 3 of the main paper. In the main paper, we found that the order of the holes has minimal to no impact on the throughput levels of a course. However, the throughput levels vary depending on whether the course is overloaded or underloaded. More specifically, in Tables 1, 2, and 3 of the main paper, we showed that $P3WU$ holes performs substantially better than the $P3$ holes – the performance improvement is more evident when the tee interval is shorter, i.e., when the golf course is more heavily loaded. Then, we showed in Table 5 of the main paper that the average waiting time of a playing group decreases substantially as the tee increases to 8.50, but we also showed that $P3$ holes still remain as a bottleneck in a golf course.

Here, we extend our results over two sub-sections. The first sub-section (Section 4.1) provides detailed statistics of the average waiting times of the 75th group with the longer tee interval 8.50. The second sub-section (Section 4.2) provides these wait time statistics for groups other than the 75th group.

4.1 Additional Wait Time Statistics for Longer Tee Intervals

Table 1: Simulation estimates of the mean waiting times for group 75 in minutes before starting play on each of the 18 holes for three course designs with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, which makes all three cases under-loaded. The course designs are (i) the base case, (ii) the par-5-holes first and (iii) the par-3 holes first.

	base case				par-5 first				par-3 first			
hole	par	P3	P3WU	SP3	par	P3	P3WU	SP3	par	P3	P3WU	SP3
1	4	0.10	0.08	0.09	5	0.09	0.09	0.08	**3**	**5.78**	**0.76**	**0.29**
2	5	0.21	0.24	0.21	5	0.31	0.28	0.28	**3**	**7.24**	**1.28**	**0.56**
3	4	0.41	0.42	0.40	5	0.43	0.42	0.38	**3**	**6.46**	**1.77**	**0.77**
4	4	0.41	0.43	0.36	**3**	**8.39**	**1.16**	**0.91**	4	0.17	*1.44*	0.43
5	**3**	**8.50**	**1.14**	**0.87**	4	0.17	*1.39*	0.47	5	0.26	0.63	0.45
6	4	0.16	*1.22*	0.43	**3**	**8.16**	**1.30**	**0.98**	4	0.46	0.81	0.59
7	4	0.33	0.58	0.51	4	0.15	*1.34*	0.49	4	0.45	0.60	0.48
8	5	0.35	0.52	0.46	**3**	**7.36**	**1.36**	**1.03**	4	0.48	0.63	0.57
9	4	0.55	0.64	0.63	4	0.17	*1.33*	0.59	4	0.46	0.55	0.54
10	4	0.49	0.54	0.51	4	0.29	0.66	0.58	4	0.44	0.59	0.53
11	**3**	**9.42**	**1.33**	**1.07**	4	0.38	0.64	0.54	5	0.46	0.51	0.47
12	4	0.15	*1.41*	0.49	4	0.40	0.57	0.51	4	0.61	0.71	0.64
13	4	0.32	0.66	0.56	4	0.43	0.59	0.50	4	0.53	0.63	0.58
14	5	0.37	0.53	0.49	4	0.44	0.58	0.51	4	0.52	0.59	0.52
15	4	0.56	0.73	0.68	4	0.47	0.57	0.53	4	0.54	0.60	0.55
16	4	0.47	0.61	0.60	4	0.47	0.59	0.62	4	0.57	0.58	0.62
17	**3**	**8.51**	**1.30**	**1.19**	4	0.46	0.57	0.60	4	0.52	0.55	0.51
18	4	0.17	*1.35*	0.53	4	0.49	0.56	0.60	5	0.65	0.72	0.64
sum		31.47	13.72	10.04		29.05	14.02	10.21		26.39	13.94	9.73

First, we include the duplicate of the Table 4 in the main paper (labeled Table 1 in this Online Companion) for readers' reference. As described in the main paper, Table 1 provides simulation estimates of the mean waiting times for group 75 on each of the 18 holes. We provide the estimates for three types of course designs and for three types of $P3$ holes. As mentioned in the main paper, the main takeaway from Table 1 is that waiting times are substantially lower for a golf course with $P3$ holes when the course is under-loaded (e.g. when tee is 8.50).

Table 2: Simulation estimates of the proportion of waiting times for group 75 (in %) before starting play on each of the 18 holes for three course designs with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, under which all three cases are underloaded. The course designs are (i) the base case, (ii) the par-5-holes first and (iii) the par-3 holes first.

	base case				par-5 first				par-3 first			
hole	par	P3	P3WU	SP3	par	P3	P3WU	SP3	par	P3	P3WU	SP3
1	4	0.3%	0.6%	0.9%	5	0.3%	0.6%	0.8%	**3**	**21.9%**	**5.4%**	**3.0%**
2	5	0.7%	1.8%	2.1%	5	1.1%	2.0%	2.8%	**3**	**27.4%**	**9.2%**	**5.7%**
3	4	1.3%	3.0%	3.9%	5	1.5%	3.0%	3.8%	**3**	**24.5%**	**12.7%**	**7.9%**
4	4	1.3%	3.1%	3.6%	**3**	**28.9%**	**8.3%**	**8.9%**	4	0.6%	10.3%	4.4%
5	**3**	**27.0%**	**8.3%**	**8.6%**	4	0.6%	9.9%	4.6%	5	1.0%	4.5%	4.6%
6	4	0.5%	8.9%	4.0%	**3**	**28.1%**	**9.3%**	**9.6%**	4	1.7%	5.8%	6.1%
7	4	1.1%	4.2%	5.1%	4	0.5%	9.6%	4.8%	4	1.5%	4.3%	4.9%
8	5	1.1%	3.8%	4.5%	**3**	**25.3%**	**9.7%**	**10.1%**	4	1.7%	4.5%	5.9%
9	4	1.7%	4.7%	6.3%	4	0.6%	9.5%	5.8%	4	1.7%	3.9%	5.5%
10	4	1.6%	3.9%	5.1%	4	1.0%	4.7%	5.6%	4	1.9%	4.3%	5.4%
11	**3**	**29.9%**	**9.7%**	**10.7%**	4	1.3%	4.6%	5.2%	5	1.7%	3.7%	4.8%
12	4	0.5%	10.3%	4.8%	4	1.4%	4.1%	5.0%	4	2.2%	5.1%	6.5%
13	4	1.0%	4.8%	5.5%	4	1.5%	4.2%	4.9%	4	2.0%	4.5%	5.9%
14	5	1.2%	3.9%	4.9%	4	1.5%	4.1%	5.0%	4	1.8%	4.2%	5.4%
15	4	1.8%	5.3%	6.7%	4	1.6%	4.1%	5.2%	4	1.9%	4.3%	5.7%
16	4	1.5%	4.5%	6.0%	4	1.6%	4.2%	6.1%	4	2.1%	4.2%	6.3%
17	**3**	**27.0%**	**9.5%**	**11.8%**	4	1.6%	4.1%	5.9%	4	2.0%	3.9%	5.2%
18	4	0.6%	9.8%	5.3%	4	1.7%	4.0%	5.9%	5	2.5%	5.2%	6.6%
sum		100%	100%	100%		100%	100%	100%		100%	100%	100%

