Statistical Analysis with Little's Law
Supplementary Material: Technical Report

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Abstract

The theory supporting Little's law ($L = \lambda W$) is now well developed, applying to both limits of averages and expected values of stationary distributions, but applications of Little's law with actual system data involve measurements over a finite time interval, which are neither of these. We advocate taking a statistical approach with such measurements. We investigate how estimates of L and λ can be used to estimate W when the waiting times are not observed. We advocate estimating confidence intervals. Given a single sample path segment, we suggest estimating confidence intervals using the method of batch means, as is often done in stochastic simulation output analysis. We show how to estimate and remove bias due to interval edge effects when the system does not begin and end empty. We illustrate the methods with data from a call center and simulation experiments. This technical report provide more details about the call center data we used as well as additional materials.

Keywords: Little's law; $L = \lambda W$; measurements; parameter estimation; confidence intervals; bias; finite-time versions of Little's law; confidence intervals with $L = \lambda W$; edge effects in $L = \lambda W$; performance analysis.

Contents

1 Introduction and Overview

In the main paper and associated e-companion Kim and Whitt (2012), we advocate taking a statistical approach with the finite-time version of Little's law for measurements from a finite time interval and show how it can be done. The importance of a finite-time perspective was emphasized by Buzen (1976), Denning and Buzen (1978) and Little (2011), but they advocated the altered definitions in §2.3 of the main paper producing an equality relation. In contrast, like Mandelbaum (2011), we emphasize the importance of the more complex relation in Theorem 2 of the main paper, because we think it is more informative.

The main contribution here is the emphasis on statistics. Even though statistics is not customarily used when applying Little's law with data over a finite time interval, there is a large literature on statistical analysis for call centers and queueing models, more generally. For call centers, statistical methods play an important role in forecasting, as illustrated by Aldor-Noiman et al. (2009) and references therein. For queueing models, many papers focus on statistics, including inference with limited data, as illustrated by Hansen and Pitts (2006) and Larson (1990).

In §2 we present additional information about the call center data used. In §3 we discuss another way to do the confidence intervals using the method of batch means. In §4 we elaborate on the bias discussed in §5 of the main paper, discussing the bias in the estimator $\bar{W}(t)$ in (1) (in the main paper) for W in a stationary setting in §4.1. In §4.2 we discuss the bias in the alternative estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper for $E[W(t)]$ in a nonstationary setting and W in a stationary setting. In both cases, we focus on the many-server and single-server paradigms in order to highlight important model-dependent properties. In §5 we introduce an alternative algorithm to estimate confidence intervals in approximately stationary intervals by batch means, exploiting Theorem 2 of the main paper. However, this algorithm did not improve the estimation for the call center example, so we do not emphasize it. The negative result itself is interesting, because it is natural to consider such alternatives. It remains to be seen if the new algorithm can be useful in other contexts. Finally, in §6 we present additional figures and tables.

2 More on the Call Center Data

In this section, we provide more details about the call center data we used. As explained in the main paper, the data are from a telephone call center of a medium-sized American bank from the data archive of Mandelbaum Mandelbaum (2012), collected from March 26, 2001 to October 26, 2003. This banking call center had sites in New York, Pennsylvania, Rhode Island, and Massachusetts, which were integrated to form a single virtual call center. The virtual call center had $900 - 1200$ agent positions on weekdays and $200 - 500$ agent positions on weekends. The center processed about $300,000$ calls per day during weekdays, with about $60,000$ (20%) handled by agents, with the rest being served by integrated voice response (IVR) technology. As in many modern call centers, in this banking call center there were multiple agent types and multiple call types, with a form of skill-based routing (SBR) used to assign calls to agents.

Since we were only concerned with estimation related to the three parameters L, λ and W, we did not get involved with the full complexity of this system. Specifically, we used data for only one class of customers, denoted by **Summit**. Furthermore, among them, only the sub-calls that had agent interactions during weekdays in May 2001 were considered.

The rest of this supplement is organized as follows. In §2.1 we briefly describe the full database of Mandelbaum Mandelbaum (2012) and how we extracted the required data in order to produce the results in the main paper. In §2.2 we describe the statistics collected for the 18 weekdays in May we used in our analysis and give an overview of the statistics.

2.1 The Data Available and Used

The full database of Mandelbaum (2012) provides nine pre-processed ACCESS tables for each day in the study period, from March 26, 2001 to October 26, 2003. In the pre-processing, issues such as midnight calls, incorrect time stamps and incorrect identifiers (id's) are already taken care of. As shown on the left pane of Figure 1, the nine database tables are titled: `calls`, `customer sub-calls`, `server sub-calls`, `queue records`, `event details`, `agent events`, `agent profile`, `agent records`, and `agent shifts`. The `calls` table includes general information on each call that enters the call center on a particular day. Each call then consists of sub-call(s) that start and end with a particular service such as IVR, agent interaction and announcement. We focus on the sub-calls of **Summit** customers that involve agent interaction, and hence use only the `customer sub-calls` table of each day.

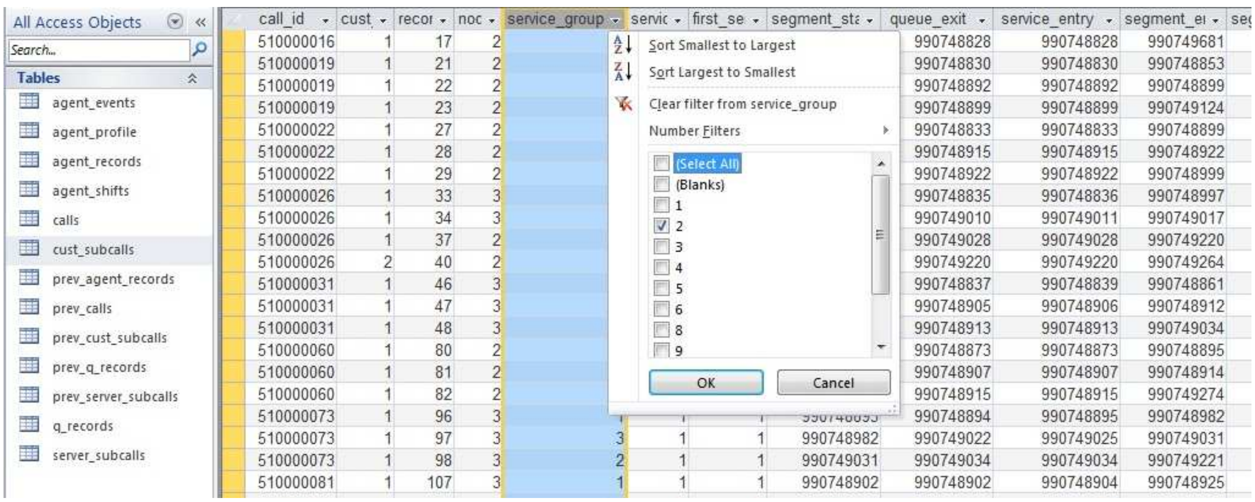

Figure 1: Example of ACCESS tables: call center data from May 25, 2001.

Figure 1 is an example of the `customer sub-calls` table. There are 23 fields in the table, which are: $call\_id$, $cust\_subcall$, $server\_subcall$, $record\_id$, $node$, $customer\_id$, $customer\_type$, $service\_group$, $service$, $first\_service$, $segment\_start$, $queue\_exit$, $service\_entry$, $segment\_end$, seg_type, $outcome$, $seg\_parties$, $wait\_time$, $queue\_time$, $preservice\_wait$, $service\_time$, $hold\_time$, and $party\_answered$. More information about the different tables, including detailed descriptions of each field, can be found at:

http://ie.technion.ac.il/Labs/Serveng/files/Model_Description_and_Introduction_to_User_Interface.pdf

To create the data set we used in Kim and Whitt (2012), we used the following steps:

- Each sub-call is served by a service group. There are five main service groups, which are IVR, Business line, non-Business line, Announcement and Message. We kept the sub-calls that were handled by the Business line ($service\_group = 2$). In the ACCESS `customer sub-calls` table, we filtered out these sub-calls by selecting $service\_group = 2$, as illustrated in Figure 1.
- We kept the sub-calls that are from **Summit** customers by keeping records with $service = 14$ (The $service$ field indicates the type of service received by the caller. For example, there are $Retail = 1$, $Premier = 2$, $Business = 3$ and $Platinum = 4$).

- We dropped the records with no agent interaction, which involve the caller hanging up (abandoning) while waiting to speak to the next agent. This was done by dropping records with $outcome = 11, 12$, or 13. The $outcome$ field indicates the cause of call termination such as whether they were handled, transferred and abandoned. $outcome = 11$ indicates the customer abandoned short (the caller abandons within an abandon threshold time), $outcome = 12$ indicates the customer abandoned (after the abandon threshold time) and $outcome = 13$ indicates the call was not handled with other reason that is not specified in the data).

- To ensure that each sub-call spent positive amount of time with an agent, we omitted records with $service\_time = 0$, where $service\_time$ is defined as the sum of talk time and hold time. It can also be defined as the difference between $segment\_end$ and $service\_entry$. Since we already dropped the records with customer abandonment, there were not many records with $service\_time = 0$, less than 5 for each day.

- In order to compute the three parameters L, λ and W, we used the time each sub-call enters the queue, leaves the queue (hence enters the service) and leaves the service. Therefore, we kept only the fields $call\_id$, $segment\_start$ (queue entry time), $queue\_exit$ (queue exit time), $service\_entry$ (service entry time) and $segment\_end$ (service exit time). (The time stamps are records in seconds, using the origin time, 00:00:00 on 01/01/1970.

- Finally, we exported the table to an EXCEL file using the "Export to Excel spreadsheet" function.

The steps above were carried out by the authors for 18 weekdays of May. (In the next section we explain how the 18 days were selected.) The combined data set for all 18 weekdays ($little\_weekdays\_in\_May.xls$) is available from the authors' web sites.

2.2 Statistics from Eighteen Weekdays in May, 2001

There were 23 weekdays in May 2001. (May 1, 2001, was a Tuesday.) Four weekdays were not normal, and so were excluded, for the following reasons:

- May 9 (Wed): shutdown from 4:53:10 AM until 11:28:54 AM

- May 10 (Thurs): shutdown from 2:59:18 PM until 11:31:24 PM

- May 28 (Mon): Memorial Day

- May 31 (Thurs): data missing

In addition, the data from May 3 were excluded because the number of arrivals was extraordinarily high. In particular, the number of arrivals was 8310 on May 3 with about 50% arriving before 9 AM, whereas for the other 18 weekdays, the average number of arrivals was 5410.5 arrivals, with a standard deviation 1080.5. For each day, there were data over a 17-hour period, from 6 AM to 11 PM, referred to as $[6, 23]$. (There were no arrivals before 6 AM and after 11 PM.)

We primarily focused on the number of Summit customers in the system, but we also considered whether they were in service or waiting (in queue). Thus we measured the numbers in the system, in service and in queue. Similarly, we measured the time that each customer spent in the system, in the queue and in service.

Using the data set $little\_weekdays\_in\_May.xls$, we collected the following statistics:

- L_{sys}: the number in system

- L_{ser}: the number in service

- L_q: the number in queue
- $A_{sys} = A_q$: number of arrivals into the system/queue
- A_{ser}: number of arrivals into service
- W_{sys}: time spent in the system
- W_{ser}: time spent in service
- W_q: time spent in queue

These are understood to be functions of the measurement interval. For example, $A_{sys} \equiv A_{sys}([9,10])$ is the number of arrivals into the system during the interval $[9,10]$. For the interval $[9,10]$, the average arrival rate per minute is

$$\bar{\lambda}(t) \equiv \frac{A_{sys}([9,10])}{m([9,10])} = \frac{A_{sys}([9,10])}{60}, \tag{1}$$

where $m([9,10]) = 60$ is the number of minutes in the interval $[9,10]$. Thus the statistics are consistent with the definitions in equation (1) of Kim and Whitt (2012).

2.2.1 The Hourly Arrival Rates

Figure 2 shows the overall average (over the full 17-hour day) and the hourly averages of the arrival rates per minute, as defined in (1), together with estimates of the 95% confidence interval (treating the daily values as i.i.d. Gaussian variables) for the 18 weekdays in May. Figure 2 shows that the arrival rate is nonstationary over the day. Figure 2 also shows that the arrival rate is highly variable from day to day, because of the wide confidence intervals for the hourly averages. Part of this day-to-day variation can be explained by day-of-week effect. Figure 3 shows that the average call volume on Mondays is the largest, followed by that of Tuesdays, and then the others. Figures 4-8 further illustrate day-to-day variation in the same day of week.

2.2.2 The Hourly Average Waiting Times

Figures 9-11 show the overall average (average of the 18 daily averages) and hourly average of the waiting time in the system, service and queue and their 95% confidence interval of 18 weekdays in May. Figure 10 suggests that the service times are approximately stationary over time. However, by comparing Figure 10 to Figure 9, we can conclude that it is hard to say that the times in system is approximately equal to the service times because the time in queue is too long in the interval [17,20], which might be due to inadequate staffing during this interval. Furthermore, Figure 11 suggests the waiting times in system is not approximately stationary over time, again possibly due to inadequate staffing.

Next, Figure 12 shows the histogram of all waiting times in the interval $[10,16]$ of Friday, May 25, 2001 in our call center example. In addition, Figure 13 and Figure 14 illustrate the histograms of all waiting times and service times in the interval $[9,17]$ over 18 weekdays in May in our call center example. As usual for call centers, the distribution is approximately lognormal, but the SCV very close to 1 indicates that an exponential approximation is reasonable.

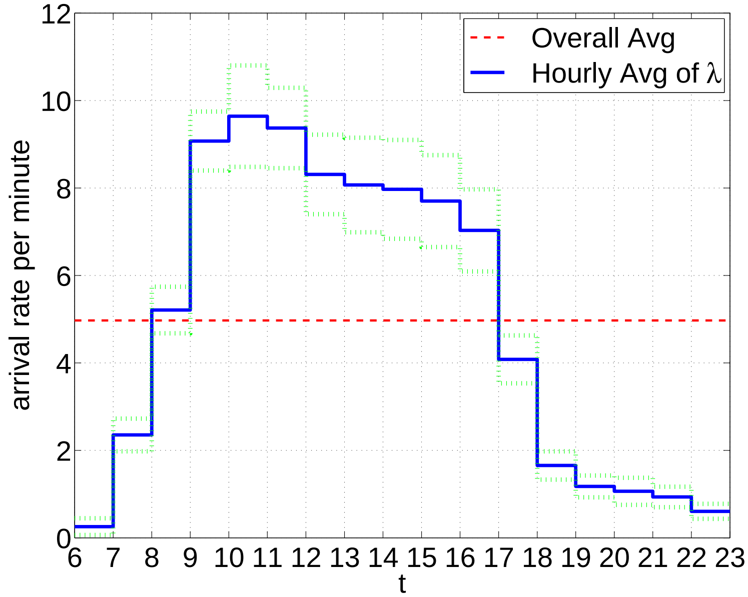

Figure 2: Overall average and hourly average of λ and its 95% confidence interval over 18 weekdays in May.

