We discussed several problems in Chapter 1 of Ross. I emphasized several key points:

1. Probability theory is a **branch of mathematics**, so it is important to pay attention to **definitions** and axioms (see the beginning of Section 1.3).

   You need to recall some elementary **set theory**. We use braces to denote a set, as in $S$ or $E$ in Problem 1.18 below. Note that $x$, $\{x\}$ and $\{\{x\}\}$ are different objects: $x$ is an element of the set $\{x\}$, while $\{x\}$ is an element of the set $\{\{x\}\}$; $x$ is not an element of the set $\{\{x\}\}$. A set containing the element $x$ is not the same as the element $x$ itself. It may help to Google “set theory.”

   Formally, a **probability measure** assigns probabilities to subsets of the sample space; those subsets are called events. See Sections 1.1-1.3 of Ross.

2. We are focusing on **problem solving**. For that purpose, a good general strategy is **divide and conquer**: break the problem into smaller pieces that are easier to analyze. Skipping steps can cause errors.

3. It is helpful to draw **pictures**.

   In particular, Chapter 1 emphasizes that a key idea overall is to remember and apply the definition of **conditional probability**:

   \[
P(A|B) \equiv \frac{P(AB)}{P(B)},
   \]

   where $AB \equiv A \cap B$ denotes the **intersection** of the sets (events) $A$ and $B$.

**The following are exercises at the end of Chapter 1 in Ross.**

1.18 Start by formalizing. Put the problem into the mathematical framework. The sample space is the set

   \[
   S = \{(g,g), (g,b), (b,g), (b,b)\},
   \]

   where $(b,g)$ means that the first (eldest or older) child is a boy and the second (younger) child is a girl. Then events are subsets of the sample space. The event we are concerned with is

   \[
   B = \{(g,g)\},
   \]

   the event that both children are girls.

   The key idea here is that the events $E$ and $L$ are different, where $E$ is the event that the eldest child is a girl, while $L$ is the event that at least one child is a girl. the events $E$ and $L$ are:

   \[
   E = \{(g,g), (g,b)\}
   \]
(eldest girl is first is girl) and
\[ L = \{(g, g), (g, b), (b, g)\} . \]
Thus \( P(B|E) = 1/2 \), while \( P(B|L) = 1/3 \). Use the definition of conditional probability:
\[ P(B|E) = \frac{P(BE)}{P(E)} . \]

Formally, a *probability measure* assigns probabilities to subsets of the sample space; those subsets are called events.

1.28 The key thing here is to formulate the problem precisely. The question can be expressed as follows: If \( P(A|B) \geq P(A) \) is it necessarily true that \( P(B|A) \geq P(B) \). Once the question has been so formulated, it is easy to see that the answer is indeed “Yes.” As a second step, apply the definition of conditional probability to write down what the conditional probabilities are. You are then left with trivial algebra.

1.29 It is convenient to draw a picture. The relevant picture here is a *Venn diagram*, as shown below. It shows the two events \( E \) and \( F \) as subsets within the sample space. When we consider the two events together, these two events determine four events: \( EF, EF^c, E^cF \) and \( E^cF^c \).

A Venn Diagram

Now turning to the specific questions, recall that “mutually exclusive” means that the intersection is empty; the two sets \( E \) and \( F \) have no elements in common; i.e., \( EF = \phi \). The answers are: (a) 0, (b) \( P(E)/P(F) = 0.6/P(F) \geq 0.6 \), (c) \( P(EF)/P(F) = P(F)/P(F) = 1.0 \).
1.33 Again we can use a Venn diagram. Let $B$ be the set of boys, with its complement $B^c$ being the set of girls. Let $F$ be the set of freshmen, with its complement $F^c$ being the set of sophomores. Let $n$ be the number of sophomore girls, i.e., the number of elements in the set $B^c F^c$. Then we have the Venn diagram below.

![Venn Diagram: Boys (B) and Girls; Freshman (F) and Sophomores](image)

There is the single unknown $n$. Make an equation with one unknown and solve it. Let the definition of independence give you the equation:

$$P(FB) = P(F)P(B)$$

so that the equation becomes

$$\frac{4}{16 + n} = \frac{10}{16 + n} \times \frac{10}{16 + n},$$

from which it is easy to see that the answer is $n = 9$. Any of the other events would yield the same answer; e.g., we could have used instead

$$P(SG) = P(S)P(G).$$
1.42 An application of Bayes Theorem, as discussed in Section 1.6. Draw a probability tree. From the root, draw three branches to show the possibilities of which coin is chosen; let the probability weight on each branch be $1/3$. Then from each of these branches, make further branches showing the outcome of the coin toss, conditional on the selected coin.

Probability tree

T = two-headed; F = fair; B = biased; H = head

\[
\begin{array}{c}
T \\
\downarrow \\
1/3 \\
\downarrow \\
F \\
\downarrow \\
1/3 \\
\downarrow \\
B \\
\downarrow \\
1/3 \\
\end{array}
\]

\[
\begin{array}{c}
\text{H} \\
\downarrow \\
1/3 \times 1 = 1/3 \\
\text{H} \\
\downarrow \\
1/2 \times 1/2 = 1/6 \\
\text{H} \\
\downarrow \\
3/4 \times 3/4 = 1/4 \\
\end{array}
\]

Use Bayes theorem to reverse the conditional probability: get $P(B|A)$ from $P(A|B)$. See examples in Section 1.6. In particular, we have

\[
P(T|H) = \frac{P(TH)}{P(H)} = \frac{P(T)P(H|T)}{P(T)P(H|T) + P(F)P(H|F) + P(B)P(H|B)} = \frac{1/3}{1/3 + 1/6 + 1/4} = \frac{4}{9}.
\]
1.45 Another application of Bayes theorem. Again, draw a probability tree. From the root, draw two branches showing the first ball drawn. Then from each of these outcomes, draw two branches showing the second ball drawn, given the first ball drawn. When inserting the probabilities, be sure to add the $c$ balls of the right color after the first draw.

An urn with black and red balls

$B_j =$ black ball drawn on draw $j$; $R_j =$ red ball drawn on draw $j$

Now we apply Bayes theorem to calculate the desired conditional probability:

$$P(B1|R2) = \frac{P(B1 \cap R2)}{P(R2)} = \frac{(br)/[(b + r)(b + r + c)]}{(br)/[(b + r)(b + r + c)] + (r(r+c))/[(b+r)(b+ r + c)]},$$

so that

$$P(B1|R2) = \frac{br}{br + r(r+c)} = \frac{b}{b + r + c},$$

as claimed.