APPENDIX

to

Approximations for Heavily-Loaded $G/GI/n+GI$ Queues

by

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A Overview

This appendix supplements the main paper by providing additional supporting material. In §B we give explicit analytical formulas for the approximation formulas developed in §3.

The remainder of this appendix gives additional numerical examples, further evaluating the performance of the Gaussian approximations DGA, TGA and TGA-G. First, in §C, we give more examples providing examples of the good performance. Then, in §D, we give examples that expose the limitations of the approximations. We have noted that the abandonment rates cannot be too much larger than 1. Thus, we give examples showing the degraded performance of the approximations when the abandonment rates that are too high in §D.1.

We then consider models with difficult loads. First we consider underloaded models with $\rho < 1$ in §D.2. As discussed in §1, Since the approximations tend to give a one-sided view, the approximations tend to be good for X_n, Q_n, W_n, PoD_n and PoA_n, but not for B_n, when $\rho > 1$, while the approximations tend to be good for X_n and B_n, but not for Q_n, W_n, PoD_n, PoA_n), when $\rho < 1$. However, in both cases all approximations except for X_n degrade as ρ approaches 1 and the system becomes critically loaded. We give examples with traffic intensities $\rho > 1$ that are too close to 1 in §D.3.

B Functions for the FCLT limits in Theorem 2.2

Theorem 2.2 directly comes from theorems 4.2, 5.1 and corollary 4.1 in [21]. The parameters functions $H(t,u)$ $I_i(t)$, $K_i(t)$ in Theorem 2.2 are defines as below:

$$
\begin{aligned}
H(t,u) &= \exp\left\{\int_u^t h(v)\mathrm{d}v\right\} = \exp\left\{\int_u^t -\frac{f(w(\infty))}{\bar{F}(w(\infty))}\mathrm{d}v\right\} = e^{-\frac{f(w(\infty))}{\bar{F}(w(\infty))}(t-u)},\\
I_1^2(t) &= \frac{c_\lambda^2 \bar{F}(w(u))b(u,0)}{\tilde{q}^2(u,w(u))} = \frac{c_\lambda^2 \bar{F}(w(\infty))s\mu}{\lambda^2\bar{F}^2(w(\infty))} = c_\lambda^2\rho^{-1},\\
I_2^2(t) &= \frac{b(u,0)}{\tilde{q}^2(u,w(u))} = \frac{s\mu}{\lambda^2\bar{F}^2(w(\infty))} = 1,\\
I_3^2(t) &= \frac{F(w(u))b(u,0)}{\tilde{q}^2(u,w(u))} = \frac{F(w(\infty))s\mu}{\lambda^2\bar{F}^2(w(\infty))} = 1-\rho^{-1},\\
I^2(t) &= I_1^2(t)+I_2^2(t)+I_3^2(t) = (c_\lambda^2-1)\rho^{-1}+2,\\
K_1(t,u) &= c_\lambda\bar{F}(t-u)\sqrt{\lambda(u)}\mathbb{1}_{\{t-w(t)u<t\}} + \tilde{q}(t,w(t))\sqrt{\lambda(u)}\bar{I}_1(L^{-1}(u))\mathbb{1}_{\{0\le u\le t-w(t)\}},\\
K_2(t,u) &= -\sqrt{b(t,0)-\dot{s}(t)}H(t,u),\\
K_3(t,u) &= -\sqrt{\lambda(u)F(t-u)\bar{F}(t-u)}\mathbb{1}_{\{t-w(t)\le u\le t\}}\\
&\quad + \tilde{q}(t,w(t))\sqrt{\lambda(u)}\bar{I}_3(L^{-1}(u))H(t,L^{-1}(u))\mathbb{1}_{\{0\le u\le t-w(t)\}},
\end{aligned}
$$

where

$$\bar{I}_1(t) \quad = \quad \frac{c\lambda \bar{F}(w(u))b(u,0)}{\tilde{q}(u,w(u))} = \frac{c\lambda \bar{F}(w(\infty))s\mu}{\lambda \bar{F}(w(\infty))}, \quad \bar{I}_3(t) = -\frac{\sqrt{\bar{F}(w(u))b(u,0)}}{\tilde{q}(u,w(u))} = -\frac{\bar{F}(w(\infty))s\mu}{\lambda \bar{F}(w(\infty))},$$

$L(t) = t - w(\infty)$, $\lambda(t) = \lambda t$ and $\mathbb{1}_A$ is an indicator random variable of A.

C More Good examples

We now give additional positive examples for Markov models in §§C.1–C.2 and non-Markov models in §C.3.

C.1 More Examples of Markov $M/M/n + M$ Models

In the main paper, we compare the exact solution with DGA's and TGA's in heavily loaded regime ($\rho = 1.2$). Here we give a even heavier regime, say $\rho = 1.5$.