2.2.3 The Hourly Average Number in System

Figures 15-17 show the overall average (average of the 18 daily averages) and hourly average number in the system, service and queue and their 95% confidence interval of 18 weekdays in May. We observe that the number in system is approximately equal to the number in service, except at the times when there are slightly longer queues, in the intervals [9,13] and [17,20].

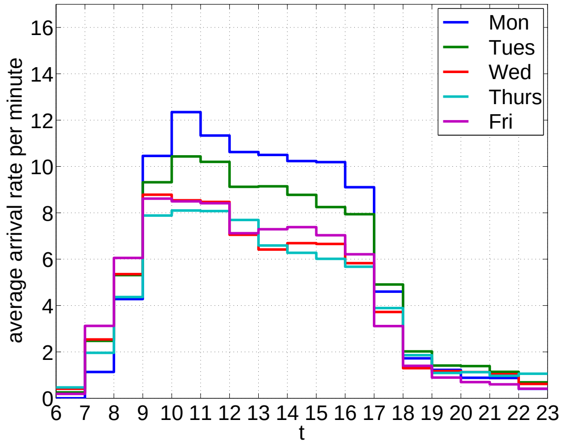

Figure 3: Average arrival rate and the day-of-week effect in May.

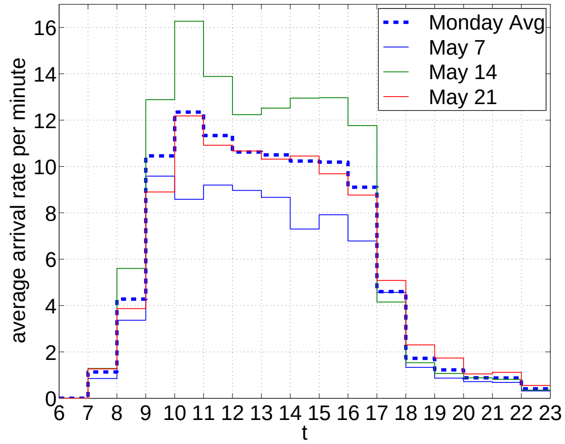

Figure 4: Average arrival rate of Mondays in May.

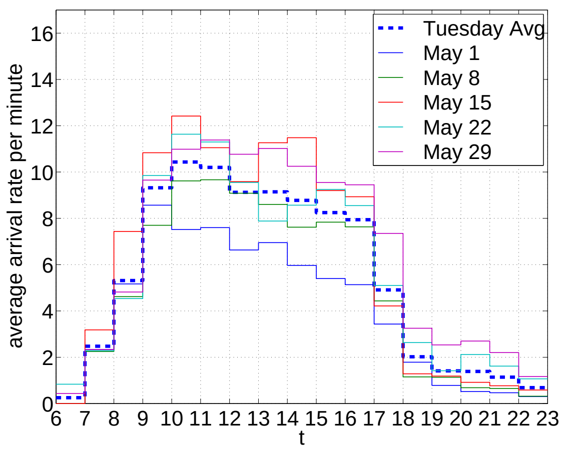

Figure 5: Average arrival rate of Mondays in May.

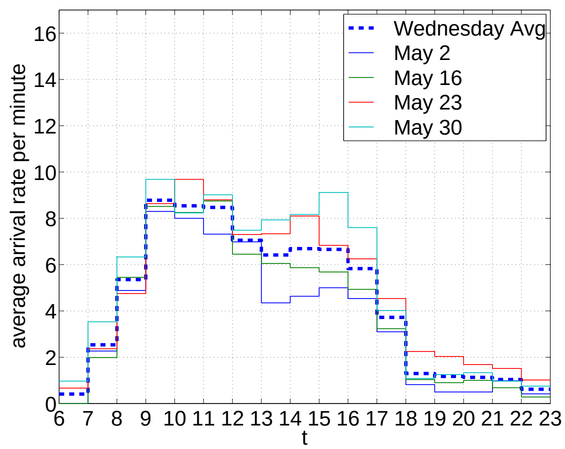

Figure 6: Average arrival rate of Wednesdays in May.

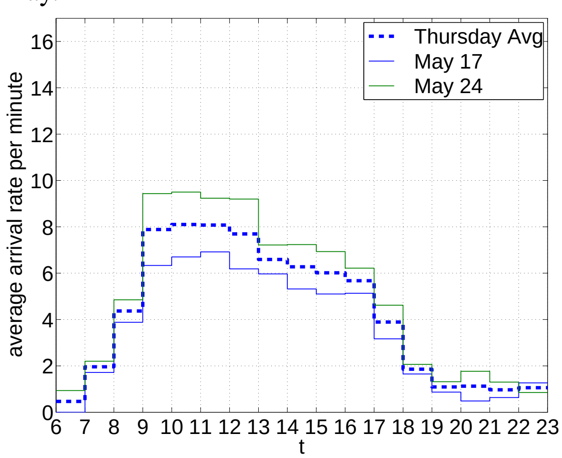

Figure 7: Average arrival rate of Thursdays in May.

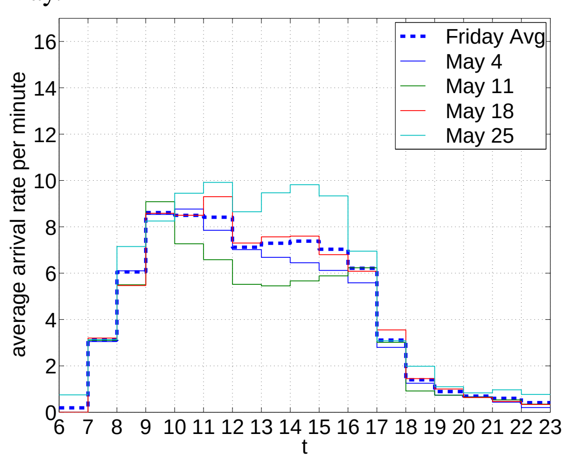

Figure 8: Average arrival rate of Fridays in May.

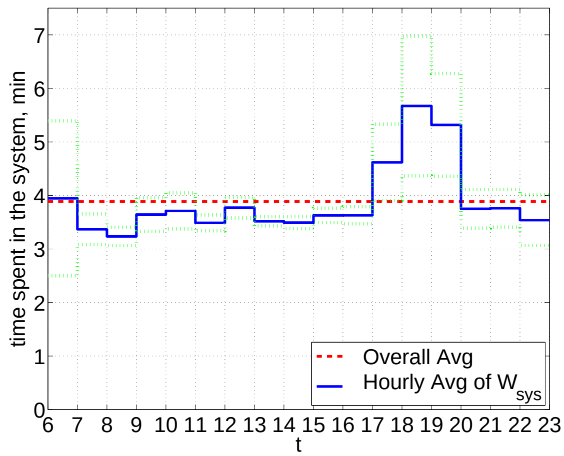

Figure 9: Overall average and hourly average of W_{sys} and its 95% confidence interval over 18 weekdays in May.

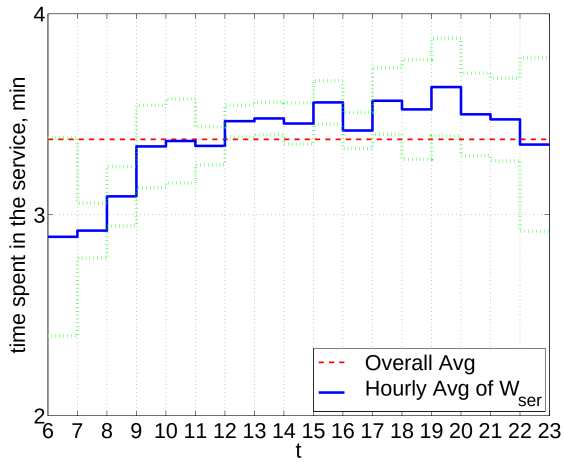

Figure 10: Overall average and hourly average of W_{ser} and its 95% confidence interval over 18 weekdays in May.

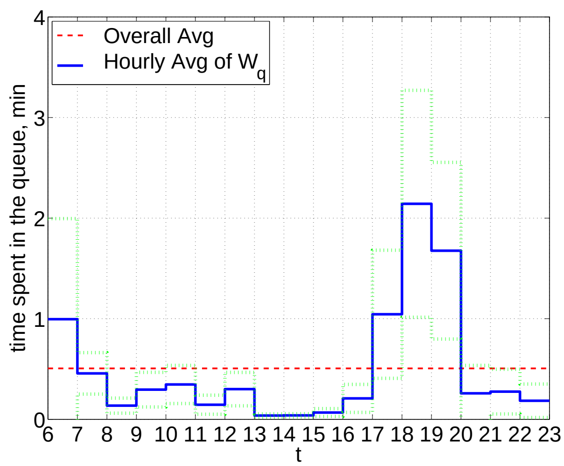

Figure 11: Overall average and hourly average of W_q and its 95% confidence interval over 18 weekdays in May.

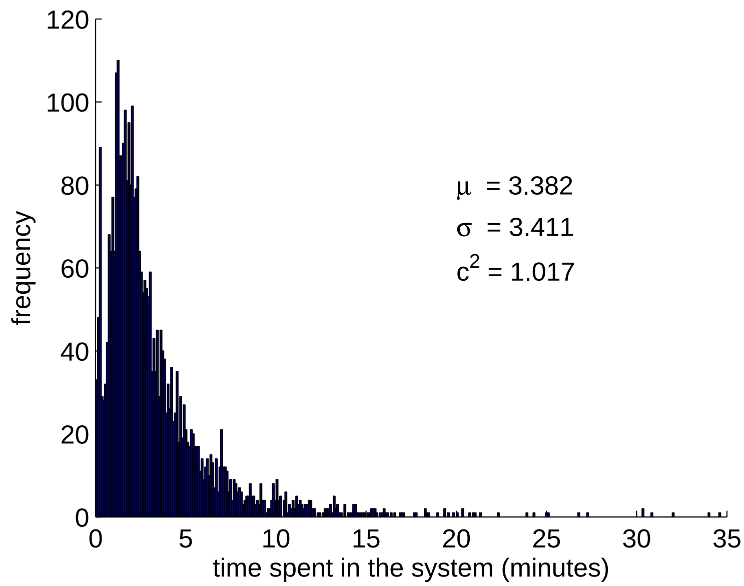

Figure 12: The histogram (empirical distribution) of the times spent in the system of all arrivals during the interval $[10, 16]$.

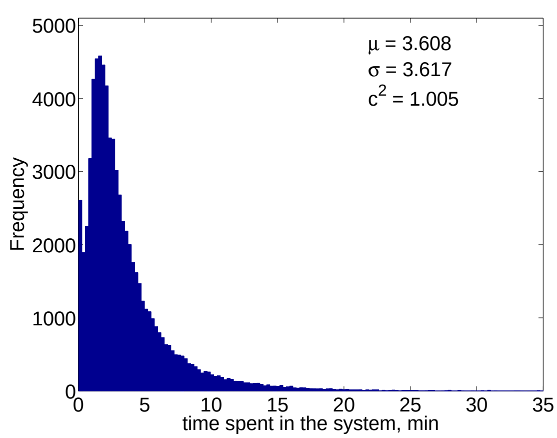

Figure 13: The histogram (empirical distribution) of the times spent in the system of all arrivals during the interval $[9, 17]$ over 18 weekdays in May ($n = 72,535$ and 41 observations that had $W_{sys} > 35$ are not represented).

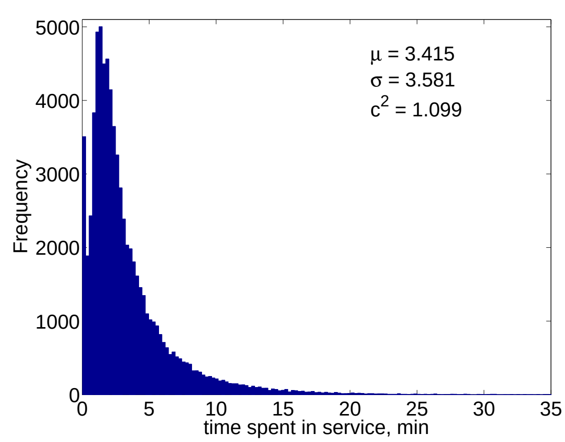

Figure 14: The histogram (empirical distribution) of the times spent in service of all arrivals to the service during the interval $[9, 17]$ over 18 weekdays in May ($n = 72,494$ and 39 observations that had $W_{ser} > 35$ are not represented).

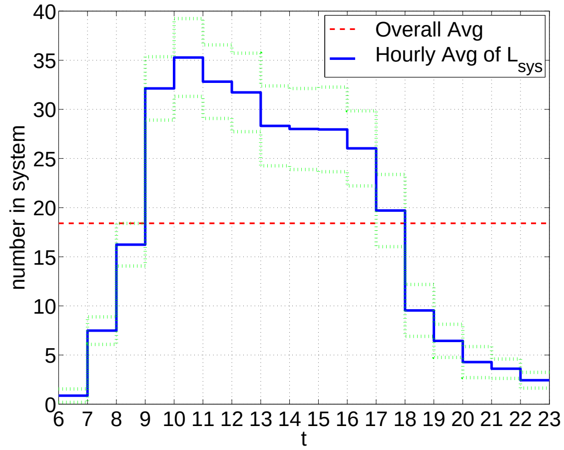

Figure 15: Overall average and hourly average of L_{sys} and its 95% confidence interval over 18 weekdays in May.

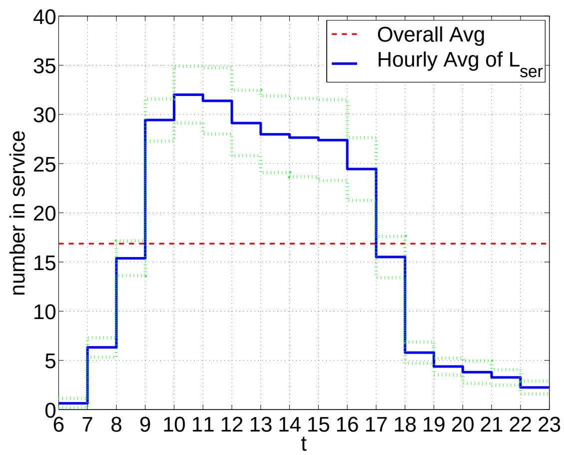

Figure 16: Overall average and hourly average of L_{ser} and its 95% confidence interval over 18 weekdays in May.