Tables 2 and 3 of the appendix provide detailed statistics in addition to Table 1 (i.e. Table 4 of the main paper). In fact, Tables 2 and 3 are analogs of Tables 2 and 3 of the main paper (please see §3.1 of the main paper in greater details).

Table 2 provides the corresponding estimated proportion of the total waiting time at each hole. The proportion in bold types indicate the proportion of each $P3$ hole with respect to the total waiting times. From the Table 2, we see that the proportion of waiting times of the 75th group is larger in the golf course with $P3$ holes than in the course with $P3WU$ or $SP3$ holes. More specifically, the three $P3$ holes together comprise of 83.9%, 82.3% and 72.3% of the total waiting times in the three golf course designs respectively. On the other hand, the combined proportion of waiting times for $P3WU$ holes are 27.6%, 27.3% and 27.3%, and those for $SP3$ holes are 32.1%, 29.6% and 16.6%. Consequently, our observation tells us that the $P3$ holes, despite the golf course being under-loaded, still remain as bottlenecks.

We further note that the waiting time proportions at the hole subsequent to $P3WU$ holes are higher than the proportions at the hole subsequent to $SP3$ holes. More specifically, the proportions at the hole after $P3WU$ holes are approximately 9 to 10 percent, while those at the hole after $SP3$ holes are 5.8% at most. The difference between the two can be primarily attributed to the difference between the number of people permitted to play at a $P3WU$ hole and at a $SP3$ hole. With the wave-up policy, two playing groups will be able to simultaneously play at $P3WU$ hole, making the interval between the two groups smaller than the interval at the $SP3$ hole. Subsequently, it is more likely that the distance between playing groups is shorter in $P3WU$ than in $SP3$, and such close distance may make it more likely for the group behind the other group to wait for a longer period of time.

Let us compare the Table 2 with its analog, Table 2 in the main paper. In Table 2 of the main paper, we see that the waiting time proportion of the first $P3$ hole is particularly high (i.e. 67.5%, 69.6% and 69.6% in the three course designs, respectively) compared to the proportion in the second and third $P3$ holes (ranges between 9.6% and 14.6%). On the other hand, in the Table 2 of the Online Companion, we see that the proportions are spread more evenly across the three $P3$ holes. Consequently, the first $P3$ hole alone becomes the primary bottleneck when the course is over-loaded but does not when the course is under-loaded.

We also compare the proportions of waiting times for $P3WU$ and $SP3$ holes. For both $P3WU$ and $SP3$ holes, we see that the proportions are spread evenly across the three holes in all three different course designs. We note that $P3WU$ and $SP3$ holes are not bottleneck holes because golf courses remain balanced even when the tee interval is 7.50.

In summary, Table 2 suggests that setting the tee interval to 8.50 relieves the congestion in golf courses with $P3$ holes but makes relatively small impact on golf courses with $P3WU$ or $SP3$ holes. Such finding is corroborated by (i) high waiting time proportions of $P3$ holes, and (ii) concentrated proportion in the first $P3$ hole when the tee interval is 7.50.

Table 3: Simulation estimates of the standard deviations of the waiting times before starting to play on each hole for group 75 for each of the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, under which all courses are stable. The course designs are (i) the base case, (ii) the par-5-holes first and (iii) the par-3 holes first. The last two rows give simulation estimates for the standard deviation of the sum of all waiting times and the half width of the 95% confidence interval for the mean.

	base case				par-5 first				par-3 first			
hole	par	P3	P3WU	SP3	par	P3	P3WU	SP3	par	P3	P3WU	SP3
1	4	0.39	0.38	0.41	5	0.45	0.40	0.36	**3**	5.77	0.92	0.99
2	5	0.68	0.71	0.70	5	0.82	0.81	0.80	**3**	6.69	1.57	1.44
3	4	0.97	0.96	1.04	5	0.99	1.03	1.00	**3**	6.28	2.07	1.62
4	4	0.88	0.97	1.12	**3**	6.85	1.39	1.76	4	0.55	2.23	1.03
5	**3**	6.89	1.43	1.71	4	0.59	2.06	1.01	5	0.79	1.28	0.93
6	4	0.64	1.91	1.02	**3**	6.84	1.51	1.75	4	1.21	1.43	1.26
7	4	0.86	1.18	1.09	4	0.56	2.08	1.17	4	1.11	1.33	1.21
8	5	0.99	1.12	1.11	**3**	6.73	1.68	1.91	4	1.07	1.22	1.16
9	4	1.10	1.44	1.31	4	0.64	2.15	1.17	4	1.13	1.28	1.07
10	4	1.04	1.20	1.19	4	0.85	1.33	1.02	4	1.11	1.28	1.13
11	**3**	7.17	1.46	1.99	4	0.95	1.27	1.20	5	1.08	1.13	1.09
12	4	0.59	2.14	1.20	4	0.96	1.27	1.22	4	1.24	1.49	1.34
13	4	0.82	1.30	1.18	4	1.01	1.30	1.15	4	1.19	1.24	1.26
14	5	0.94	1.26	1.09	4	1.10	1.28	1.25	4	1.21	1.25	1.22
15	4	1.11	1.49	1.32	4	1.09	1.26	1.19	4	1.10	1.31	1.15
16	4	1.08	1.41	1.22	4	1.11	1.24	1.28	4	1.16	1.28	1.19
17	**3**	7.06	1.64	2.01	4	1.15	1.20	1.22	4	1.10	1.14	1.20
18	4	0.62	2.12	1.19	4	1.16	1.18	1.19	5	1.36	1.37	1.33
sum		9.80	6.19	6.05		9.86	6.59	6.00		9.61	6.81	5.85
HW		0.43	0.27	0.27		0.43	0.29	0.26		0.42	0.30	0.26

Table 3 provides simulation estimates of the standard deviations of the waiting times of the 75th group; similarly as previous two tables Tables 2 and 1, the estimates are shown for three types of Par-3 holes and three types of course designs.