Table 12: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $\lambda = 100\rho = 150$ and $0.1 \leq \theta \leq 10$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Sim.	DGA	TGA	Sim.	DGA	TGA	Sim.	DGA	TGA
E[X]	5.99E+2	6.00E+2	same	3.00E+2	3.00E+2	same	2.00E+2	2.00E+2	same
rel. err.	±9.19E-1	0%		±4.01E-1	0%		±2.01E-1	0%	
Var(X)	1.49E+3	1.50E+3	same	6.00E+2	6.00E+2	same	3.00E+2	3.00E+2	same
rel. err.	±1.10E+3	1%		±2.41E+2	0%		±8.10E+1	0%	
E[Q]	4.99E+2	5.00E+2	5.00E+2	2.00E+2	2.00E+2	2.00E+2	1.00E+2	9.99E+1	9.99E+1
rel. err.	±9.19E-1	0%	0%	±4.01E-1	0%	0%	±2.01E-1	0%	0%
Var(Q)	1.49E+3	1.50E+3	1.50E+3	6.00E+2	6.00E+2	6.00E+2	3.00E+2	3.00E+2	3.00E+2
rel. err.	±9.19E+2	1%	1%	±1.61E+2	0%	0%	±4.08E+1	0%	0%
E[W]	4.05E+0	4.06E+0	4.06E+0	1.63E+0	1.62E+0	1.62E+0	8.17E-1	8.11E-1	8.11E-1
rel. err.	±7.53E-3	0%	0%	±3.28E-3	0%	0%	±1.64E-3	1%	1%
Var(W)	9.98E-2	1.00E-1	1.00E-1	4.01E-2	4.00E-2	4.00E-2	2.00E-2	2.00E-2	2.00E-2
rel. err.	±7.53E-3	0%	0%	±3.28E-3	0%	0%	±1.64E-3	0%	0%
PoD	1.00E+0	1.00E+0	same	1.00E+0	1.00E+0	same	1.00E+0	1.00E+0	same
rel. err.	±0.00E+0	0%		±0.00E+0	0%		±0.00E+0	0%	
PoA	3.33E-1	3.33E-1	same	3.33E-1	3.33E-1	same	3.34E-1	3.33E-1	same
rel. err.	±1.34E-3	0%		±1.37E-3	0%		±1.33E-3	0%	

	$\theta = 2$			$\theta = 4$			$\theta = 10$		
Perf.	Sim.	DGA	TGA	Sim.	DGA	TGA	Sim.	DGA	TGA
E[X]	1.25E+2	1.25E+2	same	1.12E+2	1.12E+2	same	1.05E+2	1.05E+2	same
rel. err.	±5.05E-2	0%		±2.64E-2	0%		±1.36E-2	0%	
Var(X)	7.53E+1	7.49E+1	same	3.79E+1	3.74E+1	same	1.71E+1	1.49E+1	same
rel. err.	±1.27E+1	1%		±5.96E+0	1%		±2.84E+0	13%	
E[Q]	2.50E+1	2.49E+1	2.49E+1	1.25E+1	1.23E+1	1.24E+1	5.03E+0	4.90E+0	5.08E+0
rel. err.	±5.05E-2	0%	0%	±2.60E-2	2%	1%	±1.17E-2	3%	1%
Var(Q)	7.52E+1	7.49E+1	7.46E+1	3.68E+1	3.74E+1	3.59E+1	1.35E+1	1.49E+1	1.24E+1
rel. err.	±2.63E+0	0%	1%	±7.08E-1	2%	2%	±1.49E-1	10%	9%
E[W]	2.08E-1	2.03E-1	2.03E-1	1.06E-1	1.01E-1	1.01E-1	4.55E-2	4.10E-2	4.25E-2
rel. err.	±4.13E-4	2%	2%	±2.14E-4	5%	5%	±1.01E-4	10%	7%
Var(W)	5.07E-3	5.00E-3	4.98E-3	2.52E-3	2.50E-3	2.40E-3	9.87E-4	1.00E-3	8.44E-4
rel. err.	±4.13E-4	1%	2%	±2.14E-4	1%	5%	±1.01E-4	2%	14%
PoD	9.98E-1	9.98E-1	same	9.79E-1	9.78E-1	same	8.62E-1	9.02E-1	same
rel. err.	±1.27E-4	0%		±4.53E-4	0%		±1.03E-3	5%	
PoA	3.32E-1	3.34E-1	same	3.34E-1	3.32E-1	same	3.35E-1	3.36E-1	same
rel. err.	±1.35E-3	0%		±1.34E-3	0%		±1.31E-3	0%	

Table 13: $M(\lambda^{-1})/M/n+M(\theta^{-1})$ with parameters $(\lambda, n, \rho) = (105, 100, 1.05)$ and low abandonment rates $\theta < 0.1$.