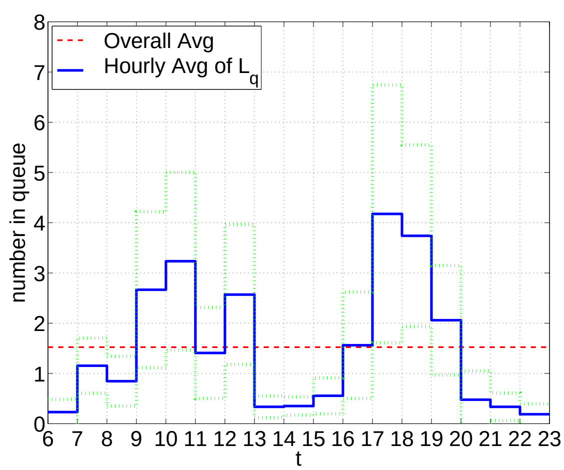

Figure 17: Overall average and hourly average of L_q and its 95% confidence interval over 18 weekdays in May.

3 An Alternative Way to Construct the Confidence Intervals

In §4.3 of the main paper, we discuss how to construct confidence intervals for the indirect estimator of the waiting time using the method of batch means. In that procedure, we first compute $\bar{W}_{L,\lambda,k}(t,m)$, which are the batch analogs of the relations in (3) of the main paper, and then directly compute the sample mean and variance based on the batches. An alternative way is to use the average of direct estimate values. That is, we can use

$$\bar{L}^{(m)}(t) \equiv \frac{1}{m}\sum_{k=1}^{m} L_k(t,m) \quad \text{and} \quad \bar{\lambda}^{(m)}(t) \equiv \frac{1}{mt}\sum_{k=1}^{m} A_k(t,m) \tag{2}$$

to construct the mean $\bar{W}'_{\lambda,L}(t) \equiv \bar{L}^{(m)}(t)/\bar{\lambda}^{(m)}(t)$ (via the relation in (3) of the main paper) and the variance via the first equation in (35) of the main paper. Table 1 demonstrates that we get similar results from the two different methods, and that both indirect methods provide reasonable estimates.

Table 1: Direct estimates of W from (1) in the main paper plus two indirect estimates using the formulas in (3) of the main paper. Both indirect estimates are constructed using the method of batch means with $m = 5, 6, ..., 20$ batches for the call center data for the time intervals $[10, 16]$ and $[14, 15]$. $\bar{W}_{\lambda,L}(t)$ is computed using the batch analogs of the relations in (3) whereas $\bar{W}'_{\lambda,L}(t)$ is computed using the average of direct estimate values.

	$[10, 16]$			$[14, 15]$		
m	$\bar{W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{W}'_{\lambda,L}(t)$	$\bar{W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{W}'_{\lambda,L}(t)$
5	3.38 ± 0.22	3.38 ± 0.19	3.37 ± 0.20	3.33 ± 0.21	3.33 ± 0.10	3.32 ± 0.10
6	3.39 ± 0.25	3.38 ± 0.23	3.37 ± 0.22	3.32 ± 0.18	3.33 ± 0.09	3.32 ± 0.09
7	3.39 ± 0.20	3.38 ± 0.17	3.37 ± 0.17	3.34 ± 0.19	3.36 ± 0.29	3.32 ± 0.24
8	3.39 ± 0.19	3.38 ± 0.21	3.37 ± 0.21	3.34 ± 0.22	3.35 ± 0.26	3.32 ± 0.25
9	3.39 ± 0.17	3.38 ± 0.17	3.37 ± 0.17	3.32 ± 0.24	3.34 ± 0.18	3.32 ± 0.17
10	3.39 ± 0.15	3.38 ± 0.16	3.37 ± 0.16	3.33 ± 0.21	3.34 ± 0.16	3.32 ± 0.16
11	3.39 ± 0.14	3.38 ± 0.15	3.37 ± 0.15	3.32 ± 0.21	3.38 ± 0.29	3.32 ± 0.26
12	3.39 ± 0.16	3.38 ± 0.15	3.37 ± 0.15	3.32 ± 0.18	3.34 ± 0.16	3.32 ± 0.15
13	3.39 ± 0.13	3.38 ± 0.11	3.37 ± 0.12	3.33 ± 0.20	3.37 ± 0.23	3.32 ± 0.24
14	3.39 ± 0.15	3.38 ± 0.14	3.37 ± 0.14	3.35 ± 0.20	3.38 ± 0.28	3.32 ± 0.23
15	3.39 ± 0.16	3.38 ± 0.14	3.37 ± 0.14	3.29 ± 0.25	3.39 ± 0.27	3.32 ± 0.24
16	3.39 ± 0.14	3.39 ± 0.16	3.37 ± 0.15	3.36 ± 0.24	3.37 ± 0.23	3.32 ± 0.22
17	3.39 ± 0.13	3.38 ± 0.14	3.37 ± 0.14	3.31 ± 0.22	3.40 ± 0.32	3.32 ± 0.25
18	3.39 ± 0.13	3.38 ± 0.13	3.37 ± 0.13	3.30 ± 0.22	3.42 ± 0.31	3.32 ± 0.24
19	3.39 ± 0.13	3.39 ± 0.14	3.37 ± 0.14	3.33 ± 0.20	3.38 ± 0.21	3.32 ± 0.21
20	3.39 ± 0.15	3.38 ± 0.11	3.37 ± 0.11	3.32 ± 0.23	3.43 ± 0.31	3.32 ± 0.25

4 More on Bias

In two subsections here we elaborate on the bias discussed in §5 of the main paper, discussing the bias in the estimator $\bar{W}(t)$ for W in a stationary setting in §4.1 and the bias of the alternative estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper for $E[W(t)]$ in a nonstationary setting in §4.2

In both cases it is important to know more about the underlying model. As in §5 of the main paper, we see that the situation is very different for many-server and single-server models. Just as it is good to have the averages in (1) of the main paper over multiple time intervals, so that we can check for stationarity, it is also good to be able to probe more fully into the model. It can be very helpful to understand if the system can be approximately represented as the many-server paradigm, the single-server paradigm or neither of these. For the single-server paradigm, it is helpful to understand the service-time distribution as well as the waiting-time distribution.

If we are considering indirect estimation of $E[\bar{W}(t)]$, it is good to have some rough understanding about the waiting times. Are successive waiting times approximately stationary? Are successive waiting times approximately independent? Can the distribution of the waiting times be regarded as approximately exponential? Are the waiting times (times in system) nearly the same as the service times or are they much greater?

4.1 Bias in the Estimator $\bar{W}(t)$ in a Stationary Setting

In §5 of the main paper we noted that, in general, the estimator $\bar{W}(t)$ in (1) of the main paper is a biased estimator of W in a stationary setting, because the mean $E[\bar{W}(t)]$ is the expected value of the ratio of two random variables. If we specify the underlying queueing model, then we can analyze that bias. As in §5 of the main paper, the model makes a big difference. For additional background on estimation in the many-server and single-server settings, see Srikant and Whitt (1996) and Whitt (1989).

4.2 More on the Bias Due to Interval Edge Effects

In §5 of the main paper we observed that if the special condition $L(t) = R(0) = 0$ does not hold, then $\bar{W}_{L,\lambda}(t)$ is typically a biased estimator of $E[W(t)]$ and we discussed various alternative estimators designed to remove that bias.

In this section we investigate the extent of the bias in the alternative estimator $\bar{W}_{L,\lambda}(t)$. We measure the relative error by the *relative root mean square error* (RMSE) and the expected relative error

$$RMSE_W(t) \equiv \frac{\sqrt{E[\Delta_W(t)^2]}}{E[W(t)]} \quad \text{and} \quad \Delta_W^{rel}(t) \equiv \frac{E[|\Delta_W(t)|]}{E[W(t)]}. \tag{3}$$

For a service system with a daily cycle, like the example in §3 of the main paper, where the number in system builds up to its steady-state level during the early morning and then later declines at the end of the day, we anticipate that $L(t) > R(0)$ at the beginning of the day, while $L(t) < R(0)$ at the end of the day. Thus we can anticipate the sign of the error ($\Delta_W(t) = \bar{W}_{L,\lambda}(t) - \bar{W}(t)$) during these non-stationary periods; i.e., we can anticipate that $\Delta_W(t) < 0$ during initial periods and $\Delta_W(t) > 0$ during final periods. To illustrate, for the call center example in §3 of the main paper, $L(t) \geq R(0)$ for 7 of the 8 half hour intervals in $[6, 10]$, 6 of the 14 half hour intervals in $[10, 17]$, and 2 of the 12 half hour intervals in $[17, 23]$; while $\Delta_W(t) < 0$ for 6 of the 8 half hour intervals in $[6, 10]$, 6 of the 14 half hour intervals in $[10, 17]$, and 3 of the 12 half hour intervals in $[17, 23]$. See Table 2 for more details.

To roughly quantify the potential benefit from refined estimation, consider the stationary $M/GI/\infty$ model with $E[S] = 1$ and arrival rate λ, starting empty. Formula (20) of Eick et al. (1993) implies that $E[W(t)] = E[L(t)] = \lambda G_e(t)$. Hence, $E[\bar{W}(t)] = E[\bar{L}(t)] = (\lambda/t)\int_0^t G_e(s)\,ds$. If we approximate $A(t)$ in the denominator of $\Delta_W(t)$ be λt, then $E[\Delta_W(t)] \approx G_e(t)/t$, which is small compared to $E[\bar{W}(t)]$ if either λ or t is large, but can be significant if neither is large.

4.2.1 Large-Time Stationary Asymptotics

For a fixed measurement interval $[0, t]$, there is a relatively clear story in a stationary regime when t is large enough. First, in a stationary regime, we replace $E[W(t)]$ in (3) by W. Let $f(t) \sim g(t)$ as $t \to \infty$ mean

Table 2: Comparison of $L(t)$, $R(0)$ and $\Delta_W(t)$ for the 34 half hour intervals in $[6, 23]$ for the data from §3 of the main paper.

Interval	$L(t)$	$R(0)$	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\Delta_W(t)$
$[06:00, 06:30]$	4	0	3.13	2.43	−0.702
$[06:30, 07:00]$	5	4	8.13	7.61	−0.517
$[07:00, 07:30]$	5	5	2.26	2.52	0.252
$[07:30, 08:00]$	21	5	3.14	2.80	−0.337
$[08:00, 08:30]$	24	21	3.17	3.14	−0.030
$[08:30, 09:00]$	32	24	3.71	3.41	−0.292
$[09:00, 09:30]$	13	32	4.29	4.64	0.348
$[09:30, 10:00]$	41	13	4.48	4.19	−0.289
$[10:00, 10:30]$	31	41	3.55	3.61	0.067
$[10:30, 11:00]$	39	31	3.64	3.42	−0.216
$[11:00, 11:30]$	32	39	3.28	3.36	0.085
$[11:30, 12:00]$	31	32	3.18	3.34	0.155
$[12:00, 12:30]$	34	31	3.70	3.49	−0.211
$[12:30, 13:00]$	27	34	3.40	3.68	0.280
$[13:00, 13:30]$	26	27	3.15	3.18	0.032
$[13:30, 14:00]$	24	26	2.90	2.82	−0.074
$[14:00, 14:30]$	31	24	3.35	3.26	−0.092
$[14:30, 15:00]$	33	31	3.29	3.40	0.111
$[15:00, 15:30]$	39	33	3.72	3.40	−0.324
$[15:30, 16:00]$	34	39	3.48	3.62	0.144
$[16:00, 16:30]$	26	34	3.38	3.51	0.130
$[16:30, 17:00]$	34	26	5.94	5.53	−0.408
$[17:00, 17:30]$	10	34	4.93	6.39	1.462
$[17:30, 18:00]$	7	10	4.44	4.51	0.076
$[18:00, 18:30]$	19	7	6.14	4.87	−1.261
$[18:30, 19:00]$	14	19	9.65	10.75	1.098
$[19:00, 19:30]$	6	14	5.20	6.27	1.069
$[19:30, 20:00]$	5	6	5.37	5.84	0.470
$[20:00, 20:30]$	2	5	4.08	4.28	0.202
$[20:30, 21:00]$	1	2	2.64	2.67	0.032
$[21:00, 21:30]$	10	1	6.82	4.59	−2.230
$[21:30, 22:00]$	3	10	4.38	5.93	1.558
$[22:00, 22:30]$	2	3	4.97	4.70	−0.267
$[22:30, 23:00]$	0	2	2.96	4.47	1.509

that $f(t)/g(t) \to 1$ as $t \to \infty$. Let $\stackrel{d}{=}$ mean equal in distribution. The following is an easy consequence of (7) of the main paper.

Theorem 4.1 (*asymptotic edge effect*) *If the system is in a stationary environment where $L = \lambda W$ is valid with $T_W^{(r)}(t)$ asymptotically independent of $T_W^{(r)}(0)$ as $t \to \infty$, then*

$$t\Delta_W(t) \Rightarrow \Delta_W^* \equiv \frac{T_{W,1}^{(r)} - T_{W,2}^{(r)}}{\lambda} \quad as \quad t \to \infty, \tag{4}$$

where $T^{(r)}_{W,1}$ and $T^{(r)}_{W,1}$ are independent random variables distributed as $T^{(r)}_W \stackrel{\mathrm{d}}{=} T^{(r)}_W(t)$, so that $E[\Delta^*_W] = 0$, $Var(\Delta^*_W) = 2Var(T^{(r)}_W)/\lambda^2$ and

$$RMSE_W(t) \equiv \frac{\sqrt{Var(\Delta_W(t))}}{W} \sim \frac{\sqrt{2Var(T^{(r)}_W)}}{t\lambda W} = \frac{\sqrt{2Var(T^{(r)}_W)}}{tL} \quad as \quad t \to \infty. \tag{5}$$

Theorem 4.1 implies that the root mean square error (RMSE) given in (5) is asymptotically inversely proportional to t. The behavior of $RMSE_W(t)$ in more detail depends on the specific model. As in §5 of the main paper, we gain insight by considering the two idealized queueing models.

4.2.2 The Single-Server Paradigm

Consider the same single-server model as in §5.5 of the main paper. Let $c^2_S \equiv Var(S)/E[S]^2$ be the squared coefficient of variation (SCV) of S and similarly for other nonnegative random variables. Let S_r be the steady-state version of $S_r(t)$, the residual service in process, if any. In steady state, $E[S_r] = \rho E[S_e] = \rho E[S](c^2_S + 1)/2 < \infty$, where $\rho \equiv \lambda E[S]$ is the traffic intensity.