In Table 3, standard deviation estimates for each of $P3$, $P3WU$ and $SP3$ holes all exhibit different behaviors. For $P3$ holes, we see that the standard deviations are noticeably higher than those for $P4$ and $P5$ holes. One reason that may explain such wider spread in waiting times at $P3$ holes is the "snowball" effect of the waiting times; that is, the waiting times caused by the slow pace of play of the earlier groups will likely grow larger for later groups (i.e. groups with larger group number). In other words, the slower the earlier groups play, the more the 75th groups wait. On the other hand, if the earlier groups play quickly, the waiting times of the 75th group will be substantially reduced. Our observation holds for all three course designs.

For $P3WU$ holes, the standard deviations are not the highest at the $P3WU$ holes, but at the holes after the $P3WU$ holes. For $SP3$ holes, similarly as $P3$ holes, standard deviations are somewhat higher at Par-3 holes, but not noticeably higher than other holes – and even lower for certain instances. For instance, we see that the estimate for the $SP3$ hole in the "Par-3 first" course design is 0.99, but the estimates for other holes in the course (except the 5th Par-5 hole) are higher than 1.

Finally, we show confidence intervals (95%; two-sided) of the (i) individual hole waiting times and (ii) total waiting times in Tables 4, 5 and 6. In all of the three tables, the tee interval is 8.50 minutes, and the confidence intervals are shown for each of the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$.

Table 4: Confidence interval (95%; two-sided) of the waiting times before starting to play on each hole for group 75 for each of the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, under which all courses are stable. The course design used in this table is "the base case". The lower and the upper confidence intervals for the total waiting times are located in the last row.

	base case						
hole	par	P3		P3WU		SP3	
		Lower	Upper	Lower	Upper	Lower	Upper
1	4	0.08	0.12	0.06	0.10	0.07	0.11
2	5	0.18	0.24	0.21	0.27	0.18	0.24
3	4	0.36	0.45	0.38	0.46	0.35	0.45
4	4	0.37	0.45	0.39	0.47	0.31	0.41
5	**3**	8.20	8.80	1.08	1.20	0.80	0.94
6	4	0.13	0.19	1.14	1.30	0.39	0.47
7	4	0.29	0.37	0.53	0.63	0.46	0.56
8	5	0.31	0.39	0.47	0.57	0.41	0.51
9	4	0.50	0.60	0.58	0.70	0.57	0.69
10	4	0.44	0.54	0.49	0.59	0.46	0.56
11	**3**	9.11	9.73	1.27	1.39	0.98	1.16
12	4	0.12	0.18	1.32	1.50	0.44	0.54
13	4	0.28	0.36	0.60	0.72	0.51	0.61
14	5	0.33	0.41	0.48	0.59	0.44	0.54
15	4	0.51	0.61	0.67	0.80	0.62	0.74
16	4	0.42	0.52	0.55	0.67	0.55	0.65
17	**3**	8.20	8.82	1.23	1.37	1.10	1.28
18	4	0.14	0.20	1.26	1.44	0.48	0.58
Sum		31.04	31.90	13.45	13.99	9.77	10.31

Table 5: Confidence Interval (95%; two-sided) of the waiting times before starting to play on each hole for group 75 for each of the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, under which all courses are stable. The course design used in this table is "par-5 first". The lower and the upper confidence intervals for the total waiting times are located in the last row.

	par-5 first						
hole	par	P3		P3WU		SP3	
		Lower	Upper	Lower	Upper	Lower	Upper
1	5	0.07	0.11	0.07	0.11	0.06	0.10
2	5	0.27	0.35	0.24	0.32	0.24	0.32
3	5	0.39	0.47	0.37	0.47	0.34	0.42
4	**3**	8.09	8.69	1.10	1.22	0.83	0.99
5	4	0.14	0.20	1.30	1.48	0.43	0.51
6	**3**	7.86	8.46	1.23	1.37	0.90	1.06
7	4	0.13	0.17	1.25	1.43	0.44	0.54
8	**3**	7.07	7.65	1.29	1.43	0.95	1.11
9	4	0.14	0.20	1.24	1.42	0.54	0.64
10	4	0.25	0.33	0.60	0.72	0.54	0.62
11	4	0.34	0.42	0.58	0.70	0.49	0.59
12	4	0.36	0.44	0.51	0.63	0.46	0.56
13	4	0.39	0.47	0.53	0.65	0.45	0.55
14	4	0.39	0.49	0.52	0.64	0.46	0.56
15	4	0.42	0.52	0.51	0.63	0.48	0.58
16	4	0.42	0.52	0.54	0.64	0.56	0.68
17	4	0.41	0.51	0.52	0.62	0.55	0.65
18	4	0.44	0.54	0.51	0.61	0.55	0.65
Sum		28.62	29.48	13.73	14.31	9.95	10.47

4.2 Comparing Waiting Times of Other Groups

In the main paper, we showed that using P3 holes with the wave-up rule significantly reduces the waiting times of the playing group. To illustrate this advantage, we displayed the waiting times of the 75th group as one example. We now expand the results to include waiting times of other groups to show that all of the other playing groups – not just the 75th group – also greatly benefit from the wave-up advantage.

Tables 7 and 8 show the average waiting times and proportion of (waiting times at the Par-3 hole to) total waiting times spent on Par-3 holes with tee 7.50, while Tables 9 and 10 show those with tee 8.50. Since the critical tee interval τ^* is 8.39, 6.53, and 6.79 for P3, P3WU and SP3, respectively, the tee interval of 7.50 minutes will make P3 over-loaded (but not P3WU and SP3), while the tee interval of 8.50 minutes will make all cases under-loaded. Base case ("4 5 4" case) is used as the hole sequence for all results.