	$\theta = 0.05$			$\theta = 0.02$			$\theta = 0.01$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	2.01E+2	2.00E+2	same	3.50E+2	3.50E+2	same	5.99E+2	6.00E+2	same
rel. err.		0%			0%			0%	
Var(X)	2.00E+3	2.10E+3	same	5.22E+3	5.25E+3	same	1.03E+4	1.05E+4	same
rel. err.		5%			1%			2%	
E[Q]	1.01E+2	1.00E+2	1.00E+2	2.50E+2	2.50E+2	2.50E+2	4.99E+2	5.00E+2	5.00E+2
rel. err.		1%	1%		0%	0%		0%	0%
Var(Q)	1.99E+3	2.10E+3	2.05E+3	5.22E+3	5.25E+3	5.25E+3	1.03E+4	1.05E+4	1.05E+4
rel. err.		5%	3%		1%	1%		2%	2%
E[W]	9.90E-1	9.76E-1	9.78E-1	2.44E+0	2.44E+0	2.44E+0	4.88E+0	4.88E+0	4.88E+0
rel. err.		1%	1%		0%	0%		0%	0%
Var(W)	1.90E-1	2.00E-1	1.95E-1	4.97E-1	5.00E-1	5.00E-1	9.85E-1	1.00E+0	1.00E+0
rel. err.		5%	3%		1%	0%		2%	2%
PoD	9.93E-1	9.85E-1	same	1.00E+0	1.00E+0	same	1.00E+0	1.00E+0	same
rel. err.		1%			0%			0%	
PoA	4.81E-2	4.75E-2	same	4.76E-2	4.75E-2	same	4.76E-2	4.76E-2	same
rel. err.		1%			0%			0%	

C.2 Low Abandonment Rates

We next consider the Markovian $M/M/n + M$ queueing system, with the arrival rate, number of servers and service rate fixed at $(\lambda, n, \mu) = (105, 100, 1)$, but decreasing abandonment rate. As studied in [34], the queueing system tends to heavily overloaded when abandonment rates decrease. The Theorem 4 in [34] states that in an $M/M/n/r + M$ model, a scaled process of number in system converges to a OU process as $s/\theta \to 0$. To reshow the results, we give the Markovian $M/M/n + M$ queueing system, with the arrival rate, number of servers and service rate fixed at $(\lambda, n, \mu) = (105, 100, 1)$ but decreasing abandonment rate θ from $0.05, 0.02$ to 0.01. Table 13 shows that our TGAs continue to work effectively for smaller abandonment rates. Notice that little difference between DGAs and TGAs are presented; it is because when $\theta \to 0$, improvements brought about by truncation become less effective as the queue tends to ED regime.

C.3 More Examples of $GI/GI/n + GI$ Models

We now consider examples with various combinations of high and low variabilities for the interarrival, service and patience times. We use simply phase-type (PH) distributions to achieve both high and low variabilities: Erlang-n (E_n) for low variabilities (with SCV $1/n$) and H_2 for high variabilities (with SCV greater than 1). Other parameters remain the same as those in Table 2.

Table 14 shows that TGA-G works well except when the SCV of service time is high (e.g. $c_s^2 = 4$).

D Examples Revealing Limitations of the Approximations

D.1 High Abandonment Rates

In the most simple $M/M/n + M$ model, both DGA and TGA can not correctly estimate the key performances when abandonment rate $\theta = 4, 10$. Moreover, the estimate of means (and probabilities) deteriorate faster than that of variances, see Table 15 for details.