To understand the implications of Theorem 4.1 in this context, we exploit the conventional heavy-traffic approximation for the number in system, arising as $\rho \uparrow 1$; see Whitt (2002). In particular, this limit tells us that (i) the first term in (38) of the main paper is asymptotically negligible (relatively) and (ii) $L(t)$ should be approximately exponentially distributed with mean

$$L \approx \left(\frac{\rho^2}{1-\rho}\right)\left(\frac{c_a + c^2_S}{2}\right), \tag{6}$$

where λc^2_a is the normalization constant in a central limit theorem for the arrival counting process, with c^2_a being the SCV of an interarrival time if the arrival process is a renewal process.

Corollary 4.1 (*RMSE for the stationary single-server model*) *In addition to the assumptions of Theorem 4.1, assume that the system is a $G/GI/1$ queue in heavy traffic. Then*

$$Var(T^{(r)}_W) \approx \left(\frac{(12L^3 - 6L^2 + L)E[S]^2c^2_S}{6} + \frac{(20L^4 - 16L^3 + 3L^2)E[S]^2}{4}\right) \approx 5E[S]^2L^4 \tag{7}$$

for L in (6), with the final approximation including only the leading term in L. Hence, for suitably large ρ and t (increasing with ρ),

$$RMSE_W(t) \approx \frac{\sqrt{10}E[S]L}{t} \approx \left(\frac{\sqrt{10}E[S]}{t}\right)\left(\frac{\rho^2}{1-\rho}\right)\left(\frac{c_a + c^2_S}{2}\right). \tag{8}$$

Proof. The heavy-traffic condition implies that we can ignore the first term in (38) of the main paper and assume that $L(t)$ is exponential with mean in (6). Thus we can apply the conditional variance formula, p. 119 of Ross (2010), to (38) of the main paper without the first term and then the formula for sums of integers and their squares. From the exponential property, $E[L(t)^k] = k!L^k$. We combine Theorem 4.1 and (7) to get the final approximation (8). ■

In contrast, suppose that we choose the interval $[0, t]$ so that $L(0) = L(t) = pL$, where $0 < p < \infty$. Then we can replace the random $L(t)$ in (38) of the main paper with the deterministic value pL, p times the deterministic mean L.

Corollary 4.2 (*reduction in RMSE for the stationary single-server model*) *In addition to the assumptions of Corollary 4.1, assume that $L(0) = L(t) = pL$. Then*

$$Var(T_W^{(r)}) \approx \left(\frac{(2(pL)^3 - 3(pL)^2 + (pL)E[S]^2 c_S^2}{6}\right) \approx E[S]^2 c_S^2 \frac{p^3 L^3}{3} \tag{9}$$

for L in (6), where the last expression is the leading term in L. Hence, Hence, for suitably large ρ and t (increasing with ρ),

$$RMSE_W(t) \approx \frac{E[S]\sqrt{c_S^2 p^3 L}}{3t} \approx \left(\frac{\sqrt{c_S^2} E[S] p^{3/2}}{t}\right) \left(\frac{\rho^2}{1-\rho}\right)^{1/2} \left(\frac{c_a + c_S^2}{2}\right)^{1/2}. \tag{10}$$

Proof. We follow the proof of Corollary 4.1. Then the second term in (7) does not appear and we need not consider higher moments in the first term, so that (7) becomes (9). ■

Since the heavy-traffic approximations in (8) and (10) are order $O(L) = O(1/(1-\rho))$ and $O(\sqrt{L}) = O(\sqrt{1/(1-\rho)})$ as $\rho \uparrow 1$, respectively, we conclude that we should significantly reduce the edge effects for a heavily loaded single-server queue by letting $L(t) \approx L(0) \approx pL$, even for $p = 1$. The p produces an additional multiplicative factor of $p^{3/2}$ in $RMSE_W(t)$.

Finally, we remark that formulas (8) and (10) provide a way to estimate the required time t to achieve a target value α for $RMSE_W(t)$ in the two settings. However, the time required for the conditions to be valid is even larger; e.g., the time for $V(t)$ to be approximately independent of $V(0)$ is $O(1/(1-\rho^2)$ as $\rho \uparrow 1$; see Abate and Whitt (1988) and Whitt (1989).

4.2.3 The Infinite-Server Paradigm

Consider the same infinite-server paradigm as in §5.3 of the main paper. As observed before, the situation is quite different for infinite-server and single-server queues. For the stationary setting, consider the $M/GI/\infty$ IS model, Let $\gamma_3(S) \equiv E[S^3]/3E[S]^3$, noting that $\gamma_3(S) = 2$ if S is exponential, and let $N(m, \sigma^2)$ denote a Gaussian random variable with mean m and variance σ^2.

Corollary 4.3 (*RMSE for the stationary infinite-server model*) *In addition to the assumptions of Theorem 4.1, assume that the system is a $M/GI/\infty$ model with service time S having cdf G, $Var(T_W^{(r)}) = \lambda E[S]E[(S_e)^2] = L(E[S])^2\gamma_3(S)$, so that*

$$RMSE_W(t) \sim (1/t)\sqrt{2\gamma_3(S)W/\lambda} \quad as \quad t \to \infty. \tag{11}$$

For λ not too small, $T_W^{(r)}(t)$ is also approximately Gaussian, so that $\Delta_W(t) \stackrel{d}{=} N(0, 2Var(T_W^{(r)})/\lambda^2 t^2)$ and

$$\Delta_W^{rel}(t) \equiv \frac{E[|\Delta_W(t)|]}{W} \approx \frac{\sqrt{Var(T_W^{(r)})}}{\lambda W t} \approx RMSE_W(t)/\sqrt{2}. \tag{12}$$

Proof. For the $M/GI/\infty$ model, $W = E[S]$ and the number in system is Poisson with mean $\lambda E[S]$. Conditional on the number, the remaining service times are distributed as i.i.d random variables distributed according to S_e in (31) of the main paper; p. 161 of Takacs (1962). Hence, $V(t) \stackrel{d}{=} S_{e,1} + \cdots + S_{e,L(t)}$, where $S_{e,i}$, $i \geq 1$, are i.i.d. random variables distributed as S_e. Hence, V has a compound Poisson distribution with mean $E[V] = \lambda E[S]E[S_e]$ and variance $Var(V) = \lambda E[S]E[(S_e)^2]$; see Ex. 3.19 of Ross (2010). By (31) of the main paper, $E[(S_e)^2] = E[S^3]/3E[S] = E[S]^2\gamma_3(S)$ for $\gamma_3(S)$ defined above. When λ is not too small, the exact compound Poisson distribution is approximately Gaussian. For (12), we use the exact formula and simplify: $E[|\sqrt{2}N(0,1)|] = 2/\sqrt{\pi} \approx 1.12 \approx 1$. ∎

From Corollary 4.3 and (31) of the main paper, we see that for the infinite-server model, $RMSE_W(t)$ depends on the first three moments of the service-time (waiting time) distribution. Otherwise, $RMSE_W(t)$ and $\Delta_W^{rel}(t)$ are inversely proportional to t and directly proportional to $\sqrt{W/\lambda}$. Hence the impact of edge effects is less in many-server systems with large λ.

Paralleling Corollary 4.2, we now see what happens if we choose the interval $[0,t]$ so that $L(0) = L(t) = pL$ for $0 < p < \infty$.

Corollary 4.4 (*reduction in RMSE for the infinite-server model*) *In addition to the assumptions of Corollary 4.3, assume that $L(0) = L(t) = L$. Then $Var(T_W^{(r)})$ is the value in Corollary 4.3 multiplied by $p\eta$, $0 < \eta < 1$, where $\eta \equiv Var(S_e)/E[S_e^2] = 1 - (c_S^2 + 1)^2/4\gamma_3(S)$. Hence, $RMSE_W(t)$ and $\Delta_W^{rel}(t)$ are the values in Corollary 4.3 multiplied by $\sqrt{p\eta}$.*

Proof. Now we have a sum of pL i.i.d. random variables distributed as S_e, which gives the variance formula as in Corollary 4.3 with $E[S_e^2$ replaced by the smaller value $Var(S_e)$ and L replaced by pL. ∎

From Corollaries 4.3 and 4.4, we see that having $L(t) = L(0) = pL$ reduces $RMSE_W(t)$ and $\Delta_W^{rel}(t)$ by a factor of $\sqrt{p\eta} \equiv pVar(S_e)/E[S_e^2]$, but in both cases $RMSE_W(t)$ and $\Delta_W^{rel}(t)$ tends to be of order $O(\sqrt{W/\lambda}/t)$, which is small as either t or λ become large. We thus conclude that the edge effects should be much less important for many-server systems than for single-server systems.

Since the call center is a many-server system, it is natural to apply Corollary 4.3 to estimate the relative error $\Delta_W^{rel}(t)$ in (3), estimating W. Assuming that S is exponential because the estimated SCV for the waiting times over $[10,16]$ was 1.02, we apply Corollary 4.3 to estimate $Var(V) = 2LW^2$. We thus obtain estimated values for $\Delta_W^{rel}(t) \approx \sqrt{2}W/t\sqrt{L}$ based on the estimates $\bar{W}(t)$ in Table 3 of the main paper and associated estimates $\bar{L}(t)$ of 32.6, 29.8 and 31.7. The estimates of 27.4% for $t = 3$ minutes, 3.3% for $t = 30$ minutes and 0.29% for $t = 300$ minutes provide good rough estimates of the observed values 20.3%, 5.6% and 0.5%, respectively.

5 A New Algorithm to Estimate Confidence Intervals by Batch Means

In §4 of the main paper we showed how to estimate confidence intervals for the parameters L, λ and W for an approximately stationary interval based on the averages in (1) of the main paper and (3) of the main paper using the method of batch means. Our approach for doing so was quite standard, except for the estimation of W based on $\bar{L}(t)$ and $\bar{\lambda}(t)$. It is natural to consider if the finite-time version of Little's law in Theorem 2 of the main paper can be used to create improved estimators of confidence intervals based on batch means. As a guide we can draw on §5 of the main paper, where we showed how the bias can be reduced by applying Theorem 2 of the main paper and using the values $R(0)$ and $L(t)$ at the ends of the interval, i.e., drawing on formulas (5), (7), and (27) of the main paper.

First, Theorem 1 of the main paper suggests that it could be advantageous to start and end each batch with an empty system. That is especially true if the queueing model regenerates when the system empties. If that is the case, then the successive batches become i.i.d. and we are in the setting of regenerative simulation, as in Asmussen and Glynn (2007). However, as we noted in §2, most large-scale service systems never empty completely during stationary intervals of operation. We could achieve these end effects by using the altered definitions in Theorem 3 of the main paper, but we already saw in §3.4.2. of the main paper these alternative definitions cause problems.

Even if we do not alter the definitions, formulas (5), (7), and (27) of the main paper suggest then it should be good to have $L(t) = R(0)$, if possible. To meet that goal, we developed a new algorithm for estimating confidence intervals exploiting that property. However, as illustrated by Table 3 below, our experiments indicated that this new algorithm did not provide any improvement for the call center example. Hence, we do not highlight it here. We include it here for two reasons: First, the negative results for the call center example are good to know, indicating that the approach is less promising than might be anticipated. Second, the new algorithm may possibly be useful in other contexts.

In summary, the new algorithm is based on the following ideas:

- We should be able to reduce the impact of edge effects within each batch subinterval by trying to have the number of arrivals equal to the number of departures in each subinterval. That is, we should try to end each subinterval with the same number in system that we start with, so that the edge effect errors have the best chance of canceling.
- We should start and end all measurement subintervals at the instants of arrival into a fixed state; i.e., we should start and end at times such that $L(t) = k^*$, To successfully carry this out, the state k^* should be a frequently visited state.
- We should make sure that the beginning and end times are sufficiently separated; i.e., the measurement subintervals are long enough. Furthermore, we advocate selecting them to be of nearly equal length.

The algorithm requires specifying 4 parameters:

t length of the interval,

k^* boundary state,

m number of measurement subintervals,

p proportion of each interval to allocate to slack at the end.

We will create m measurement subintervals and an extra slack interval so that the m intervals falls within $[0, T]$ nearly. Let each of m measurement subintervals have length at least $x = \frac{t}{m(1+p)}$. Then the final slack has length $s = mpx = \frac{pt}{1+p}$. To verify, we have $mx + s = \frac{t}{1+p} + \frac{pt}{1+p} = t$. As an illustrating example, we consider the interval $[10, 16]$ with $t = 360$, $k^* = 33$, $m = 10$, and $p = 1/8$. By the reasoning above, $x = 360/(10 \times (9/8)) = 32$ minutes (the total measurement period is about $8/9$ of the total) and $s = 40$ minutes.

ALGORITHM

1. Pick a frequently visited state, e.g, $L_{sys}(t) = 33$.
2. Start the first measurement subinterval at the first time $t_{1,L}$ at which $L_{sys}(t_{1,L}) = 33$.
3. For fixed x, e.g., $x = 32$, end the measurement subinterval at the first time $t_{1,U} \geq t_{1,L} + x$ such that $L_{sys}(t_{1,U}) = 33$.
4. Let the right endpoint of each interval be the left endpoint of the next interval.
5. Apply estimate confidence intervals using the Student t distribution with $m - 1$ degrees of freedom.
6. Record subinterval endpoints and produce a record of the batch statistics. Use these results to adjust the parameters (k^*, m, p) to get the best estimates.

We conducted several experiments with our new algorithm using the call center data, but the new algorithm did not prove to be advantageous; i.e., it did not provide significant improvement. Indeed, it often performed slightly worse, evidently because it failed to use some data, because of the slack intervals. Table 3 illustrates some of the results. For example, for the interval $[10, 16]$, we used $k^* = 32$, $m = \{5, 10, 20\}$ and $p = 1/8$. The results for the direct estimates, $\bar{L}_{alg}(t)$, $\bar{\lambda}_{alg}(t)$ and $\bar{W}_{alg}(t)$, as well as the indirect estimate for W, $\bar{W}_{alg;\lambda,L}(t)$, can be compared to the results in Table 1 of the main paper. We conclude that the lack of improvement is due to the high arrival rates and short service times (9.44 arrivals per minute and 3.39 minutes for average service time in the interval $[10, 16]$), which changes the value of $L_{sys}(t)$ every second; ending each subinterval with the same number in system that we started with is less important.

Table 3: Estimate results using the new algorithm.