Table 6: Confidence Interval (95%; two-sided) of the waiting times before starting to play on each hole for group 75 for each of the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, under which all courses are stable. The course design used in this table is “par-3 first”. The lower and the upper confidence intervals for the total waiting times are located in the last row.

	par-3 first						
hole	par	P3		P3WU		SP3	
		Lower	Upper	Lower	Upper	Lower	Upper
1	**3**	5.53	6.03	0.72	0.80	0.25	0.33
2	**3**	6.95	7.53	1.21	1.35	0.50	0.62
3	**3**	6.18	6.74	1.68	1.86	0.70	0.84
4	4	0.15	0.19	1.34	1.54	0.38	0.48
5	5	0.23	0.29	0.57	0.69	0.41	0.49
6	4	0.41	0.51	0.75	0.87	0.53	0.65
7	4	0.40	0.50	0.54	0.66	0.43	0.53
8	4	0.43	0.53	0.58	0.68	0.52	0.62
9	4	0.41	0.51	0.49	0.61	0.49	0.59
10	4	0.39	0.49	0.53	0.65	0.48	0.58
11	5	0.41	0.51	0.46	0.56	0.42	0.52
12	4	0.56	0.66	0.64	0.78	0.58	0.70
13	4	0.48	0.58	0.58	0.68	0.52	0.64
14	4	0.47	0.57	0.54	0.64	0.47	0.57
15	4	0.49	0.59	0.54	0.66	0.50	0.60
16	4	0.52	0.62	0.52	0.64	0.57	0.67
17	4	0.47	0.57	0.50	0.60	0.46	0.56
18	5	0.59	0.71	0.66	0.78	0.58	0.70
Sum		25.97	26.81	13.64	14.24	9.47	9.99

Table 7: Simulation estimates of the mean waiting times for a range of different playing groups (e.g. 5, 10, 15, ...) before starting play on each of the Par-3 holes with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 7.50 minutes, which makes P3 over-loaded, but not P3WU and SP3.

	base case - P3				base case - P3WU				base case - SP3			
group	1st	2nd	3rd	**3-hole Sum**	1st	2nd	3rd	**3-hole Sum**	1st	2nd	3rd	**3-hole Sum**
5	3.7	2.2	1.7	**7.6**	1.4	1.0	0.9	**2.6**	1.0	0.9	0.7	**3.2**
10	8.1	4.1	3.2	**15.4**	1.7	1.5	1.3	**4.2**	1.6	1.4	1.2	**4.5**
15	12.6	5.6	4.2	**22.4**	1.9	1.7	1.5	**5.3**	2.0	1.8	1.5	**5.1**
20	17.1	6.8	5.2	**29.1**	2.0	2.0	1.7	**5.9**	2.1	2.1	1.7	**5.7**
25	21.5	7.7	5.8	**35.0**	2.0	2.0	1.8	**6.4**	2.3	2.2	2.0	**5.8**
30	25.9	8.7	6.5	**41.1**	2.0	2.0	1.9	**6.9**	2.4	2.3	2.2	**5.9**
35	30.3	9.4	7.2	**47.0**	2.0	2.0	1.9	**7.0**	2.3	2.5	2.2	**6.0**
40	34.8	10.1	7.7	**52.5**	2.0	2.1	2.0	**7.3**	2.3	2.6	2.4	**6.0**
45	39.2	10.8	8.4	**58.4**	2.0	2.2	2.1	**7.7**	2.4	2.8	2.5	**6.3**
50	43.6	11.4	8.6	**63.6**	2.0	2.1	2.2	**7.6**	2.3	2.8	2.4	**6.3**
55	48.0	12.1	9.1	**69.2**	2.0	2.1	2.1	**7.5**	2.4	2.7	2.4	**6.2**
60	52.6	12.8	9.5	**74.9**	2.0	2.0	2.2	**7.6**	2.4	2.6	2.6	**6.3**
65	57.0	13.3	10.0	**80.3**	1.9	2.1	2.1	**7.8**	2.2	2.8	2.8	**6.1**
70	61.4	13.9	10.0	**85.4**	1.9	2.1	2.2	**7.8**	2.3	2.8	2.6	**6.2**
75	65.0	14.3	10.4	**89.7**	1.9	2.1	2.1	**7.5**	2.2	2.7	2.6	**6.2**
80	70.7	14.6	10.9	**96.2**	1.9	2.1	2.2	**7.6**	2.3	2.7	2.6	**6.3**
85	75.1	15.2	11.2	**101.5**	2.0	2.2	2.3	**7.7**	2.3	2.7	2.6	**6.5**
90	79.7	15.6	11.5	**106.8**	2.0	2.1	2.2	**7.8**	2.3	2.8	2.7	**6.4**
95	84.1	15.9	11.8	**111.8**	2.0	2.2	2.2	**7.6**	2.2	2.7	2.8	**6.4**
100	88.5	16.4	12.1	**117.0**	2.0	2.2	2.1	**7.6**	2.2	2.7	2.7	**6.3**

Table 7 shows the average waiting times spent on three types of Par-3 holes, $P3$, $P3WU$ and $SP3$, with tee 7.50. Since the critical tee interval τ^* is 8.39, 6.53, and 6.79 for these holes respectively, the tee interval of 7.50 minutes will make $P3$ over-loaded (but not P3WU and SP3), while the tee interval of 8.50 minutes will make all cases under-loaded. Base case ("4 5 4" case) is used as the hole sequence for all results.

Table 7 provides simulation estimates of the mean waiting times for a number of groups, from the 5-th group to the 100-th group (i.e., the last group). For the purpose of brevity, Table 7 (i) displays waiting times for the base case only, and (ii) does not display waiting times for all 100 groups; Table 7 shows waiting times for every 5th group from the 5-th group to the 100-th group, skipping the groups in between. However, since we see that the waiting times increase monotonically as the group number increases, we can roughly interpolate the estimates for the groups not shown in the table.

As shown in Table 1 of the main paper, the waiting times at $P3$ holes are noticeably higher than the waiting times at $P3WU$ or $SP3$ holes. In addition, the gaps between the waiting times at $P3$ holes and those at $P3WU$ or $SP3$ holes widen as the group number increases. In particular, we even see that the 100-th group (i.e. last group) waits for almost 2 hours.