Table 14: Examples on $PH/PH/n+PH$ models

Perf. Meas.	SCV		c_s^2					
			0.25			0.5		
	c_λ^2	c_{ab}^2	Sim	CI	TGA-GA	Sim	CI	TGA-GA
E[Q]	0.5	0.5	3.40E+1	±2.15E-1	3.57E+1	3.34E+1	±2.50E-1	3.57E+1
		2	8.67E+0	±6.46E-2	8.42E+0	9.00E+0	±7.46E-2	8.67E+0
	2	0.5	3.37E+1	±3.14E-1	3.63E+1	3.37E+1	±3.23E-1	3.64E+1
		2	1.07E+1	±9.32E-2	1.01E+1	1.09E+1	±1.02E-1	1.03E+1
Var(Q)	0.5	0.5	1.84E+2	±1.44E+1	2.14E+2	2.30E+2	±1.66E+1	2.51E+2
		2	5.14E+1	±1.48E+0	5.55E+1	6.10E+1	±1.84E+0	6.36E+1
	2	0.5	4.30E+2	±2.29E+1	4.89E+2	4.55E+2	±2.38E+1	5.18E+2
		2	1.12E+2	±3.01E+0	1.12E+2	1.21E+2	±3.44E+0	1.19E+2
E[W]	0.5	0.5	3.33E-1	±2.04E-3	3.46E-1	3.28E-1	±2.40E-3	3.47E-1
		2	8.86E-2	±6.32E-4	8.30E-2	9.21E-2	±7.41E-4	8.55E-2
	2	0.5	3.26E-1	±2.90E-3	3.51E-1	3.26E-1	±3.00E-3	3.52E-1
		2	1.06E-1	±8.77E-4	9.84E-2	1.07E-1	±9.71E-4	1.00E-1
Var(W)	0.5	0.5	1.66E-2	±2.04E-3	1.97E-2	2.13E-2	±2.40E-3	2.34E-2
		2	4.96E-3	±6.32E-4	5.32E-3	6.01E-3	±7.41E-4	6.13E-3
	2	0.5	3.68E-2	±2.90E-3	4.32E-2	3.94E-2	±3.00E-3	4.62E-2
		2	1.00E-2	±8.77E-4	1.05E-2	1.10E-2	±9.71E-4	1.12E-2
PoD	0.5	0.5	9.90E-1	±8.51E-4	9.93E-1	9.79E-1	±1.37E-3	9.87E-1
		2	8.18E-1	±2.62E-3	7.97E-1	7.96E-1	±2.89E-3	7.76E-1
	2	0.5	9.30E-1	±2.49E-3	9.43E-1	9.22E-1	±2.66E-3	9.36E-1
		2	7.28E-1	±3.18E-3	7.04E-1	7.19E-1	±3.31E-3	6.97E-1
PoA	0.5	0.5	4.82E-2	±7.19E-4	4.78E-2	4.81E-2	±7.62E-4	4.79E-2
		2	5.54E-2	±7.12E-4	5.36E-2	5.84E-2	±7.76E-4	5.52E-2
	2	0.5	5.16E-2	±8.66E-4	4.90E-2	5.26E-2	±8.83E-4	4.93E-2
		2	6.54E-2	±8.09E-4	6.32E-2	6.66E-2	±8.67E-4	6.44E-2

Perf. Meas.	SCV		c_s^2					
			2			4		
	c_λ^2	c_{ab}^2	Sim	CI	TGA-GA	Sim	CI	TGA-GA
E[Q]	0.5	0.5	3.27E+1	±3.38E-1	3.59E+1	3.27E+1	±4.14E-1	3.62E+1
		2	9.73E+0	±1.03E-1	9.44E+0	1.03E+1	±1.36E-1	1.00E+1
	2	0.5	3.35E+1	±3.76E-1	3.68E+1	3.39E+1	±4.38E-1	3.71E+1
		2	1.13E+1	±1.29E-1	1.09E+1	1.13E+1	±1.52E-1	1.14E+1
Var(Q)	0.5	0.5	4.03E+2	±2.31E+1	3.70E+2	5.13E+2	±2.97E+1	4.60E+2
		2	9.28E+1	±3.04E+0	8.96E+1	1.14E+2	±4.38E+0	1.10E+2
	2	0.5	5.84E+2	±2.91E+1	6.15E+2	6.74E+2	±3.50E+1	6.91E+2
		2	1.47E+2	±4.69E+0	1.43E+2	1.61E+2	±5.75E+0	1.62E+2
E[W]	0.5	0.5	3.23E-1	±3.34E-3	3.49E-1	3.25E-1	±4.14E-3	3.52E-1
		2	1.01E-1	±1.06E-3	9.33E-2	1.07E-1	±1.43E-3	9.91E-2
	2	0.5	3.26E-1	±3.61E-3	3.56E-1	3.31E-1	±4.28E-3	3.60E-1
		2	1.13E-1	±1.28E-3	1.07E-1	1.14E-1	±1.54E-3	1.11E-1
Var(W)	0.5	0.5	3.91E-2	±3.34E-3	3.53E-2	5.10E-2	±4.14E-3	4.42E-2
		2	9.72E-3	±1.06E-3	8.74E-3	1.22E-2	±1.43E-3	1.08E-2
	2	0.5	5.31E-2	±3.61E-3	5.60E-2	6.30E-2	±4.28E-3	6.37E-2
		2	1.42E-2	±1.28E-3	1.36E-2	1.60E-2	±1.54E-3	1.55E-2
PoD	0.5	0.5	9.25E-1	±3.13E-3	9.63E-1	8.89E-1	±4.13E-3	9.41E-1
		2	7.32E-1	±3.74E-3	7.27E-1	7.13E-1	±4.75E-3	7.02E-1
	2	0.5	8.80E-1	±3.66E-3	9.12E-1	8.57E-1	±4.36E-3	8.94E-1
		2	6.87E-1	±4.03E-3	6.77E-1	6.69E-1	±4.64E-3	6.64E-1
PoA	0.5	0.5	5.17E-2	±9.27E-4	4.84E-2	5.46E-2	±1.11E-3	4.91E-2
		2	6.28E-2	±8.88E-4	6.01E-2	6.58E-2	±1.03E-3	6.37E-2
	2	0.5	5.49E-2	±1.03E-3	5.03E-2	5.81E-2	±1.13E-3	5.12E-2
		2	6.80E-2	±9.75E-4	6.83E-2	6.92E-2	±1.10E-3	7.13E-2