$Interval$	k^*	m	p	x	$\bar{L}_{alg}(t)$	$\bar{\lambda}_{alg}(t)$	$\bar{W}_{alg}(t)$	$\bar{W}_{alg;\lambda,L}(t)$
$[10, 16]$	32	5	$1/8$	64.0	31.8 ± 1.9	9.48 ± 0.80	3.37 ± 0.26	3.36 ± 0.30
	32	10	$1/8$	32.0	31.8 ± 1.9	9.46 ± 0.50	3.37 ± 0.19	3.38 ± 0.21
	32	20	$1/8$	16.0	32.0 ± 1.1	9.45 ± 0.33	3.41 ± 0.14	3.39 ± 0.12
$[10, 11]$	33	5	$1/8$	10.7	32.8 ± 2.1	9.36 ± 1.06	3.50 ± 0.33	3.51 ± 0.24
$[14, 15]$	33	5	$1/8$	10.7	32.8 ± 3.4	9.75 ± 1.76	3.36 ± 0.33	3.39 ± 0.35
$[7, 8]$	8	5	$1/2$	8.0	7.5 ± 1.8	2.92 ± 0.39	2.65 ± 0.72	2.55 ± 0.37
$[20, 21]$	3	5	$1/6$	10.3	2.9 ± 2.2	0.83 ± 0.28	3.36 ± 1.73	3.33 ± 1.25

Even though the algorithm did not prove to be advantageous in our call center example, we think that there may be contexts (possibly with lower arrival rates and longer average service time in a stationary setting) in which our new algorithm could prove to be helpful.

6 Supplementary Tables

Table 4: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/\infty$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	31.5 ± 2.0	209.28	9.33 ± 0.42	9.29	3.38 ± 0.15	1.24	95.1%	31.5 ± 2.0	218.92	9.33 ± 0.45	10.56	3.38 ± 0.15	1.25	95.4%
6	31.5 ± 1.8	203.82	9.33 ± 0.39	9.23	3.38 ± 0.15	1.27	95.7%	31.5 ± 1.9	219.01	9.33 ± 0.42	10.52	3.38 ± 0.14	1.25	95.3%
7	31.5 ± 1.8	200.57	9.33 ± 0.37	9.20	3.38 ± 0.14	1.24	95.0%	31.5 ± 1.8	214.96	9.33 ± 0.41	10.77	3.38 ± 0.14	1.24	94.0%
8	31.5 ± 1.7	199.86	9.33 ± 0.37	9.31	3.38 ± 0.13	1.26	95.0%	31.5 ± 1.8	217.66	9.33 ± 0.40	11.08	3.38 ± 0.13	1.25	95.1%
9	31.5 ± 1.6	195.36	9.33 ± 0.36	9.20	3.38 ± 0.13	1.23	95.2%	31.5 ± 1.7	214.77	9.33 ± 0.39	11.08	3.38 ± 0.13	1.22	95.2%
10	31.5 ± 1.6	193.72	9.33 ± 0.35	9.38	3.38 ± 0.13	1.23	95.0%	31.5 ± 1.7	212.88	9.33 ± 0.39	11.42	3.38 ± 0.13	1.25	95.7%
11	31.5 ± 1.6	192.82	9.33 ± 0.35	9.39	3.38 ± 0.13	1.23	95.1%	31.5 ± 1.7	214.65	9.33 ± 0.39	11.52	3.38 ± 0.13	1.23	94.9%
12	31.5 ± 1.6	191.55	9.33 ± 0.35	9.31	3.38 ± 0.13	1.25	95.3%	31.5 ± 1.7	217.78	9.33 ± 0.39	11.72	3.38 ± 0.13	1.23	95.4%
13	31.5 ± 1.5	187.81	9.33 ± 0.34	9.24	3.38 ± 0.13	1.24	95.0%	31.5 ± 1.6	215.15	9.33 ± 0.39	11.85	3.38 ± 0.12	1.23	94.9%
14	31.5 ± 1.5	184.65	9.33 ± 0.34	9.35	3.38 ± 0.12	1.24	95.1%	31.5 ± 1.6	215.27	9.33 ± 0.39	12.08	3.38 ± 0.12	1.22	95.2%
15	31.5 ± 1.5	181.88	9.33 ± 0.34	9.29	3.38 ± 0.12	1.23	95.4%	31.5 ± 1.6	214.10	9.34 ± 0.39	12.12	3.38 ± 0.12	1.23	94.6%
16	31.5 ± 1.5	179.89	9.33 ± 0.34	9.21	3.38 ± 0.12	1.24	94.5%	31.5 ± 1.6	215.85	9.34 ± 0.38	12.14	3.38 ± 0.12	1.23	95.4%
17	31.5 ± 1.5	179.00	9.33 ± 0.33	9.23	3.38 ± 0.12	1.23	94.7%	31.5 ± 1.6	213.15	9.34 ± 0.39	12.56	3.38 ± 0.12	1.24	95.0%
18	31.5 ± 1.5	175.75	9.33 ± 0.33	9.27	3.38 ± 0.12	1.24	94.7%	31.5 ± 1.6	213.48	9.34 ± 0.39	12.64	3.38 ± 0.12	1.23	94.7%
19	31.5 ± 1.4	173.95	9.33 ± 0.33	9.39	3.38 ± 0.12	1.24	94.4%	31.5 ± 1.6	215.97	9.34 ± 0.39	12.76	3.38 ± 0.12	1.22	94.9%
20	31.5 ± 1.4	170.79	9.33 ± 0.33	9.25	3.38 ± 0.12	1.23	94.4%	31.5 ± 1.6	211.56	9.34 ± 0.39	12.96	3.38 ± 0.12	1.25	95.3%

Table 5: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/s_t$ with staffing based on the square-root-staffing formula using QoS parameter $\beta = 2.5$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	31.5 ± 2.0	211.40	9.33 ± 0.42	9.29	3.38 ± 0.15	1.25	95.3%	31.5 ± 2.0	220.94	9.33 ± 0.45	10.55	3.38 ± 0.15	1.26	95.9%
6	31.5 ± 1.8	205.80	9.33 ± 0.39	9.23	3.38 ± 0.15	1.29	95.7%	31.5 ± 1.9	221.01	9.33 ± 0.42	10.52	3.38 ± 0.15	1.26	95.7%
7	31.5 ± 1.8	202.50	9.33 ± 0.37	9.20	3.38 ± 0.14	1.25	95.2%	31.5 ± 1.8	216.88	9.33 ± 0.41	10.77	3.38 ± 0.14	1.25	94.2%
8	31.5 ± 1.7	201.84	9.33 ± 0.37	9.31	3.38 ± 0.14	1.27	95.6%	31.5 ± 1.8	219.71	9.33 ± 0.40	11.08	3.38 ± 0.13	1.26	95.3%
9	31.5 ± 1.7	197.48	9.33 ± 0.36	9.20	3.38 ± 0.13	1.24	95.5%	31.5 ± 1.7	216.95	9.33 ± 0.39	11.08	3.38 ± 0.13	1.23	95.2%
10	31.5 ± 1.6	195.82	9.33 ± 0.35	9.38	3.38 ± 0.13	1.24	95.2%	31.5 ± 1.7	214.86	9.33 ± 0.39	11.41	3.38 ± 0.13	1.27	95.8%
11	31.5 ± 1.6	194.68	9.33 ± 0.35	9.39	3.38 ± 0.13	1.24	95.2%	31.5 ± 1.7	216.60	9.33 ± 0.39	11.52	3.38 ± 0.13	1.24	95.0%
12	31.5 ± 1.6	193.53	9.33 ± 0.35	9.31	3.38 ± 0.13	1.26	95.4%	31.5 ± 1.7	219.79	9.33 ± 0.39	11.72	3.38 ± 0.13	1.24	95.2%
13	31.5 ± 1.5	189.73	9.33 ± 0.34	9.24	3.38 ± 0.13	1.25	95.0%	31.5 ± 1.7	217.08	9.33 ± 0.39	11.84	3.38 ± 0.13	1.24	95.2%
14	31.5 ± 1.5	186.45	9.33 ± 0.34	9.35	3.38 ± 0.12	1.24	95.4%	31.5 ± 1.6	217.00	9.33 ± 0.39	12.08	3.38 ± 0.12	1.24	95.6%
15	31.5 ± 1.5	183.67	9.33 ± 0.34	9.29	3.38 ± 0.12	1.24	95.6%	31.5 ± 1.6	215.93	9.34 ± 0.39	12.12	3.38 ± 0.12	1.24	94.9%
16	31.5 ± 1.5	181.71	9.33 ± 0.34	9.21	3.38 ± 0.12	1.25	94.9%	31.5 ± 1.6	217.72	9.34 ± 0.38	12.14	3.38 ± 0.12	1.24	95.4%
17	31.5 ± 1.5	180.84	9.33 ± 0.33	9.23	3.38 ± 0.12	1.24	95.0%	31.5 ± 1.6	214.92	9.34 ± 0.39	12.56	3.38 ± 0.12	1.25	94.8%
18	31.5 ± 1.5	177.48	9.33 ± 0.33	9.27	3.38 ± 0.12	1.25	94.6%	31.5 ± 1.6	215.28	9.34 ± 0.39	12.63	3.38 ± 0.12	1.24	94.8%
19	31.5 ± 1.4	175.61	9.33 ± 0.33	9.39	3.38 ± 0.12	1.24	94.4%	31.5 ± 1.6	217.73	9.34 ± 0.39	12.75	3.38 ± 0.12	1.23	94.9%
20	31.5 ± 1.4	172.52	9.33 ± 0.33	9.25	3.38 ± 0.12	1.24	95.0%	31.5 ± 1.6	213.20	9.34 ± 0.39	12.95	3.38 ± 0.12	1.26	95.3%

Table 6: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/s_t$ with staffing based on the square-root-staffing formula using QoS parameter $\beta = 2.0$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	31.5 ± 2.0	218.90	9.33 ± 0.42	9.29	3.38 ± 0.16	1.29	95.2%	31.5 ± 2.1	228.43	9.33 ± 0.45	10.55	3.38 ± 0.16	1.31	95.7%
6	31.5 ± 1.9	212.87	9.33 ± 0.39	9.23	3.38 ± 0.15	1.33	95.8%	31.5 ± 1.9	228.17	9.33 ± 0.42	10.52	3.38 ± 0.15	1.31	95.7%
7	31.5 ± 1.8	209.39	9.33 ± 0.37	9.20	3.38 ± 0.14	1.29	95.5%	31.5 ± 1.9	223.67	9.33 ± 0.41	10.78	3.38 ± 0.14	1.29	94.4%
8	31.5 ± 1.7	208.75	9.33 ± 0.37	9.31	3.38 ± 0.14	1.31	95.4%	31.5 ± 1.8	226.69	9.33 ± 0.40	11.08	3.38 ± 0.14	1.31	95.6%
9	31.5 ± 1.7	204.49	9.33 ± 0.36	9.20	3.38 ± 0.13	1.28	95.4%	31.5 ± 1.8	224.06	9.33 ± 0.39	11.07	3.38 ± 0.13	1.28	95.2%
10	31.5 ± 1.6	202.78	9.33 ± 0.35	9.38	3.38 ± 0.13	1.28	95.3%	31.5 ± 1.7	221.72	9.33 ± 0.39	11.41	3.38 ± 0.13	1.31	95.6%
11	31.5 ± 1.6	201.13	9.33 ± 0.35	9.39	3.38 ± 0.13	1.27	95.4%	31.5 ± 1.7	223.28	9.33 ± 0.39	11.52	3.38 ± 0.13	1.28	94.9%
12	31.5 ± 1.6	200.17	9.33 ± 0.35	9.31	3.38 ± 0.13	1.30	95.2%	31.5 ± 1.7	226.51	9.33 ± 0.39	11.71	3.38 ± 0.13	1.29	95.2%
13	31.5 ± 1.6	196.09	9.33 ± 0.34	9.24	3.38 ± 0.13	1.28	94.9%	31.5 ± 1.7	223.43	9.33 ± 0.39	11.83	3.38 ± 0.13	1.28	95.1%
14	31.5 ± 1.5	192.62	9.33 ± 0.34	9.35	3.38 ± 0.13	1.27	95.3%	31.5 ± 1.7	222.92	9.33 ± 0.39	12.08	3.38 ± 0.13	1.27	95.4%
15	31.5 ± 1.5	189.62	9.33 ± 0.34	9.29	3.38 ± 0.13	1.27	95.2%	31.5 ± 1.7	221.94	9.34 ± 0.39	12.11	3.38 ± 0.13	1.28	94.8%
16	31.5 ± 1.5	187.60	9.33 ± 0.34	9.21	3.38 ± 0.12	1.28	94.8%	31.5 ± 1.6	223.77	9.34 ± 0.38	12.13	3.38 ± 0.12	1.28	95.5%
17	31.5 ± 1.5	186.85	9.33 ± 0.33	9.23	3.38 ± 0.12	1.27	95.0%	31.5 ± 1.6	220.90	9.34 ± 0.39	12.54	3.38 ± 0.12	1.29	94.6%
18	31.5 ± 1.5	183.13	9.33 ± 0.33	9.27	3.38 ± 0.12	1.28	94.8%	31.5 ± 1.6	221.06	9.34 ± 0.39	12.62	3.38 ± 0.12	1.28	94.6%
19	31.5 ± 1.5	181.11	9.33 ± 0.33	9.39	3.38 ± 0.12	1.27	94.4%	31.5 ± 1.6	223.46	9.34 ± 0.39	12.73	3.38 ± 0.12	1.27	95.0%
20	31.5 ± 1.4	178.03	9.33 ± 0.33	9.25	3.38 ± 0.12	1.26	95.0%	31.5 ± 1.6	218.79	9.34 ± 0.39	12.92	3.38 ± 0.12	1.29	95.5%