With $P3$ holes, we see that the waiting times continue to increase until the 100-th group without converging. The waiting times with $P3$ holes are increasing and concave. On the other hand, with $P3WU$ and $SP3$ holes, we see that the waiting times converge to roughly 7.6 and 6.3 minutes (with minor variations), respectively, approximately at the 45-th group. The convergence that emerges in the base case with $P3WU$ and $SP3$ holes explains that these two courses are stable in terms of traffic, unlike the base case with $P3$ holes.

Table 8: Simulation estimates of the proportion (in %) of the waiting times at Par-3 holes (to the waiting time of the entire 18-hole course) for a range of different playing groups (e.g. 5, 10, 15, ...) before starting play on each of the Par-3 holes with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 7.50 minutes, which makes P3 over-loaded, but not P3WU and SP3.

	base case - P3 (%)			base case - P3WU (%)			base case - SP3 (%)		
group	3-hole Sum	18-hole Sum	%	3-hole Sum	18-hole Sum	%	3-hole Sum	18-hole Sum	%
5	7.6	12.4	**61%**	2.6	11.2	**28%**	3.2	9.4	**27%**
10	15.4	21.7	**71%**	4.2	17.2	**26%**	4.5	14.4	**29%**
15	22.4	28.9	**78%**	5.3	20.0	**26%**	5.1	17.3	**31%**
20	29.1	36.0	**81%**	5.9	21.7	**26%**	5.7	19.1	**31%**
25	35.0	41.9	**84%**	6.4	22.9	**25%**	5.8	20.1	**32%**
30	41.1	47.9	**86%**	6.9	23.5	**25%**	5.9	21.2	**32%**
35	47.0	53.9	**87%**	7.0	23.9	**25%**	6.0	21.5	**33%**
40	52.5	59.7	**88%**	7.3	24.2	**25%**	6.0	22.2	**33%**
45	58.4	65.3	**89%**	7.7	24.9	**25%**	6.3	22.7	**34%**
50	63.6	70.6	**90%**	7.6	25.2	**25%**	6.3	22.5	**34%**
55	69.2	76.2	**91%**	7.5	24.9	**25%**	6.2	22.8	**33%**
60	74.9	81.9	**91%**	7.6	24.9	**25%**	6.3	23.3	**33%**
65	80.3	87.3	**92%**	7.8	24.8	**25%**	6.1	23.3	**33%**
70	85.4	92.3	**93%**	7.8	24.8	**25%**	6.2	23.1	**34%**
75	90.7	97.7	**93%**	7.5	24.9	**25%**	6.2	23.1	**32%**
80	96.2	103.3	**93%**	7.6	24.8	**25%**	6.3	23.3	**33%**
85	101.5	108.3	**94%**	7.7	24.9	**26%**	6.5	23.1	**33%**
90	106.8	113.8	**94%**	7.8	25.0	**25%**	6.4	23.1	**34%**
95	111.8	118.7	**94%**	7.6	25.2	**25%**	6.4	23.2	**33%**
100	117.0	123.8	**94%**	7.6	25.1	**25%**	6.3	23.0	**33%**

Table 8 shows the proportion of waiting times spent at the Par-3 holes to the total waiting times of each group. Again, for the purpose of brevity, the table only shows waiting time proportions for every 5th group for the base case only. Similarly as Table 7, because the proportions for all three cases are either increases as the group number increases (for $P3$ holes) or converges (for $P3WU$ and $SP3$ holes), we can roughly interpolate the proportions for the groups not displayed in the table.

With $P3$ holes, the proportion of the waiting times ranges from 61% (5th group) to 94% (100th group). For the 20th group and beyond, we see that the waiting times at $P3$ holes account for more than 80% of the total waiting times, and for the 50th group and beyond, we see that the waiting times at $P3$ holes account for more than 90% of the total. Overall, Table 8 shows that the impact of $P3$ holes is significant across all groups.

On the other hand, we see that the proportion of waiting times remain constant across all groups for $P3WU$ and $SP3$ holes. The proportion ranges from 25% to 28% for $P3WU$ holes, and from 27% to 34% for $SP3$ holes. In addition, we see that the proportion of waiting times is lower at $P3WU$ holes than at $SP3$ holes. The primary reason is that, although the waiting times of the playing groups at other holes (e.g. $P4$, $P5$) are roughly the same for golf courses with $P3WU$ holes and those with $SP3$ holes, the waiting times at holes subsequent to $P3WU$ holes tend to be higher than those at holes subsequent to $SP3$ holes.

Table 9: Simulation estimates of the mean waiting times for a range of different playing groups (e.g. 5, 10, 15, ...) before starting play on each of the Par-3 holes with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, which makes all three cases ($P3$, $P3WU$, and $SP3$) under-loaded.

	base case - P3				base case - P3WU				base case - SP3			
group	1st	2nd	3rd	**3-hole Sum**	1st	2nd	3rd	**3-hole Sum**	1st	2nd	3rd	**3-hole Sum**
5	2.5	1.9	1.5	**5.8**	1.0	0.9	0.8	**2.8**	0.7	0.7	0.6	**2.0**
10	4.0	3.4	2.6	**9.9**	1.1	1.2	1.1	**3.4**	0.8	1.0	0.9	**2.7**
15	4.9	4.2	3.7	**12.8**	1.1	1.3	1.3	**3.7**	0.9	1.1	1.0	**3.0**
20	5.7	5.1	4.4	**15.1**	1.1	1.3	1.3	**3.7**	0.9	1.2	1.1	**3.2**
25	6.2	5.8	4.9	**16.8**	1.2	1.2	1.3	**3.8**	0.9	1.1	1.2	**3.1**
30	6.5	6.2	5.3	**18.0**	1.2	1.3	1.4	**3.9**	0.9	1.0	1.1	**3.0**
35	6.9	6.6	5.9	**19.5**	1.1	1.3	1.3	**3.6**	0.9	1.0	1.1	**3.1**
40	7.2	7.1	6.3	**20.7**	1.2	1.3	1.4	**3.8**	0.9	1.0	1.1	**3.1**
45	7.5	7.7	6.7	**22.0**	1.1	1.3	1.4	**3.8**	0.9	1.1	1.1	**3.1**
50	7.7	8.0	7.1	**22.8**	1.1	1.3	1.3	**3.7**	0.9	1.0	1.2	**3.0**
55	7.7	8.4	7.3	**23.5**	1.2	1.3	1.4	**3.9**	0.8	1.0	1.1	**3.0**
60	8.0	8.4	7.7	**24.1**	1.2	1.4	1.3	**3.8**	0.9	1.0	1.2	**3.1**
65	8.3	8.5	8.1	**25.0**	1.1	1.3	1.3	**3.8**	0.9	1.0	1.1	**3.0**
70	8.6	8.9	8.1	**25.6**	1.1	1.3	1.4	**3.8**	0.9	1.0	1.1	**3.1**
75	8.7	9.1	8.5	**26.3**	1.2	1.2	1.4	**3.8**	0.9	1.1	1.1	**3.1**
80	8.8	9.4	8.7	**26.9**	1.1	1.4	1.4	**3.9**	0.9	1.1	1.1	**3.1**
85	8.8	9.6	8.9	**27.3**	1.2	1.3	1.4	**3.8**	0.9	1.2	1.1	**3.2**
90	9.1	9.9	9.0	**27.9**	1.1	1.3	1.4	**3.7**	0.9	1.1	1.1	**3.1**
95	9.1	10.1	9.2	**28.4**	1.2	1.3	1.4	**3.8**	0.9	1.1	1.1	**3.1**
100	9.2	10.4	9.4	**28.9**	1.2	1.3	1.3	**3.7**	0.9	1.1	1.1	**3.1**