D.2 Underloaded systems

In this section we present the performance of the underloaded $M/M/n+M$ in QED regime, of which the Halfin-Whitt factor β, defined in [11] as $\beta=\sqrt{n}(1-\rho)$ to show grade of congestion of queue in heavy traffic, is fixed at 0.5. During underloaded intervals, it is easy to check that PoD_n is 0 and PoA_n is not well defined since the fluid and diffusion limit of waiting time in underloaded intervals implies that $w(\infty)=v(\infty)=0$ and $\sigma_W=0$. In order to improve the performance, we replace $w(\infty)=0$ by $\mathrm{E}\left[W^{TGA}\right]=\mathrm{E}\left[Q^{TGA}\right]/\mu$ and $\sigma_W=0$ by $\mathrm{Var}\left(W^{TGA}\right)=$

Table 15: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $n = 100, \rho = 1.05$ and $\theta = 4, 10$

	$\theta = 4$			$\theta = 10$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA
E[X] rel. err	96.6	100 4%	-	95.0	100 5%	-
Var(X) rel. err	56.1	25.1 55%	-	44.9	10.0 78%	-
E[Q] rel. err	1.47	0.100 83%	2.05 39%	0.665	0 100%	1.26 90%
Var(Q) rel. err	7.77	25.1 223%	8.76 13%	2.47	10.0 303%	3.41 38%
E[W] rel. err	1.66E-2	2.40E-3 86%	2.11E-2 27%	8.20E-3	1.00E-3 88%	1.31E-2 61%
Var(W) rel. err	8.80E-4	2.50E-3 182%	9.00E-4 2%	3.10E-4	1.00E-3 219%	3.50E-4 13%
PoD rel. err	0.379	-	0.508 34%	0.282	-	0.500 77%
PoA rel. err	5.83E-2	9.50E-3 84%	8.12E-2 39%	6.58E-2	9.95E-2 85%	1.23E-1 87%

$\text{Var}\left(Q^{TGA}\right) + \text{E}\left[Q^{TGA}\right]$, then design PoD^{TGA} and PoA^{TGA} as follows.

$$\begin{aligned} PoD^{TGA} &= \Phi\left(-a'_W(n)\right), \\ PoA^{TGA} &= \int_0^\infty \Phi\left(a'_W(n)\left(\frac{x}{w} - 1\right)\right) f(x)\mathrm{d}x, \end{aligned} \tag{33}$$

where $a'_W(n) = \sqrt{n}\text{E}[W^{TGA}]/\text{Var}\left(\hat{W}^{TGA}\right)$.

The main idea of (33) is to use $W^{TGA} = \sum_{i=1}^{Q^{TGA}} S_i$ to replace the zero waiting time; here S_i is the processing time of the i^{th} customers. We omit the impact of abandonments since according to the numerical results on underloaded intervals, probability of abandonments are about 10^{-2}. To verify (33), it is suffice to prove that

$$\text{E}\left[W^{TGA}\right] = \text{E}\left[Q^{TGA}\right]/\mu, \quad \text{and} \quad \text{Var}\left(W^{TGA}\right) = \text{Var}\left(Q^{TGA}\right) + \text{E}\left[Q^{TGA}\right].$$

Proof. It is obvious for the expression of $\text{E}[W^{TGA}]$ so we only focus on $\text{Var}\left(W^{TGA}\right)$ here.

$$\begin{aligned} \text{E}\left[\left(W^{TGA}\right)^2\right] &= \text{E}\left[\sum_{i=1}^{Q^{TGA}} S_i^2 + 2\sum_{j<i}^{Q^{TGA}} S_i S_j\right] = \text{E}\left[\text{E}\left[\sum_{i=1}^{Q^{TGA}} S_i^2 + 2\sum_{j<i}^{Q^{TGA}} S_i S_j \middle| Q^{TGA}\right]\right] \\ &= \text{E}\left[2Q^{TGA} + Q^{TGA}(Q^{TGA} - 1)\right] = \text{E}\left[Q^{TGA}\right] + \text{E}\left[\left(Q^{TGA}\right)^2\right], \end{aligned}$$

which implies the expression of $\text{Var}(W^{TGA})$. □

D.2.1 Underloaded with a Range of Abandonment Rates

Table 16 shows the performance of an underloaded $M/M/n + M$ model with $\rho = 0.95$. In particular, with parameters $\lambda = 100$, $\rho = 1.05$ and $0.1 \le \theta \le 4.0$. Table 16 shows good performance for X_n and B_n for $0.5 \le \theta \le 2.0$, but poor performance otherwise.