Table 7: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/s_t$ with staffing based on the square-root-staffing formula using QoS parameter $\beta = 1.5$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	31.6 ± 2.2	248.68	9.33 ± 0.42	9.29	3.39 ± 0.17	1.48	95.8%	31.6 ± 2.2	258.61	9.33 ± 0.45	10.54	3.39 ± 0.17	1.50	95.9%
6	31.6 ± 2.0	241.54	9.33 ± 0.39	9.23	3.39 ± 0.16	1.51	95.9%	31.6 ± 2.1	256.95	9.33 ± 0.42	10.52	3.39 ± 0.16	1.50	95.5%
7	31.6 ± 1.9	237.36	9.33 ± 0.37	9.20	3.39 ± 0.15	1.46	95.4%	31.6 ± 2.0	251.23	9.33 ± 0.41	10.78	3.39 ± 0.15	1.48	94.0%
8	31.6 ± 1.8	236.18	9.33 ± 0.37	9.31	3.39 ± 0.14	1.47	94.7%	31.6 ± 1.9	254.43	9.33 ± 0.40	11.06	3.39 ± 0.15	1.49	95.4%
9	31.6 ± 1.8	231.72	9.33 ± 0.36	9.20	3.39 ± 0.14	1.44	95.0%	31.6 ± 1.9	251.50	9.33 ± 0.39	11.07	3.39 ± 0.14	1.45	95.3%
10	31.6 ± 1.7	229.75	9.33 ± 0.35	9.38	3.39 ± 0.14	1.44	94.9%	31.6 ± 1.8	248.72	9.33 ± 0.39	11.40	3.39 ± 0.14	1.48	95.1%
11	31.6 ± 1.7	226.64	9.33 ± 0.35	9.39	3.39 ± 0.14	1.42	94.5%	31.6 ± 1.8	249.41	9.33 ± 0.39	11.52	3.39 ± 0.14	1.45	93.9%
12	31.6 ± 1.7	226.06	9.33 ± 0.35	9.31	3.39 ± 0.14	1.45	94.6%	31.6 ± 1.8	252.69	9.33 ± 0.39	11.69	3.39 ± 0.14	1.45	94.8%
13	31.6 ± 1.7	220.98	9.33 ± 0.34	9.24	3.39 ± 0.13	1.42	94.3%	31.6 ± 1.8	248.65	9.33 ± 0.39	11.80	3.39 ± 0.13	1.44	94.6%
14	31.6 ± 1.6	216.99	9.33 ± 0.34	9.35	3.39 ± 0.13	1.40	94.2%	31.6 ± 1.7	246.76	9.33 ± 0.39	12.05	3.39 ± 0.13	1.43	95.4%
15	31.6 ± 1.6	212.92	9.33 ± 0.34	9.29	3.39 ± 0.13	1.40	94.6%	31.6 ± 1.7	245.60	9.34 ± 0.39	12.07	3.39 ± 0.13	1.43	94.5%
16	31.6 ± 1.6	210.35	9.33 ± 0.34	9.21	3.39 ± 0.13	1.41	94.8%	31.6 ± 1.7	247.22	9.34 ± 0.38	12.10	3.39 ± 0.13	1.43	94.5%
17	31.6 ± 1.6	209.73	9.33 ± 0.33	9.23	3.39 ± 0.13	1.39	94.5%	31.6 ± 1.7	243.96	9.34 ± 0.39	12.50	3.39 ± 0.13	1.44	94.9%
18	31.6 ± 1.6	204.82	9.33 ± 0.33	9.27	3.39 ± 0.13	1.40	94.0%	31.6 ± 1.7	243.25	9.34 ± 0.39	12.57	3.39 ± 0.13	1.43	93.9%
19	31.6 ± 1.5	202.30	9.33 ± 0.33	9.39	3.39 ± 0.13	1.39	93.9%	31.6 ± 1.7	245.51	9.34 ± 0.39	12.67	3.39 ± 0.13	1.41	94.0%
20	31.6 ± 1.5	198.84	9.33 ± 0.33	9.25	3.39 ± 0.13	1.38	94.0%	31.6 ± 1.7	240.24	9.34 ± 0.39	12.87	3.40 ± 0.13	1.43	94.9%

Table 8: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/s_t$ with staffing based on the square-root-staffing formula using QoS parameter $\beta = 1.0$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	32.1 ± 2.6	375.14	9.33 ± 0.42	9.29	3.44 ± 0.21	2.40	95.0%	32.1 ± 2.6	386.77	9.33 ± 0.45	10.53	3.44 ± 0.21	2.45	95.3%
6	32.1 ± 2.4	366.51	9.33 ± 0.39	9.23	3.44 ± 0.19	2.42	94.7%	32.1 ± 2.5	383.00	9.33 ± 0.42	10.51	3.44 ± 0.19	2.44	94.7%
7	32.1 ± 2.3	359.64	9.33 ± 0.37	9.20	3.44 ± 0.18	2.30	94.9%	32.1 ± 2.3	371.66	9.33 ± 0.41	10.79	3.44 ± 0.18	2.39	94.0%
8	32.1 ± 2.2	352.84	9.33 ± 0.37	9.31	3.44 ± 0.18	2.30	93.8%	32.1 ± 2.3	373.15	9.33 ± 0.40	11.02	3.44 ± 0.18	2.36	94.1%
9	32.1 ± 2.1	347.24	9.33 ± 0.36	9.20	3.44 ± 0.17	2.24	94.0%	32.1 ± 2.2	368.56	9.33 ± 0.39	11.04	3.44 ± 0.17	2.30	93.4%
10	32.1 ± 2.1	341.21	9.33 ± 0.35	9.38	3.44 ± 0.17	2.20	93.2%	32.1 ± 2.1	361.20	9.33 ± 0.39	11.37	3.44 ± 0.17	2.30	93.5%
11	32.1 ± 2.0	333.14	9.33 ± 0.35	9.39	3.44 ± 0.16	2.16	92.6%	32.1 ± 2.1	359.59	9.33 ± 0.39	11.49	3.44 ± 0.17	2.24	92.2%
12	32.1 ± 2.0	332.42	9.33 ± 0.35	9.31	3.44 ± 0.16	2.17	93.0%	32.1 ± 2.1	361.79	9.33 ± 0.39	11.61	3.44 ± 0.16	2.24	93.0%
13	32.1 ± 2.0	322.04	9.33 ± 0.34	9.24	3.44 ± 0.16	2.10	93.3%	32.1 ± 2.1	352.45	9.33 ± 0.38	11.73	3.44 ± 0.16	2.20	93.3%
14	32.1 ± 1.9	317.28	9.33 ± 0.34	9.35	3.44 ± 0.16	2.06	92.7%	32.1 ± 2.0	347.15	9.33 ± 0.39	11.95	3.44 ± 0.16	2.19	93.7%
15	32.1 ± 1.9	308.05	9.33 ± 0.34	9.29	3.44 ± 0.15	2.03	92.0%	32.1 ± 2.0	342.09	9.34 ± 0.38	11.97	3.44 ± 0.16	2.16	92.8%
16	32.1 ± 1.9	302.03	9.33 ± 0.34	9.21	3.44 ± 0.15	2.02	92.1%	32.1 ± 2.0	342.51	9.34 ± 0.38	12.02	3.44 ± 0.16	2.13	92.3%
17	32.1 ± 1.9	300.46	9.33 ± 0.33	9.23	3.44 ± 0.15	1.99	92.1%	32.1 ± 2.0	337.11	9.34 ± 0.39	12.38	3.44 ± 0.16	2.14	92.6%
18	32.1 ± 1.8	292.74	9.33 ± 0.33	9.27	3.44 ± 0.15	1.97	92.5%	32.1 ± 2.0	332.76	9.34 ± 0.39	12.44	3.44 ± 0.16	2.11	92.4%
19	32.1 ± 1.8	286.85	9.33 ± 0.33	9.39	3.44 ± 0.15	1.94	92.1%	32.1 ± 2.0	333.75	9.34 ± 0.39	12.53	3.44 ± 0.15	2.07	91.7%
20	32.1 ± 1.8	281.32	9.33 ± 0.33	9.25	3.44 ± 0.15	1.90	91.4%	32.1 ± 1.9	324.56	9.34 ± 0.39	12.73	3.44 ± 0.15	2.08	92.5%

Table 9: Supplement to Table 1 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[10, 16]$, constructed using batch means for varying number of batches for the call center example.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$
5	31.9 ± 1.9	167.56	9.44 ± 0.49	11.31	3.38 ± 0.22	2.31	31.9 ± 2.2	234.63	9.42 ± 0.48	10.76	3.38 ± 0.19	1.76
6	31.9 ± 2.0	219.47	9.44 ± 0.47	12.06	3.39 ± 0.25	3.41	31.9 ± 2.1	244.28	9.42 ± 0.58	18.54	3.38 ± 0.23	2.76
7	31.9 ± 1.6	151.42	9.44 ± 0.45	12.22	3.39 ± 0.20	2.34	31.9 ± 1.7	182.94	9.42 ± 0.52	16.05	3.38 ± 0.17	1.72
8	31.9 ± 1.6	171.79	9.44 ± 0.52	17.33	3.39 ± 0.19	2.29	31.9 ± 1.9	224.80	9.42 ± 0.41	10.98	3.38 ± 0.21	2.97
9	31.9 ± 1.5	146.79	9.44 ± 0.45	13.74	3.39 ± 0.17	1.98	31.9 ± 1.6	165.50	9.42 ± 0.43	12.57	3.38 ± 0.17	2.02
10	31.9 ± 1.3	119.36	9.44 ± 0.36	9.23	3.39 ± 0.15	1.50	31.9 ± 1.3	124.41	9.42 ± 0.35	8.78	3.38 ± 0.16	1.79
11	31.9 ± 1.1	84.44	9.44 ± 0.38	10.67	3.39 ± 0.14	1.51	31.9 ± 1.2	102.84	9.42 ± 0.38	10.46	3.38 ± 0.15	1.72
12	31.9 ± 1.2	114.24	9.44 ± 0.39	11.43	3.39 ± 0.16	1.82	31.9 ± 1.7	214.92	9.43 ± 0.39	11.46	3.38 ± 0.15	1.60
13	31.9 ± 1.0	78.66	9.44 ± 0.33	8.32	3.39 ± 0.13	1.27	31.9 ± 1.3	129.34	9.42 ± 0.32	7.74	3.38 ± 0.11	0.97
14	31.9 ± 1.1	90.69	9.44 ± 0.34	8.97	3.39 ± 0.15	1.77	31.9 ± 1.3	123.82	9.43 ± 0.38	10.91	3.38 ± 0.14	1.55
15	31.9 ± 1.3	126.33	9.44 ± 0.31	7.74	3.39 ± 0.16	1.92	31.9 ± 1.5	166.61	9.43 ± 0.38	11.11	3.38 ± 0.14	1.48
16	31.9 ± 1.2	123.16	9.44 ± 0.35	9.98	3.39 ± 0.14	1.46	31.9 ± 1.3	138.84	9.42 ± 0.33	8.43	3.39 ± 0.16	1.90
17	31.9 ± 1.1	97.69	9.44 ± 0.31	7.87	3.39 ± 0.13	1.43	31.9 ± 1.2	124.75	9.43 ± 0.33	8.94	3.38 ± 0.14	1.47
18	31.9 ± 1.1	99.74	9.44 ± 0.34	9.14	3.39 ± 0.13	1.45	31.9 ± 1.5	178.32	9.43 ± 0.34	9.21	3.38 ± 0.13	1.37
19	31.9 ± 0.9	71.37	9.44 ± 0.32	8.34	3.39 ± 0.13	1.34	31.9 ± 1.2	108.72	9.42 ± 0.29	7.05	3.39 ± 0.14	1.55
20	31.9 ± 1.0	76.47	9.44 ± 0.30	7.30	3.39 ± 0.15	1.78	31.9 ± 1.4	158.73	9.45 ± 0.40	13.21	3.38 ± 0.11	1.03

Table 10: Supplement to Table 2 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[14, 15]$, constructed using batch means for varying number of batches for the idealized simulation model, $M_t/M/\infty$. Estimated confidence interval coverage is shown for the two waiting time estimates for the simulations based on 1000 replications.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	cov.	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$	cov.
5	31.4 ± 4.0	143.73	9.32 ± 1.04	9.50	3.37 ± 0.37	1.19	95.6%	31.4 ± 4.9	207.65	9.36 ± 1.32	15.40	3.38 ± 0.37	1.23	94.8%
6	31.4 ± 3.6	132.78	9.32 ± 0.97	9.49	3.37 ± 0.35	1.21	95.9%	31.4 ± 4.6	211.63	9.37 ± 1.24	15.49	3.38 ± 0.35	1.22	95.5%
7	31.4 ± 3.4	126.29	9.32 ± 0.93	9.55	3.37 ± 0.34	1.24	95.9%	31.4 ± 4.4	215.16	9.39 ± 1.21	16.16	3.38 ± 0.33	1.22	95.5%
8	31.4 ± 3.2	118.40	9.32 ± 0.91	9.55	3.37 ± 0.32	1.22	95.5%	31.4 ± 4.3	211.32	9.41 ± 1.19	16.27	3.39 ± 0.33	1.26	95.0%
9	31.4 ± 3.0	111.49	9.32 ± 0.88	9.37	3.37 ± 0.32	1.25	95.5%	31.4 ± 4.2	213.03	9.43 ± 1.16	16.47	3.39 ± 0.32	1.26	95.3%
10	31.4 ± 2.9	104.33	9.32 ± 0.87	9.44	3.37 ± 0.32	1.25	95.8%	31.4 ± 4.1	210.51	9.45 ± 1.15	16.62	3.40 ± 0.32	1.29	95.4%
11	31.4 ± 2.8	99.07	9.32 ± 0.86	9.37	3.37 ± 0.31	1.26	95.2%	31.4 ± 4.1	210.26	9.46 ± 1.14	16.52	3.40 ± 0.32	1.28	95.1%
12	31.4 ± 2.7	93.70	9.32 ± 0.86	9.52	3.37 ± 0.31	1.24	96.1%	31.4 ± 4.0	211.25	9.48 ± 1.12	16.39	3.41 ± 0.32	1.31	94.9%
13	31.4 ± 2.6	89.38	9.32 ± 0.84	9.30	3.37 ± 0.31	1.26	95.4%	31.4 ± 4.0	210.34	9.50 ± 1.11	16.44	3.41 ± 0.31	1.31	95.2%
14	31.4 ± 2.5	85.31	9.32 ± 0.84	9.42	3.37 ± 0.31	1.26	95.4%	31.4 ± 4.0	214.17	9.52 ± 1.11	16.52	3.42 ± 0.31	1.34	95.3%
15	31.4 ± 2.4	80.83	9.32 ± 0.83	9.39	3.37 ± 0.30	1.25	95.5%	31.4 ± 4.0	211.70	9.54 ± 1.09	16.24	3.42 ± 0.31	1.36	95.9%
16	31.4 ± 2.4	77.56	9.32 ± 0.83	9.50	3.37 ± 0.30	1.26	94.9%	31.4 ± 3.9	211.65	9.56 ± 1.09	16.36	3.43 ± 0.31	1.38	95.3%
17	31.4 ± 2.3	74.11	9.32 ± 0.83	9.50	3.37 ± 0.30	1.26	95.4%	31.4 ± 3.9	213.05	9.58 ± 1.09	16.47	3.44 ± 0.32	1.40	95.6%
18	31.4 ± 2.2	71.05	9.32 ± 0.82	9.40	3.37 ± 0.30	1.25	94.9%	31.4 ± 3.9	211.56	9.60 ± 1.08	16.27	3.44 ± 0.32	1.41	95.1%
19	31.4 ± 2.2	68.08	9.32 ± 0.82	9.50	3.37 ± 0.30	1.27	95.0%	31.4 ± 3.9	213.77	9.63 ± 1.09	16.70	3.45 ± 0.32	1.45	94.7%
20	31.4 ± 2.1	65.69	9.32 ± 0.82	9.38	3.37 ± 0.30	1.28	95.9%	31.4 ± 3.9	212.17	9.64 ± 1.07	16.34	3.46 ± 0.32	1.44	94.3%

