IEOR 3106, Fall 2006, Professor Whitt
Solutions for Visit to the Post Office.
The Exponential Distribution

1. The Lack-Of-Memory Property

A nonnegative random variable X is said to have the *lack of memory* (LOM) *property* of

$$P(X > x + y | X > y) = P(X > x) \quad \text{for all} \quad x > 0, \, y > 0 \; .$$

Theorem 0.1 *A nonnegative random variable has the LOM property if and only if it has an exponential distribution.*

2. A Trip to the Post Office

Here is a variant of the Post Office Problem with different names: Five students – Ruxian, Shiqian, Yori, Yotam and Yu Hang – simultaneously enter an empty post office with three clerks. Ruxian, Shiqian and Yori begin to receive service immediately, while Yotam and Yu Hang wait in a single line, ready to be served by the first free clerk, with Yotam at the head of the line (to be served first when a server becomes free). Suppose that the service times of the three clerks are independent exponential random variables, each with mean 2 minutes.

(a) What is the expected time (from the moment the students enter the post office) until Yotam completes his service?

Yotam must wait until one of the first three finish service. We use the fact that a minimum of independent exponential random variables is again exponentially distributed with a rate equal to the sum of the component rates. Here the minimum has a rate equal to three times an individual service rate. Since the mean is the reciprocal of the rate, the mean time for the first service completion is 2/3 minute. We must add to that Yotam's own expected service time. Hence, the expected time until Yotam finishes service is $(2/3) + 2 = (8/3) = 2.33$ minutes.

(b) What is the expected time (again since entering the post office) until *all* five students finish service?

All three clerks are working for the first three service completions; for the fourth service completion two clerks are working; for the fifth (last) service completion one clerk is working. Hence the overall expected time is: $(2/3) + (2/3) + (2/3) + (2/2) + (2/1) = 5$ minutes. We use the lack of memory property; at each service completion, the remaining service time of each service not completed is exponential, just as if the service began at that point.

(c) What is the variance of the time until *all* five students finish service?

The total time, for which the expectation is given in part (b) above, is the sum of five independent exponential random variables, with the specified means. The sum of exponential random variables is not exponentially distributed. The variance of the sum is the sum of the variances, however, since the random variables are independent. The variance of an exponential random variable is the square of its mean. Hence the desired variance is $(2/3)^2 + (2/3)^2 + (2/3)^2 + (2/2)^2 + (2/1)^2 = 19/3 = 6.33$.

(d) What is the probability that Yotam is the *third* student to finish service?

2/9. Of course, Yotam cannot be the first to finish. Since he is one of three in service when he starts service (after the first service completion), Yotam is not the second to finish with probability 2/3. Conditional on not being the second to finish, Yotam is the third to finish with probability 1/3, since he is then one of three in service. Hence the probability is $(2/3 \times (1/3) = (2/9)$.

(e) Suppose that you wanted to calculate the probability that the time required for all five students to complete service will exceed 10 minutes. What computational tool makes that calculation easy to perform? Briefly explain why.

The time required for all five students to complete service is the sum of five independent exponential random variables, but with different means. The sum of the first three with identical means has a gamma (or Erlang) distribution, as indicated on p. 37. However, the sum of all five random times has a complicated distribution. The desired probability can be expressed as a multidimensional convolution integral. That multidimensional convolution integral could be computed by MATLAB. It would be made easier by reducing it to a three-dimensional integral, exploiting the gamma distribution for the first three service completions. Convolution is discussed in Example 1.5 (d) on p. 25 of Ross. For example the density of $X+Y$, when X and Y are independent nonnegative random variables with densities f_X and f_Y is

$$f_{X+Y}(x) = \int_0^x f_X(y) f_Y(x-y)\, dy \ .$$

You just iterate those integrals to compute the density of sums of more independent random variables. The computation gets harder, though, as the number of integrals increases.

The desired computation is easy to perform, however, by numerically inverting the Laplace transform; see the papers listed on the course computational tools web page.