Similarly, we also show the results of our waiting time study with a larger tee time interval, 8.50, in Tables 9 and 10. Table 9 shows the waiting times of different playing groups at Par-3 holes (same list of groups as in Table 7). The results shown below also reflect the waiting time study results for the base case ("4 5 4" case).

When we compare Table 7 with Table 9, we notice that the waiting times of the playing group decrease significantly—especially for $P3$ holes. For instance, we see that the respective waiting times at $P3$ holes for groups 25, 50 and 100 decrease from 35.0, 63.6 and 117.0 minutes (in Table 7) to 16.8, 22.8 and 28.9 minutes (in Table 7). These translate to roughly 51%, 64% and 77% decrease in percentage points, respectively. On the other hand, we see that the waiting times at $P3WU$ holes for groups 25, 50 and 100 decrease only slightly, from 6.4, 7.6 and 7.6 minutes to 3.8, 3.7 and 3.7 minutes. Similarly, the waiting times at $SP3$ holes also undergo only a slight decrease. We note that, however, the waiting times with $P3$ holes still remain substantially higher than waiting times with $P3WU$ or $SP3$ holes.

Table 10: Simulation estimates of the proportion (in %) of the waiting times at Par-3 holes (to the waiting time of the entire 18-hole course) for a range of different playing groups (e.g. 5, 10, 15, ...) before starting play on each of the Par-3 holes with the three kinds of par-3 holes: $P3$, $P3WU$ and $SP3$. In all cases the tee interval is 8.50 minutes, which makes all three cases ($P3$, $P3WU$, and $SP3$) under-loaded.

	base case - P3 (%)			base case - P3WU (%)			base case - SP3 (%)		
group	3-hole Sum	18-hole Sum	%	3-hole Sum	18-hole Sum	%	3-hole Sum	18-hole Sum	%
5	5.8	9.3	**63%**	2.8	9.2	**30%**	2.0	7.0	**29%**
10	9.9	14.3	**69%**	3.4	12.4	**28%**	2.7	9.1	**30%**
15	12.8	17.4	**74%**	3.7	13.5	**27%**	3.0	9.8	**31%**
20	15.1	19.7	**77%**	3.7	13.6	**27%**	3.2	10.2	**31%**
25	16.8	21.7	**78%**	3.8	13.7	**27%**	3.1	10.1	**31%**
30	18.0	22.8	**79%**	3.9	13.9	**28%**	3.0	9.9	**30%**
35	19.5	24.2	**80%**	3.6	13.5	**27%**	3.1	10.0	**31%**
40	20.7	25.6	**81%**	3.8	13.7	**28%**	3.1	10.1	**31%**
45	22.0	27.1	**81%**	3.8	13.9	**28%**	3.1	9.9	**31%**
50	22.8	27.8	**82%**	3.7	13.8	**27%**	3.0	10.1	**30%**
55	23.5	28.5	**82%**	3.9	13.9	**28%**	3.0	9.8	**30%**
60	24.1	29.0	**83%**	3.8	13.8	**28%**	3.1	10.0	**31%**
65	25.0	29.9	**83%**	3.8	13.8	**27%**	3.0	9.8	**31%**
70	25.6	30.6	**84%**	3.8	13.8	**27%**	3.1	9.8	**32%**
75	26.3	31.2	**84%**	3.8	14.0	**27%**	3.1	10.1	**31%**
80	26.9	31.8	**85%**	3.9	14.2	**28%**	3.1	10.1	**31%**
85	27.3	32.3	**85%**	3.8	14.0	**27%**	3.2	10.0	**32%**
90	27.9	32.8	**85%**	3.7	13.7	**27%**	3.1	10.0	**31%**
95	28.4	33.5	**85%**	3.8	13.7	**28%**	3.1	10.0	**31%**
100	28.9	34.0	**85%**	3.7	13.5	**28%**	3.1	10.3	**31%**

On the other hand, Tables 8 and 10 suggest that there is little to no difference between proportion of waiting times. From Table 8, we see that the waiting time proportions for groups 25, 50 and 100 are 84%, 90% and 94%, respectively; from Table 10, we see that the proportions are 78%, 82% and 85%. For $P3WU$ and $SP3$ holes, when the tee interval is 7.50, we pointed out earlier that the proportion for $P3WU$ holes ranges from 25 to 28%, and the proportion for $SP3$ holes ranges from 27 to 34%. When the tee interval is 8.50, as is the case in Table 10, the proportion of waiting times for $P3WU$ ranges from 27 to 30%, and the proportion for $SP3$ holes ranges from 29 to 31%, suggesting that there is no significance in proportion of waiting times in Tables 8 and 10.