Table 11: Supplement to Table 2 of the main paper: Direct estimates of L, λ and W from (1) of the main paper plus indirect estimate $\bar{W}_{L,\lambda}(t)$ from (3) of the main paper with associated 95% confidence intervals for the approximately stationary time interval $[14, 15]$, constructed using batch means for varying number of batches for the call center example.

m	$\bar{L}(t)$	$\bar{\sigma}^2_L(t)$	$\bar{\lambda}(t)$	$\bar{\sigma}^2_\lambda(t)$	$\bar{W}(t)$	$\bar{\sigma}^2_W(t)$	$\bar{L}_{\lambda,W}(t)$	$\bar{\sigma}^2_{L;\lambda,W}(t)$	$\bar{\lambda}_{L,W}(t)$	$\bar{\sigma}^2_{\lambda;L,W}(t)$	$\bar{W}_{\lambda,L}(t)$	$\bar{\sigma}^2_{W;L,\lambda}(t)$
5	32.6 ± 1.9	28.30	9.82 ± 0.82	5.18	3.33 ± 0.21	0.34	32.6 ± 2.3	42.21	9.84 ± 0.97	7.35	3.33 ± 0.10	0.09
6	32.6 ± 2.1	39.92	9.82 ± 0.81	5.98	3.32 ± 0.18	0.30	32.6 ± 3.0	80.12	9.85 ± 0.91	7.48	3.33 ± 0.09	0.07
7	32.6 ± 2.2	49.22	9.82 ± 1.22	14.87	3.34 ± 0.19	0.38	32.6 ± 3.3	112.30	9.84 ± 1.15	13.21	3.36 ± 0.29	0.85
8	32.6 ± 1.6	26.57	9.82 ± 0.95	9.64	3.34 ± 0.22	0.51	32.6 ± 2.5	64.95	9.84 ± 0.91	8.98	3.35 ± 0.26	0.73
9	32.6 ± 1.7	31.37	9.82 ± 0.78	6.94	3.32 ± 0.24	0.65	32.6 ± 3.5	138.99	9.92 ± 0.94	9.97	3.34 ± 0.18	0.36
10	32.6 ± 1.6	30.34	9.82 ± 0.79	7.35	3.33 ± 0.21	0.50	32.6 ± 2.9	95.52	9.90 ± 1.04	12.72	3.34 ± 0.16	0.31
11	32.6 ± 1.5	26.96	9.82 ± 1.04	13.14	3.32 ± 0.21	0.51	32.6 ± 4.1	199.81	9.92 ± 0.89	9.60	3.38 ± 0.29	1.01
12	32.6 ± 1.6	32.34	9.82 ± 0.70	6.13	3.32 ± 0.18	0.40	32.6 ± 2.9	102.77	9.90 ± 0.78	7.63	3.34 ± 0.16	0.31
13	32.6 ± 1.4	25.70	9.82 ± 0.84	8.86	3.33 ± 0.20	0.51	32.6 ± 3.2	126.40	9.92 ± 0.84	8.95	3.37 ± 0.23	0.69
14	32.6 ± 1.6	31.59	9.82 ± 0.91	10.61	3.35 ± 0.20	0.49	32.6 ± 2.5	81.73	9.88 ± 0.89	10.12	3.38 ± 0.28	0.99
15	32.6 ± 1.4	24.98	9.82 ± 0.90	10.55	3.29 ± 0.25	0.80	32.6 ± 4.3	242.40	10.09 ± 0.84	9.18	3.39 ± 0.27	0.92
16	32.6 ± 1.5	30.47	9.82 ± 0.76	7.65	3.36 ± 0.24	0.75	32.6 ± 2.2	65.86	9.93 ± 1.00	13.18	3.37 ± 0.23	0.70
17	32.6 ± 1.4	24.85	9.82 ± 0.74	7.26	3.31 ± 0.22	0.63	32.6 ± 3.4	156.18	10.04 ± 0.84	9.39	3.40 ± 0.32	1.34
18	32.6 ± 1.4	25.46	9.82 ± 0.90	10.93	3.30 ± 0.22	0.67	32.6 ± 3.8	192.93	10.07 ± 0.82	9.17	3.42 ± 0.31	1.29
19	32.6 ± 1.5	29.28	9.82 ± 0.75	7.67	3.33 ± 0.20	0.53	32.6 ± 3.0	118.58	9.95 ± 0.78	8.26	3.38 ± 0.21	0.58
20	32.6 ± 1.3	22.49	9.82 ± 0.81	9.00	3.32 ± 0.23	0.71	32.6 ± 3.6	179.68	10.09 ± 0.91	11.35	3.43 ± 0.31	1.36

Table 12: Supplement to Table 3 of the main paper: Comparison of the direct and indirect estimators $\bar{W}(t)$ and $\bar{W}_{L,\lambda}(t)$ for the 20 subintervals of $[14, 15]$ for $t = 3$ minutes for the data from §3 of the main paper.

Interval	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\|\Delta_W(t)\|$	$\Delta_W^{rel}(t)$	$\|U\|$	$\|A\|$	$\|B\|$	$\|C\|$	$\|D\|$	$\|E\|$	$\|F\|$	$\|(\|F\| - \|B\|)\|$
[14 : 00, 14 : 03]	3.82	2.93	0.895	30.6%	302.7	30.0%	15.8%	12.5%	3.8%	12.3%	25.6%	9.8%
[14 : 03, 14 : 06]	3.55	3.30	0.254	7.7%	342.4	33.1%	19.1%	14.6%	3.1%	8.6%	21.4%	2.4%
[14 : 06, 14 : 09]	3.13	4.05	0.929	22.9%	308.9	35.7%	22.1%	17.9%	4.7%	4.8%	14.9%	7.2%
[14 : 09, 14 : 12]	3.66	3.46	0.198	5.7%	319.1	35.0%	18.7%	13.0%	1.9%	10.9%	20.5%	1.8%
[14 : 12, 14 : 15]	2.66	3.20	0.543	17.0%	327.4	38.1%	21.4%	11.3%	4.3%	9.5%	15.4%	6.0%
[14 : 15, 14 : 18]	2.86	3.19	0.333	10.4%	300.6	39.6%	20.5%	8.6%	3.0%	11.5%	16.8%	3.7%
[14 : 18, 14 : 21]	4.22	3.25	0.970	29.8%	298.1	33.3%	18.8%	6.8%	2.6%	10.2%	28.2%	9.4%
[14 : 21, 14 : 24]	2.98	3.20	0.227	7.1%	266.2	29.4%	21.3%	18.0%	3.2%	9.2%	18.9%	2.4%
[14 : 24, 14 : 27]	3.94	2.81	1.132	40.3%	336.1	27.4%	16.9%	12.3%	4.2%	9.8%	29.4%	12.5%
[14 : 27, 14 : 30]	2.64	3.47	0.828	23.8%	308.0	27.9%	21.1%	24.4%	4.7%	9.2%	12.8%	8.3%
[14 : 30, 14 : 33]	3.10	2.91	0.197	6.8%	324.2	31.2%	18.9%	16.4%	1.9%	10.6%	21.1%	2.1%
[14 : 33, 14 : 36]	3.09	4.66	1.564	33.6%	297.4	39.4%	20.4%	20.4%	3.8%	5.6%	10.4%	10.0%
[14 : 36, 14 : 39]	4.06	2.56	1.504	58.8%	311.3	28.8%	12.9%	16.5%	2.4%	10.9%	28.4%	15.5%
[14 : 39, 14 : 42]	2.92	5.23	2.310	44.2%	309.5	37.9%	22.4%	22.8%	2.3%	5.7%	8.9%	13.4%
[14 : 42, 14 : 45]	3.75	3.59	0.155	4.3%	318.8	41.0%	19.5%	11.3%	1.4%	6.1%	20.7%	1.2%
[14 : 45, 14 : 48]	2.78	3.15	0.367	11.7%	339.4	44.6%	18.7%	11.3%	3.1%	6.9%	15.4%	3.3%
[14 : 48, 14 : 51]	3.39	2.73	0.667	24.5%	309.3	34.5%	17.5%	11.8%	2.9%	8.7%	24.6%	7.1%
[14 : 51, 14 : 54]	3.30	3.53	0.230	6.5%	302.3	31.1%	21.9%	15.3%	1.0%	11.0%	19.7%	2.2%
[14 : 54, 14 : 57]	3.60	4.55	0.951	20.9%	313.1	39.7%	22.7%	11.2%	3.5%	7.3%	15.7%	7.0%
[14 : 57, 15 : 00]	2.87	2.88	0.009	0.3%	284.8	37.2%	20.4%	9.1%	4.8%	8.2%	20.3%	0.1%
Average	3.32	3.43	0.713	20.3%	311.0	34.7%	19.5%	14.3%	3.1%	8.8%	19.4%	6.3%

Table 13: Supplement to Table 3 of the main paper: Comparison of the direct and indirect estimators $\bar{W}(t)$ and $\bar{W}_{L,\lambda}(t)$ for the 20 subintervals of $[8, 18]$ for $t = 30$ minutes for the data from §3 of the main paper.

Interval	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\|\Delta_W(t)\|$	$\Delta_W^{rel}(t)$	$\|U\|$	$\|A\|$	$\|B\|$	$\|C\|$	$\|D\|$	$\|E\|$	$\|F\|$	$\|(\|F\| - \|B\|)\|$
[08 : 00, 08 : 30]	3.17	3.14	0.030	1.0%	701.0	9.4%	6.1%	0%	67.9%	9.6%	6.9%	0.8%
[08 : 30, 09 : 00]	3.71	3.41	0.292	8.6%	1013.1	6.7%	4.8%	0%	61.8%	15.0%	11.8%	7.0%
[09 : 00, 09 : 30]	4.29	4.64	0.348	7.5%	1279.7	11.9%	9.3%	0%	69.8%	6.1%	2.9%	6.4%
[09 : 30, 10 : 00]	4.48	4.19	0.289	6.9%	1279.5	6.1%	2.9%	0%	70.5%	11.7%	8.8%	5.9%
[10 : 00, 10 : 30]	3.55	3.61	0.067	1.9%	1219.8	12.2%	9.2%	0%	61.6%	9.2%	7.8%	1.5%
[10 : 30, 11 : 00]	3.64	3.42	0.216	6.3%	1287.3	8.7%	7.3%	0%	62.4%	9.2%	12.3%	5.0%
[11 : 00, 11 : 30]	3.28	3.36	0.085	2.5%	1270.4	9.3%	12.5%	0%	59.4%	8.3%	10.5%	2.0%
[11 : 30, 12 : 00]	3.18	3.34	0.155	4.6%	1167.9	9.1%	11.4%	0%	62.6%	9.5%	7.5%	3.9%
[12 : 00, 12 : 30]	3.70	3.49	0.211	6.0%	1175.8	9.4%	7.5%	0%	60.3%	10.6%	12.2%	4.7%
[12 : 30, 13 : 00]	3.40	3.68	0.280	7.6%	1135.0	11.0%	12.6%	0%	61.6%	8.4%	6.3%	6.3%
[13 : 00, 13 : 30]	3.15	3.18	0.032	1.0%	1003.2	9.5%	7.2%	0%	68.5%	8.4%	6.3%	0.8%
[13 : 30, 14 : 00]	2.90	2.82	0.074	2.6%	1025.5	8.2%	6.2%	0%	68.3%	8.8%	8.4%	2.2%
[14 : 00, 14 : 30]	3.35	3.26	0.092	2.8%	1221.2	7.4%	7.0%	0%	67.9%	8.3%	9.4%	2.3%
[14 : 30, 15 : 00]	3.29	3.40	0.111	3.3%	1126.9	9.0%	10.2%	0%	65.1%	8.4%	7.4%	2.7%
[15 : 00, 15 : 30]	3.72	3.40	0.324	9.5%	1253.2	7.5%	6.7%	0%	61.1%	10.5%	14.2%	7.5%
[15 : 30, 16 : 00]	3.48	3.62	0.144	4.0%	1251.8	10.5%	14.2%	0%	53.9%	10.4%	11.1%	3.1%
[16 : 00, 16 : 30]	3.38	3.51	0.130	3.7%	1086.8	11.9%	12.7%	0%	58.5%	7.0%	9.8%	2.9%
[16 : 30, 17 : 00]	5.94	5.53	0.408	7.4%	1223.3	6.3%	8.7%	0%	57.4%	13.1%	14.6%	5.8%
[17 : 00, 17 : 30]	4.93	6.39	1.462	22.9%	840.9	19.0%	21.2%	0%	50.2%	6.1%	3.5%	17.7%
[17 : 30, 18 : 00]	4.44	4.51	0.076	1.7%	449.0	11.5%	6.5%	0%	67.1%	9.8%	5.1%	1.4%
Average	3.75	3.80	0.241	5.6%	1100.6	9.7%	9.2%	0%	62.8%	9.4%	8.8%	4.5%

Table 14: Supplement to Table 3 of the main paper: Comparison of the direct and indirect estimators $\bar{W}(t)$ and $\bar{W}_{L,\lambda}(t)$ for the 4 overlapping 5-hour subintervals of $[9, 17]$ for $t = 300$ minutes for the data from §3 of the main paper.

Interval	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\|\Delta_W(t)\|$	$\Delta_W^{rel}(t)$	$\|U\|$	$\|A\|$	$\|B\|$	$\|C\|$	$\|D\|$	$\|E\|$	$\|F\|$	$\|(\|F\| - \|B\|)\|$
[09 : 00, 14 : 00]	3.53	3.54	0.012	0.3%	9961.5	1.5%	1.2%	0%	95.5%	0.9%	0.9%	0.3%
[10 : 00, 15 : 00]	3.34	3.35	0.010	0.3%	9736.2	1.5%	1.2%	0%	95.5%	1.0%	0.9%	0.3%
[11 : 00, 16 : 00]	3.34	3.35	0.007	0.2%	9730.2	1.2%	1.6%	0%	94.4%	1.3%	1.4%	0.2%
[12 : 00, 17 : 00]	3.54	3.51	0.034	1.0%	9587.7	1.2%	0.9%	0%	94.4%	1.7%	1.9%	0.9%
Average	3.44	3.44	0.016	0.5%	9753.9	1.4%	1.2%	0%	94.9%	1.2%	1.3%	0.4%

Table 15: Supplement to Table 5 of the main paper: Comparison of the refined estimator $\bar{W}_{L,\lambda,r}(t)$ in (28) of the main paper to the direct estimator $\bar{W}(t)$ in (1) and the unrefined estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper: Average absolute errors (AAE) and average squared errors (ASE) for each hour over 18 weekdays in the call center example.