To give a quick brief explanation. Let X be a random variable and let f be the pdf (density) of X. The Laplace transform $\hat{f}$ is defined as

$$\hat{f}(s) \equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \ .$$

From elementary Laplace transform theory, the Laplace transform of the complementary cumulative distribution function (ccdf) $F^c(x) \equiv 1 - F(x)$, where the cdf is $F(x) \equiv \int_0^x f(t)\, dt$, is

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s \ .$$

To be more concrete, let X be an exponentially distributed random variable with mean $1/\lambda$ (and thus rate λ). In this context, $f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$. The Laplace transform here is

$$\begin{aligned}
\hat{f}(s) &\equiv E[e^{-sX}] \equiv \int_0^\infty e^{-sx} f(x)\, dx \\
&= \int_0^\infty e^{-sx} \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{-(s+\lambda)x}\, dx \\
&= \lambda/(\lambda + s) \ .
\end{aligned}$$

Directly, the Laplace transform of the ccdf in this exponential case is

$$\begin{aligned}
\hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\
&= \int_0^\infty e^{-sx} e^{-\lambda x}\, dx \\
&= \int_0^\infty e^{-(s+\lambda)x}\, dx \\
&= 1/(\lambda + s) \ .
\end{aligned}$$

In this exponential case we can easily verify that

$$\hat{F^c}(s) = (1 - \hat{f}(s))/s \ .$$

But why are these transforms so useful? They are useful here, as in many other cases, because the transform of $X_1 + \cdots + X_n$, say $\hat{f}$, where $X_1, \ldots, X_n$ are independent random variables such that X_i has Laplace transform $\hat{f}_i$, is

$$\hat{f}(s) = \hat{f}_1(s) \times \hat{f}_2(s) \times \cdots \times \hat{f}_n(s) \ .$$

Thus, we can easily compute the Laplace transform of the ccdf of the time required for all students to complete service. In particular, the Laplace transform is

$$\begin{aligned}
\hat{F^c}(s) &\equiv \int_0^\infty e^{-sx} F^c(x)\, dx \\
&= (1 - \hat{f}(s))/s \\
&= (1 - [\hat{f}_1(s) \times \cdots \times \hat{f}_5(s)])/s \\
&= (1 - [(\lambda_1/(\lambda_1 + s)) \times \cdots \times (\lambda_5/(\lambda_5 + s))])/s \\
&= (1 - [((3/2)/(3/2 + s))^3 \times (1/(1 + s)) \times ((1/2)/(1/2 + s))])/s \ .
\end{aligned}$$

The final expression is not so pleasing for humans to look at, but the computer is happy.

We can thus calculate the desired ccdf value $F^c(10)$ by numerically inverting that Laplace transform $\hat{F^c}(s)$.

Here is yet another problem on the exponential distribution. A bonus problem to help you with your understanding.

3. Yiping's flashlight

Yiping's flashlight needs two batteries to be operational. Suppose that, in addition to his flashlight, Yiping has a set of 12 functioning batteries – battery 1, battery 2, and so forth. Initially, Yiping puts batteries 1 and 2 in his flashlight. Whenever a battery fails, Yiping immediately replaces the failed battery by the lowest-numbered functioning battery that has not yet been put in use. Suppose that the batteries remain like new until they are installed in the flashlight. Suppose that the lifetimes of the different batteries (in use in the flashlight) are independent random variables, each with an exponential distribution having mean 4 months. Let T be the time that the flashlight ceases to work, i.e., the time that a battery fails and Yiping's stockpile of spares is empty. At that moment, exactly one of the 12 original batteries – which we will call battery N – will not yet have failed. (It will be the one working battery in the flashlight, even though the flashlight no longer works.)

(a) What is the expected value of T?

T is the time (measures in time that the flashlight is in use) until 11 batteries fail, but 2 batteries are working at any one time. The minimum of two IID exponential random variables is again exponential with half the mean. The time between successive failures is thus an exponential random variable with mean $4/2 = 2$ months. Hence $ET = 11 \times 2 = 22$ months.

(b) What is the variance of T?

The variance of T is the variance of the sum of 11 exponential random variables, each with mean 2. Since the exponential variables are independent, the variance of the sum is the sum of the variances. The variance of an exponential random variable is the square of the mean. Hence

$$Var(T) = 11 \times (2)^2 = 11 \times 4 = 44 \ .$$

(c) What is $P(N = 12)$?

Each of the two batteries in use has probability 1/2 of being the one to fail. That is independent of the history because of the lack-of-memory property. Hence, $P(N = 12) = 1/2$.

(d) What is $P(N = 1)$?

The first battery must survive 11 successive independent trials, each with probability 1/2 of failure. Hence, $P(N = 1) = (1/2)^{11} = 1/2048$.