Overall, comparing Tables 9 and 10 with their earlier analogs Tables 7 and 8, we se a decrease in waiting times of the playing groups for all three types of Par-3 holes when the tee interval becomes longer. The difference is especially stark for $P3$ holes (and becomes more noticeable as the group

Group / Par #	4	5	4	4	3	4	4	5	4	4	3	4	4	5	4	4	3	4	Sum
005	0.3	0.5	0.7	0.6	3.7	0.2	0.3	0.3	0.5	0.4	2.2	0.1	0.2	0.2	0.3	0.3	1.7	0.1	12.4
010	0.3	0.6	1.0	0.9	8.1	0.2	0.3	0.4	0.5	0.4	4.1	0.2	0.3	0.3	0.4	0.4	3.2	0.1	21.7
015	0.3	0.6	0.9	0.8	12.6	0.2	0.3	0.4	0.6	0.5	5.6	0.2	0.3	0.3	0.5	0.4	4.2	0.1	28.9
020	0.3	0.6	1.0	0.9	17.1	0.2	0.3	0.4	0.6	0.5	6.8	0.2	0.3	0.4	0.5	0.5	5.2	0.2	36.0
025	0.3	0.6	1.0	0.9	21.5	0.2	0.3	0.4	0.6	0.5	7.7	0.2	0.3	0.4	0.5	0.5	5.8	0.1	41.9
030	0.4	0.6	1.0	0.9	25.9	0.2	0.3	0.4	0.6	0.5	8.7	0.2	0.3	0.4	0.5	0.5	6.5	0.2	47.9
035	0.3	0.6	1.0	1.0	30.3	0.2	0.3	0.3	0.6	0.5	9.4	0.2	0.3	0.4	0.5	0.5	7.2	0.2	53.9
040	0.3	0.6	1.0	1.0	34.8	0.2	0.4	0.4	0.6	0.5	10.1	0.2	0.3	0.4	0.5	0.5	7.7	0.2	59.7
045	0.3	0.6	1.0	0.9	39.2	0.2	0.4	0.4	0.6	0.5	10.8	0.1	0.3	0.4	0.5	0.5	8.4	0.2	65.3
050	0.3	0.6	1.0	1.0	43.6	0.2	0.4	0.4	0.6	0.5	11.4	0.2	0.3	0.4	0.6	0.5	8.6	0.1	70.6
055	0.3	0.6	1.0	0.9	48.0	0.2	0.3	0.4	0.6	0.6	12.1	0.2	0.4	0.4	0.5	0.5	9.1	0.2	76.2
060	0.3	0.6	0.9	0.9	52.6	0.2	0.3	0.4	0.6	0.5	12.8	0.2	0.4	0.4	0.6	0.5	9.5	0.2	81.9
065	0.3	0.6	1.0	0.9	57.0	0.2	0.3	0.4	0.6	0.5	13.3	0.2	0.3	0.4	0.5	0.5	10.0	0.2	87.3
070	0.3	0.6	1.0	1.0	61.4	0.2	0.3	0.4	0.6	0.5	13.9	0.2	0.3	0.4	0.6	0.4	10.0	0.2	92.3
075	0.3	0.6	0.9	1.0	66.0	0.2	0.3	0.4	0.6	0.5	14.3	0.2	0.3	0.4	0.6	0.5	10.4	0.2	97.7
080	0.3	0.6	1.0	1.0	70.7	0.2	0.4	0.4	0.5	0.5	14.6	0.2	0.3	0.4	0.6	0.5	10.9	0.2	103.3
085	0.3	0.6	1.0	1.0	75.1	0.2	0.3	0.4	0.5	0.5	15.2	0.2	0.3	0.4	0.5	0.5	11.2	0.2	108.3
090	0.3	0.6	1.0	0.9	79.7	0.2	0.4	0.4	0.6	0.5	15.6	0.2	0.3	0.4	0.5	0.5	11.5	0.2	113.8
095	0.3	0.6	0.9	0.9	84.1	0.2	0.4	0.4	0.6	0.5	15.9	0.2	0.3	0.4	0.6	0.5	11.8	0.2	118.7
100	0.3	0.6	1.0	0.9	88.5	0.2	0.3	0.4	0.6	0.5	16.4	0.2	0.3	0.4	0.6	0.5	12.1	0.2	123.8

Figure 5: Above figure shows waiting times of different playing groups at different holes. $P3$ holes are used, with the base case chosen as the course design for this particular case. The redder the colors are, the higher the waiting times are.

number increases), but not so much for $P3WU$ and $SP3$ holes. On the other hand, we see that there is little to no difference in proportion of waiting times for all three types of Par-3 holes, implying that Par-3 holes, regardless of the type, still remain as a bottleneck.

Finally, Figure 5 presents a visual representation of the waiting times of different playing groups, with unscaled $P3$ holes on the base case. Figure 5 suggests that a substantial proportion of the total waiting times is concentrated on the three Par-3 holes—and more specifically on the first of the three Par-3 holes. Furthermore, Figure 5 also suggests that, even though the three par-3 holes shown in the figure all have equal capacity, the waiting times gradually thin out in the later par-3 holes. This kind of behavior is exactly what is expected with 3 bottleneck holes in standard queues in series.

5 Extension of the Simulation Optimization Study

In this section, we extend the results from Table 5 in §4 of the main paper, Simulation Optimization of the Tee Schedule.

Table 11: the maximum number of groups that can play each day as a function of (i) the tee interval on the first hole, τ (ii) the hole order, and (ii) the type of par-3 hole used. Two hole orders are used: the "base case", and the "par-3 first" case. The optimal tee intervals are in bold, while the critical tee interval τ^* from Whitt (2015) for that kind of par-3 hole is shown at the bottom.

	throughput for base case			throughput for "3 3 3" case		
tee interval	P3	P3WU	SP3	P3	P3WU	SP3
5.00	11	10	16	11	10	16
5.50	13	12	19	13	12	20
6.00	15	15	26	15	15	26
6.50	18	21	41	19	21	40
7.00	23	42	**87**	23	41	**87**
7.10	24	63	**87**	25	62	**87**
7.20	25	**84**	86	27	**84**	**87**
7.30	27	**84**	86	29	**84**	86
7.40	30	83	85	31	83	86
7.50	33	82	85	33	82	85
7.60	34	82	84	36	82	84
7.70	37	81	83	40	81	83
7.80	41	80	82	45	80	82
7.90	46	79	81	51	79	81
8.00	53	78	81	59	78	81
8.10	62	78	80	71	78	80
8.20	**74**	77	79	**74**	77	79
8.30	**74**	76	78	**74**	76	78
8.40	**74**	75	77	**74**	76	77
8.50	**74**	75	76	**74**	75	76
9.00	71	71	72	71	71	72
9.50	68	68	69	68	68	69
τ^*	8.39	6.53	6.79	8.39	6.53	6.79

5.1 Comparing Throughput Levels

Here, Table 11 is the duplicate of Table 5 in §4.1 of the main paper, and Table 12 is an extension of the results in Table 5 in §4.1 of the main paper. Similarly as Table 11, Table 12 in this companion shows maximum number of groups (i.e. throughput) that can play each day as a function of (i) the tee interval on the first hole, τ, (ii) the hole order and (iii) the type of Par-3 hole used. In Table 11, we show throughput levels for two course designs, namely "base case" and "par-3 first case".