Interval	$\bar{W}(t)$ in (1)	unrefined in (3)			refined in (28)		
		$\bar{W}_{L,\lambda}(t)$	AAE	ASE	$\bar{W}_{L,\lambda,r}(t)$	AAE	ASE
$[6, 7]$	3.95 ± 1.45	3.48 ± 1.24	0.471	0.337	3.92 ± 1.50	0.132	0.049
$[7, 8]$	3.37 ± 0.29	3.15 ± 0.27	0.250	0.083	3.37 ± 0.30	0.097	0.014
$[8, 9]$	3.24 ± 0.17	3.09 ± 0.16	0.155	0.034	3.21 ± 0.16	0.060	0.005
$[9, 10]$	3.64 ± 0.31	3.57 ± 0.32	0.087	0.011	3.64 ± 0.32	0.040	0.003
$[10, 11]$	3.71 ± 0.34	3.74 ± 0.35	0.077	0.009	3.69 ± 0.33	0.043	0.003
$[11, 12]$	3.49 ± 0.15	3.49 ± 0.15	0.075	0.007	3.50 ± 0.14	0.049	0.004
$[12, 13]$	3.77 ± 0.19	3.80 ± 0.20	0.073	0.008	3.77 ± 0.20	0.057	0.005
$[13, 14]$	3.52 ± 0.08	3.50 ± 0.10	0.078	0.015	3.50 ± 0.10	0.050	0.007
$[14, 15]$	3.49 ± 0.11	3.51 ± 0.09	0.082	0.015	3.49 ± 0.09	0.080	0.010
$[15, 16]$	3.63 ± 0.14	3.61 ± 0.14	0.074	0.009	3.61 ± 0.13	0.070	0.008
$[16, 17]$	3.63 ± 0.16	3.69 ± 0.14	0.099	0.014	3.64 ± 0.15	0.060	0.005
$[17, 18]$	4.62 ± 0.71	4.89 ± 0.77	0.268	0.146	4.52 ± 0.65	0.120	0.028
$[18, 19]$	5.67 ± 1.30	5.62 ± 1.28	0.239	0.092	5.66 ± 1.33	0.158	0.037
$[19, 20]$	5.32 ± 0.96	5.56 ± 1.09	0.487	0.460	5.16 ± 0.88	0.294	0.190
$[20, 21]$	3.75 ± 0.36	3.81 ± 0.40	0.290	0.181	3.73 ± 0.36	0.180	0.061
$[21, 22]$	3.76 ± 0.35	3.81 ± 0.36	0.184	0.057	3.76 ± 0.34	0.106	0.017
$[22, 23]$	3.54 ± 0.47	3.87 ± 0.50	0.331	0.170	3.48 ± 0.41	0.152	0.062
Average $[6, 10]$	3.55 ± 0.55	3.32 ± 0.50	0.241	0.117	3.54 ± 0.57	0.082	0.018
Average $[10, 16]$	3.60 ± 0.17	3.61 ± 0.17	0.076	0.010	3.60 ± 0.17	0.058	0.006
Average $[16, 23]$	4.33 ± 0.62	4.46 ± 0.65	0.271	0.160	4.28 ± 0.59	0.153	0.057
Average All	3.89 ± 0.44	3.89 ± 0.44	0.195	0.097	3.86 ± 0.44	0.103	0.030

Table 16: Supplement to Table 5 of the main paper: Comparison of the refined estimator $\bar{W}_{L,\lambda,r}(t)$ in (28) of the main paper to the direct estimator $\bar{W}(t)$ in (1) and the unrefined estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper: Average absolute errors (AAE) and average squared errors (ASE) for each half hour over 18 weekdays in the call center example.

Interval	$\bar{W}(t)$ in (1)	unrefined in (3)			refined in (28)		
		$\bar{W}_{L,\lambda}(t)$	AAE	ASE	$\bar{W}_{L,\lambda,r}(t)$	AAE	ASE
$[06:00, 06:30]$	2.51 ± 0.35	2.22 ± 0.24	0.291	0.181	2.40 ± 0.34	0.162	0.050
$[06:30, 07:00]$	5.05 ± 2.12	4.39 ± 1.96	0.714	0.788	5.12 ± 2.22	0.416	0.301
$[07:00, 07:30]$	3.11 ± 0.30	2.88 ± 0.25	0.344	0.181	3.14 ± 0.27	0.161	0.045
$[07:30, 08:00]$	3.55 ± 0.37	3.34 ± 0.40	0.317	0.166	3.51 ± 0.40	0.175	0.062
$[08:00, 08:30]$	3.27 ± 0.26	3.13 ± 0.22	0.232	0.127	3.26 ± 0.24	0.157	0.041
$[08:30, 09:00]$	3.22 ± 0.21	3.07 ± 0.19	0.207	0.058	3.19 ± 0.18	0.106	0.016
$[09:00, 09:30]$	3.77 ± 0.38	3.68 ± 0.38	0.149	0.043	3.78 ± 0.39	0.095	0.016
$[09:30, 10:00]$	3.54 ± 0.30	3.49 ± 0.28	0.169	0.038	3.53 ± 0.30	0.080	0.011
$[10:00, 10:30]$	3.62 ± 0.33	3.56 ± 0.31	0.119	0.032	3.57 ± 0.31	0.097	0.015
$[10:30, 11:00]$	3.84 ± 0.49	4.00 ± 0.59	0.246	0.185	3.83 ± 0.46	0.099	0.016
$[11:00, 11:30]$	3.49 ± 0.19	3.47 ± 0.16	0.131	0.037	3.50 ± 0.17	0.081	0.013
$[11:30, 12:00]$	3.47 ± 0.18	3.50 ± 0.18	0.128	0.027	3.49 ± 0.17	0.083	0.010
$[12:00, 12:30]$	3.88 ± 0.28	3.84 ± 0.26	0.145	0.031	3.84 ± 0.29	0.095	0.013
$[12:30, 13:00]$	3.67 ± 0.15	3.78 ± 0.20	0.173	0.047	3.71 ± 0.15	0.102	0.014
$[13:00, 13:30]$	3.50 ± 0.10	3.51 ± 0.13	0.131	0.023	3.47 ± 0.12	0.059	0.006
$[13:30, 14:00]$	3.54 ± 0.15	3.51 ± 0.14	0.172	0.060	3.55 ± 0.16	0.116	0.030
$[14:00, 14:30]$	3.61 ± 0.12	3.56 ± 0.10	0.160	0.039	3.56 ± 0.10	0.154	0.032
$[14:30, 15:00]$	3.38 ± 0.16	3.46 ± 0.12	0.183	0.053	3.43 ± 0.13	0.147	0.030
$[15:00, 15:30]$	3.64 ± 0.16	3.61 ± 0.18	0.167	0.042	3.60 ± 0.16	0.140	0.033
$[15:30, 16:00]$	3.62 ± 0.17	3.61 ± 0.17	0.174	0.046	3.62 ± 0.16	0.146	0.029
$[16:00, 16:30]$	3.47 ± 0.11	3.58 ± 0.14	0.137	0.035	3.55 ± 0.14	0.106	0.020
$[16:30, 17:00]$	3.84 ± 0.39	3.85 ± 0.34	0.181	0.051	3.80 ± 0.38	0.125	0.024
$[17:00, 17:30]$	4.46 ± 0.60	4.48 ± 0.70	0.385	0.347	4.26 ± 0.56	0.216	0.070
$[17:30, 18:00]$	4.94 ± 1.18	5.67 ± 1.41	0.738	0.997	4.77 ± 0.93	0.363	0.355
$[18:00, 18:30]$	5.40 ± 1.33	4.93 ± 1.11	0.571	0.700	5.00 ± 1.23	0.431	0.344
$[18:30, 19:00]$	5.94 ± 1.49	6.43 ± 1.70	0.842	1.106	6.15 ± 1.52	0.351	0.260
$[19:00, 19:30]$	5.89 ± 1.50	5.79 ± 1.77	0.841	1.475	5.29 ± 1.25	0.671	1.226
$[19:30, 20:00]$	4.90 ± 1.04	5.49 ± 1.12	0.805	1.330	5.01 ± 1.08	0.604	0.782
$[20:00, 20:30]$	3.80 ± 0.51	3.94 ± 0.66	0.401	0.422	3.77 ± 0.52	0.294	0.165
$[20:30, 21:00]$	3.70 ± 0.48	3.69 ± 0.41	0.377	0.208	3.67 ± 0.42	0.237	0.105
$[21:00, 21:30]$	3.93 ± 0.55	3.78 ± 0.47	0.605	0.644	3.77 ± 0.48	0.295	0.162
$[21:30, 22:00]$	3.66 ± 0.45	3.93 ± 0.54	0.486	0.442	3.76 ± 0.45	0.183	0.067
$[22:00, 22:30]$	3.49 ± 0.58	3.71 ± 0.55	0.335	0.225	3.51 ± 0.54	0.232	0.099
$[22:30, 23:00]$	3.74 ± 0.73	4.49 ± 0.91	0.754	1.442	3.22 ± 0.54	0.654	0.834
Average $[6, 10]$	3.50 ± 0.53	3.27 ± 0.49	0.303	0.198	3.49 ± 0.54	0.169	0.068
Average $[10, 16]$	3.60 ± 0.21	3.62 ± 0.21	0.161	0.052	3.60 ± 0.20	0.110	0.020
Average $[16, 23]$	4.37 ± 0.78	4.55 ± 0.84	0.533	0.673	4.25 ± 0.72	0.340	0.322
Average All	3.90 ± 0.52	3.92 ± 0.54	0.347	0.342	3.84 ± 0.49	0.219	0.156

Table 17: Supplement to Table 6 of the main paper: Estimating $E[\bar{W}(t)]$ and its associated 95% confidence interval of the time interval $[6, 10]$ over 18 weekdays in the call center example: comparison of the refined estimator $\bar{W}_{L,\lambda,r}(t)$ in (28) of the main paper to the unrefined estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper

Date	$\bar{L}(t)$	$\bar{\lambda}(t)$	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$R(0)$	$L(t)$
1	14.1	4.02	3.62	3.51	3.63	0	32
2	12.8	3.86	3.42	3.32	3.42	0	28
4	17.3	4.42	4.04	3.91	4.05	0	39
7	12.0	3.45	3.64	3.47	3.62	0	36
8	11.5	3.64	3.35	3.17	3.31	0	39
11	13.3	4.42	3.17	3.01	3.14	0	48
14	10.7	4.94	2.23	2.17	2.22	0	26
15	20.5	5.36	4.01	3.82	4.00	0	63
16	16.0	3.99	4.12	4.02	4.14	0	29
17	9.2	2.98	3.14	3.07	3.17	0	22
18	13.9	4.31	3.29	3.22	3.31	0	27
21	11.6	3.51	3.46	3.30	3.42	0	29
22	14.6	4.38	3.44	3.33	3.45	0	37
23	12.6	4.10	3.18	3.07	3.17	0	30
24	12.7	4.35	3.06	2.92	3.04	0	43
25	18.1	4.82	3.85	3.76	3.89	0	41
29	15.4	4.31	3.66	3.58	3.69	0	30
30	18.9	5.13	3.81	3.69	3.78	0	30
95% CI	14.18 ± 1.50	4.22 ± 0.30	3.47 ± 0.22	3.35 ± 0.23	3.47 ± 0.23	0.0	34.9

Table 18: Supplement to Table 6 of the main paper: Estimating $E[\bar{W}(t)]$ and its associated 95% confidence interval of the time interval $[10, 16]$ over 18 weekdays in the call center example: comparison of the refined estimator $\bar{W}_{L,\lambda,r}(t)$ in (28) of the main paper to the unrefined estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper

Date	$\bar{L}(t)$	$\bar{\lambda}(t)$	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$R(0)$	$L(t)$
1	26.0	6.68	3.89	3.90	3.87	32	15
2	20.8	6.05	3.45	3.44	3.43	28	22
4	24.3	7.15	3.38	3.40	3.37	39	18
7	31.6	8.44	3.72	3.74	3.73	36	31
8	31.6	8.74	3.59	3.61	3.60	39	28
11	22.1	6.06	3.61	3.65	3.61	48	21
14	43.5	13.47	3.24	3.23	3.24	25	42
15	44.3	10.83	4.05	4.08	4.07	62	43
16	25.6	6.84	3.73	3.74	3.72	29	21
17	20.8	6.03	3.47	3.45	3.46	21	23
18	27.5	7.84	3.51	3.50	3.50	28	21
21	38.3	10.70	3.58	3.58	3.59	29	36
22	36.9	9.70	3.82	3.81	3.80	35	30
23	26.8	8.01	3.33	3.35	3.34	30	18
24	28.1	8.22	3.39	3.41	3.39	43	21
25	31.9	9.44	3.38	3.37	3.37	41	34
29	40.1	10.66	3.78	3.77	3.77	30	33
30	32.2	8.33	3.87	3.87	3.88	30	40
95% CI	30.68 ± 3.67	8.51 ± 1.00	3.60 ± 0.11	3.61 ± 0.11	3.60 ± 0.11	34.7	27.6

Table 19: Supplement to Table 6 of the main paper: Estimating $E[\bar{W}(t)]$ and its associated 95% confidence interval of the time interval $[16, 23]$ over 18 weekdays in the call center example: comparison of the refined estimator $\bar{W}_{L,\lambda,r}(t)$ in (28) of the main paper to the unrefined estimator $\bar{W}_{L,\lambda}(t)$ in (3) of the main paper

Date	$\bar{L}(t)$	$\bar{\lambda}(t)$	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$R(0)$	$L(t)$
1	8.8	1.77	4.83	4.93	4.83	16	0
2	5.5	1.55	3.35	3.52	3.40	22	0
4	7.9	1.67	4.66	4.76	4.64	18	0
7	7.5	2.18	3.35	3.46	3.34	32	0
8	8.9	2.29	3.81	3.90	3.79	28	1
11	7.8	1.78	4.33	4.42	4.29	21	0
14	13.0	2.94	4.32	4.43	4.28	42	0
15	12.9	2.55	4.95	5.06	4.86	42	0
16	6.5	1.72	3.67	3.79	3.68	21	0
17	8.0	1.89	4.17	4.26	4.15	23	1
18	7.9	1.93	3.95	4.07	3.96	21	0
21	12.2	2.94	4.02	4.13	4.01	36	0
22	15.2	3.21	4.62	4.73	4.62	30	0
23	11.2	2.75	4.02	4.05	3.99	18	0
24	11.9	2.59	4.53	4.59	4.51	21	0
25	11.4	2.24	4.96	5.10	4.92	34	0
29	16.4	4.09	3.94	4.02	3.94	33	0
30	12.1	2.43	4.82	4.99	4.80	40	0
95% CI	10.29 ± 1.52	2.36 ± 0.33	4.24 ± 0.26	4.35 ± 0.26	4.22 ± 0.25	27.7	0.1

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