D.2.2 Smaller systems

We next consider systems with smaller arrival rates and numbers of servers $n = 20, 10, 5, 3$ and 1. For different n, we choose different values for the traffic intensity, following (24), with the QoS factor fixed at $\beta = 0.5$. Just as in Table 16, Table 17 shows good performance for X_n and B_n for $0.5 \leq \theta \leq 2.0$, but poor performance otherwise.

Table 17: $M(\lambda^{-1})/M(1)/n + M(2)$ with $n = 50, 20, 10, 5, 3$ and 1, $\rho = 1 - \beta/\sqrt{n}$, $\lambda = n\rho$

	$n = 50, \rho = 0.93$			$n = 20, \rho = 0.88$			$n = 10, \rho = 0.84$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	4.75E+1	4.65E+1	same	1.84E+1	1.78E+1	same	8.84E+0	8.42E+0	same
rel. err.		2%			3%			5%	
Var(X)	5.90E+1	4.65E+1	same	2.26E+1	1.78E+1	same	1.08E+1	8.42E+0	same
rel. err.		21%			22%			22%	
E[B]	4.55E+1	4.65E+1	4.52E+1	1.72E+1	1.78E+1	1.70E+1	8.00E+0	8.42E+0	7.89E+0
rel. err.		2%	1%		4%	1%		5%	1%
Var(B)	2.48E+1	4.65E+1	2.61E+1	9.39E+0	1.78E+1	1.01E+1	4.39E+0	8.42E+0	4.79E+0
rel. err.		87%	5%		89%	7%		92%	9%
E[Q]	2.00E+0	0.00E+0	1.31E+0	1.23E+0	0.00E+0	7.95E-1	8.34E-1	0.00E+0	5.35E-1
rel. err.		100%	35%		100%	35%		100%	36%
Var(Q)	1.60E+1	0.00E+0	7.69E+0	6.27E+0	0.00E+0	2.88E+0	3.06E+0	0.00E+0	1.33E+0
rel. err.		100%	52%		100%	54%		100%	56%
E[V]	4.54E-2	0.00E+0	2.62E-2	7.51E-2	0.00E+0	3.97E-2	1.12E-1	0.00E+0	5.35E-2
rel. err.		100%	42%		100%	47%		100%	52%
Var(V)	7.48E-3	0.00E+0	3.60E-3	2.02E-2	0.00E+0	9.19E-3	4.42E-2	0.00E+0	1.87E-2
rel. err.		100%	52%		100%	54%		100%	58%
PoD	3.67E-1	0.00E+0	6.69E-1	3.73E-1	0.00E+0	6.61E-1	3.80E-1	0.00E+0	6.52E-1
rel. err.		100%	82%		100%	77%		100%	72%
PoA	2.16E-2	NaN	1.30E-2	3.45E-2	NaN	1.97E-2	4.95E-2	NaN	2.64E-2
rel. err.		NaN	40%		NaN	43%		NaN	47%

	$n = 5, \rho = 0.78$			$n = 3, \rho = 0.71$			$n = 1, \rho = 0.5$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	4.16E+0	3.88E+0	same	2.33E+0	2.13E+0	same	5.82E-1	5.00E-1	same
rel. err.		7%			9%			14%	
Var(X)	5.00E+0	3.88E+0	same	2.77E+0	2.13E+0	same	6.61E-1	5.00E-1	same
rel. err.		22%			23%			24%	
E[B]	3.61E+0	3.88E+0	3.55E+0	1.94E+0	2.13E+0	1.93E+0	4.18E-1	5.00E-1	5.00E-1
rel. err.		8%	2%		10%	0%		20%	20%
Var(B)	1.99E+0	3.88E+0	2.11E+0	1.08E+0	2.13E+0	1.03E+0	2.43E-1	5.00E-1	1.60E-1
rel. err.		95%	6%		98%	4%		106%	34%
E[Q]	5.53E-1	0.00E+0	3.50E-1	3.97E-1	0.00E+0	2.49E-1	1.64E-1	0.00E+0	9.98E-2
rel. err.		100%	37%		100%	37%		100%	39%
Var(Q)	1.47E+0	0.00E+0	5.93E-1	8.45E-1	0.00E+0	3.12E-1	2.27E-1	0.00E+0	6.00E-2
rel. err.		100%	60%		100%	63%		100%	74%
E[V]	1.70E-1	0.00E+0	7.01E-2	2.36E-1	0.00E+0	8.31E-2	5.20E-1	0.00E+0	9.98E-2
rel. err.		100%	59%		100%	65%		100%	81%
Var(V)	1.01E-1	0.00E+0	3.77E-2	1.93E-1	0.00E+0	6.24E-2	9.27E-1	0.00E+0	1.60E-1
rel. err.		100%	63%		100%	68%		100%	83%
PoD	3.88E-1	0.00E+0	6.41E-1	3.96E-1	0.00E+0	6.30E-1	4.18E-1	0.00E+0	5.99E-1
rel. err.		100%	65%		100%	59%		100%	43%
PoA	7.12E-2	NaN	3.44E-2	9.31E-2	NaN	4.07E-2	1.64E-1	NaN	4.87E-2
rel. err.		NaN	52%		NaN	56%		NaN	70%