Table 12: the maximum number of groups that can play each day as a function of (i) the tee interval on the first hole, τ (ii) the hole order, and (ii) the type of Par-3 hole used. Three hole orders are used: “5 5 5 case”, “3 3 3 mid” case, and “3 3 3 end” case. The optimal tee intervals are in bold, while the critical tee interval τ^* from Whitt (2015) for that kind of par-3 hole is shown at the bottom.

	throughput for “5 5 5” case			“3 3 3”-mid case			“3 3 3”-end case		
tee interval	P3	P3WU	SP3	P3	P3WU	SP3	P3	P3WU	SP3
5.00	12	10	16	12	10	16	11	11	16
5.50	13	12	20	13	12	19	13	13	19
6.00	15	15	26	16	15	26	15	16	26
6.50	19	21	39	19	21	39	18	22	39
7.00	24	42	**87**	24	42	**87**	23	49	**87**
7.10	25	63	**87**	25	66	**87**	24	**85**	**87**
7.20	26	**84**	86	27	**84**	86	26	**85**	86
7.30	28	**84**	86	29	**84**	86	28	84	86
7.40	30	83	85	31	83	85	30	83	85
7.50	33	82	84	33	82	85	32	83	85
7.60	35	82	84	36	82	84	35	82	84
7.70	39	81	83	40	81	83	38	81	83
7.80	43	80	82	44	80	82	43	80	82
7.90	49	79	81	50	79	81	48	79	81
8.00	56	78	80	58	78	81	57	78	81
8.10	67	78	80	68	78	80	65	78	80
8.20	**74**	77	79	**74**	77	79	**74**	77	79
8.30	**74**	76	78	**74**	76	78	**74**	76	78
8.40	**74**	75	77	**74**	75	77	**74**	76	77
8.50	**74**	75	76	**74**	75	76	**74**	75	76
9.00	71	71	72	71	71	72	71	71	72
9.50	68	68	69	68	67	69	68	68	69
τ^*	8.39	6.53	6.79	8.39	6.53	6.79	8.39	6.53	6.79

In Table 12, we show the throughput results for three additional course designs: “5 5 5 case”, “3 3 3 mid case” and “3 3 3 end case”.

The hole sequence for these three designs, as introduced in Fig 3 of the main paper, are:

“5 5 5” case: 555 – 343 – 434 – 444 – 444 – 444

“3 3 3 mid” case: 444 – 454 – 444 – 454 – 444 – 454 (concentrated in the middle)

“3 3 3 end” case: 454 – 444 – 454 – 444 – 454 – 333 (concentrated in the end)

We include Table 12 in this companion primarily to confirm that our key findings in the main paper hold true for other golf course designs. In §4.2 of the main paper, we presented five insights gathered from Table 11. We now describe how each of the five findings hold true for the extension of our findings in Table 12.

The first (two-fold) conclusion that we drew in §4.2 of the main paper was that (i) the optimal tee interval for the unbalanced course with three $P3$ holes – and for the balanced course with three $SP3$ holes – is approximately equal to the limiting cycle (i.e. the course is critically loaded), (ii) the optimal tee interval for the balanced course with $P3WU$ holes is slightly greater than the limiting cycle (i.e. the course is under-loaded). As noted in Table 11, the limiting cycle times for $P3$, $P3WU$ and $SP3$ holes are 8.39, 6.53 and 6.79, respectively. Here, in Table 12, we observe that the optimal tee intervals for the "5 5 5 case", "3 3 3 mid case" and "3 3 3 end case" are the same as the optimal tee intervals in Table 11, in which the optimal tee intervals are approximately $[8.20, 8.50]$, $[7.20, 7.30]$ and $[7.00, 7.10]$ for the $P3$, $P3WU$ and $SP3$ holes, respectively. We confirm that the optimal tee for $P3WU$ holes remains noticeably larger than its limiting cycle regardless of the hole order, and we consequently confirm that the wave-up rule contributes to the deviation of the tee interval from critical load.

Our second conclusion was that the wave-up rule indeed contributes to a higher level of throughput. From Tables 11 and 12, we see that the respective maximum throughput levels are 74, 84 and 87 with $P3$, $P3WU$ and $SP3$ holes in all of the five course designs (with the exception of "3 3 3 end case", which shows that the max throughput with $P3WU$ holes is 85). Thus, we confirm that the advantage of the wave-up rule exists regardless of the hole sequence. Subsequently, we confirm our third conclusion, in which we asserted that the $SP3$ holes allow 3 more groups to play than $P3WU$ holes. Additionally, because the maximum throughput level and its optimal tee interval for $P3$, $P3WU$ and $SP3$ holes are essentially the same across all five course designs in Tables 11 and 12, we confirm our fourth conclusion that the hole sequence is relatively unimportant.

Finally, we confirm our fifth and final conclusion: there is a greater penalty from having too small of an interval (i.e. over-loaded) than having an interval that is too large (i.e. under-loaded). We substantiate our assertion by noting the noticeable similarity among the throughput trends across the three course designs in Table 12. More specifically, we see that when the tee is 5.00, the throughput levels are approximately 12, 10 and 16 for $P3$, $P3WU$ and $SP3$ holes. Then, the throughput levels increase at a steep rate as the tee interval increases until they reach the maximum, but the throughput levels decrease at a much slower pace after reaching their peak. Although the throughput levels at each tee interval for $P3$, $P3WU$ and $SP3$ holes are not exactly the same, we see that the trends for all Par-3 hole types and course designs are essentially tantamount to each other.

References

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Choi M, Fu Q, Whitt W (2017) Using simulation to manage the pace of play in golf, the main paper.

Whitt W (2015) The maximum throughput on a golf course. *Production and Operations Management* 24(5):685–703.