D.3 Critically Loaded Systems with $\rho > 1$

So far, we have concentrated on $G/GI/n + GI$ models with $(\rho, n) = (1.05, 100)$, or models with smaller scale n but the same QoS factor $\beta = \sqrt{n}(1 - \rho) = -0.5$. We now consider systems with

lighter loading, closer to the critical loading at $\rho = 1.000$. These include systems with $n = 100$ but $1.00 < \rho < 1.05$ and systems with smaller n but $-0.5 < \beta < 0.0$.

Table 18 provides results for $M(\lambda^{-1})/M/n + M(\theta^{-1})$ (Erlang-A) models with parameter triples $(\lambda, \theta, n) = (n\rho, 0.5, 100)$ and traffic intensity ρ ranges among $\{1.01, 1.005, 1.001\}$. Table 18 shows that the TGA performs well in the extreme cases ($\rho = 1.001$) and surprisingly does not degenerate even the system become (critically) underloaded, when the abandonment rate $\theta = 0.5$. However, as θ decreases in light traffic models, the performance of DGA and TGA degenerates, see results of $\theta \leq 0.25$ in Tables 22 and 23. More experiments for ligher loading models with different queues are presented in Tables 19-23 showing performance of TGA for the $M/M/n + M$ model with abandonment rate $0.1 \leq \theta \leq 4$ and different traffic intensity 1.1, 1.03, 1.02, 1.01 and 1.001.

Table 18: $M(\lambda^{-1})/M(1)/n + M(\theta^{-1})$ with $(n, \theta, \lambda) = (100, 0.5, 100\rho)$ and $\rho \to 1$

	$\rho = 1.01$			$\rho = 1.005$			$\rho = 1.001$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X] rel. err.	1.05E+2	1.02E+2 3%	same	1.04E+2	1.01E+2 3%	same	1.03E+2	1.00E+2 3%	same
Var(X) rel. err.	1.55E+2	2.02E+2 30%	same	1.52E+2	2.01E+2 32%	same	1.49E+2	2.00E+2 34%	same
E[Q] rel. err.	7.63E+0	1.91E+0 75%	6.67E+0 13%	7.11E+0	9.02E-1 87%	6.12E+0 14%	6.70E+0	1.00E-1 99%	5.69E+0 15%
Var(Q) rel. err.	8.68E+1	2.02E+2 133%	7.99E+1 8%	8.12E+1	2.01E+2 147%	7.36E+1 9%	7.68E+1	2.00E+2 161%	6.88E+1 10%
E[V] rel. err.	7.92E-2	2.00E-2 75%	6.70E-2 15%	7.40E-2	1.00E-2 86%	6.16E-2 17%	7.01E-2	2.00E-3 97%	5.74E-2 18%
Var(V) rel. err.	8.90E-3	2.00E-2 125%	7.98E-3 10%	8.38E-3	2.00E-2 139%	7.39E-3 12%	7.97E-3	2.00E-2 151%	6.93E-3 13%
PoD rel. err.	6.42E-1	5.56E-1 13%	same same	6.20E-1	5.28E-1 15%	same same	6.01E-1	5.06E-1 16%	same same
PoA rel. err.	3.78E-2	3.20E-2 15%	same same	3.54E-2	2.94E-2 17%	same same	3.35E-2	2.75E-2 18%	same same

D.4 Comparison with Approximations in [35]

A numerical approximation algorithm for the $M/GI/n+GI$ model was developed and evaluated in [35]. It was based on an application of an exact analysis of an associated state-dependent basic $M/M/n + M(n)$ queue, after approximating the GI abandonment by the state-dependent $M(n)$ abandonment. (The GI service was simply approximated by M.) That numerical procedure has the advantage that it applies to all loadings (underloaded, critically loaded and overloaded), but it is much more computationally intensive. That approximation was shown to be quite effective. A shortcoming of [35] that we address here is that it does not describe the impact of non-M arrival processes and service times.

We now compare our new TGA-G approximation to the approximation developed in [35] by comparing to the displayed reults in Tables 6 and 7 of [35], which are also for $n = 100$, but for the relatively light loading $\rho = 1.02$, which is at the edge of the range of effectiveness for TGA-G.

As expected for this relatively light loading, Tables 24 and 25 show that the engineering approximations in [35] are more accurate. These are labeled as *Eng. Approx. (W05)*. First, Table 24 shows that TGA-G performs reasonably well for the Erlang E_2 patience distribution, except for the variance $var(Q)$, even if not as accurate as Table 7 of [35]. On the other hand, Table 25 shows that the performance of TGA-G degrades significantly for the $LN(1, 1)$ patience distribution.

Nevertheless, the TGA-G (=DGA) approximation for EN remains good.

In summary, in this paper and appendix we have seen that our proposed TGA-G approximation for heavily-loaded $G/GI/n + GI$ model is remarkably effective for a wide class of models. Nevertheless, there are limitations, as we have exposed in this section. Breakdown is most likely as the loading decreases toward critical loading. That breakdown is likely to occur sooner (for higher ρ) if the component model elements deviate more from M. These final examples showed problems for low loading ($\rho = 1.02$) and non-M patience distributions.

Table 20: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $(\lambda, \rho) = (103, 1.03)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.36E+2	1.30E+2	4%	1.16E+2	1.12E+2	3%	1.08E+2	1.06E+2	2%
Var(X)	7.63E+2	1.03E+3	35%	3.04E+2	4.12E+2	36%	1.68E+2	2.06E+2	23%
E[Q]	3.63E+1	3.30E+1		1.73E+1	1.55E+1	11%	9.97E+0	9.21E+0	8%
Var(Q)	7.10E+2	7.50E+2	6%	2.45E+2	2.44E+2	0%	1.10E+2	1.07E+2	3%
E[V]	3.63E-1	3.25E-1	10%	1.75E-1	1.52E-1	13%	1.02E-1	9.08E-2	11%
Var(V)	6.93E-2	7.28E-2	5%	2.41E-2	2.37E-2	2%	1.10E-2	1.04E-2	6%
PoD	9.13E-1	8.25E-1	10%	8.18E-1	7.23E-1	12%	7.28E-1	6.62E-1	9%
PoA	3.53E-2	3.17E-2	10%	4.20E-2	3.67E-2	13%	4.84E-2	4.32E-2	11%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.03E+2	1.03E+2	0%	9.98E+1	1.01E+2	2%	9.77E+1	1.01E+2	3%
Var(X)	1.03E+2	1.03E+2	0%	7.02E+1	5.15E+1	27%	5.29E+1	2.57E+1	51%
E[Q]	5.70E+0	5.72E+0	0%	3.22E+0	3.67E+0	14%	1.78E+0	2.42E+0	36%
Var(Q)	4.94E+1	4.78E+1	3%	2.17E+1	2.20E+1	1%	9.33E+0	1.03E+1	11%
E[V]	5.94E-2	5.64E-2	5%	3.44E-2	3.62E-2	5%	1.98E-2	2.39E-2	20%
Var(V)	5.04E-3	4.65E-3	8%	2.29E-3	2.14E-3	7%	1.03E-3	1.00E-3	3%
PoD	6.29E-1	6.16E-1	2%	5.29E-1	5.83E-1	10%	4.34E-1	5.59E-1	29%
PoA	5.54E-2	5.27E-2	5%	6.25E-2	6.61E-2	6%	6.92E-2	8.41E-2	22%

Table 16: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $n = 100, \rho = 0.95$ and $0.1 \leq \theta \leq 2$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	1.00E+2	9.50E+1	same	9.80E+1	9.50E+1	same	9.64E+1	9.50E+1	same
rel. err.		5%			3%			1%	
Var(X)	2.18E+2	9.50E+1	same	1.56E+2	9.50E+1	same	1.20E+2	9.50E+1	same
rel. err.		56%			39%			21%	
E[B]	9.44E+1	9.50E+1	9.31E+1	9.40E+1	9.50E+1	9.31E+1	9.36E+1	9.50E+1	9.31E+1
rel. err.		1%	1%		1%	1%		2%	0%
Var(B)	4.91E+1	9.50E+1	5.31E+1	5.03E+1	9.50E+1	5.31E+1	5.11E+1	9.50E+1	5.31E+1
rel. err.		94%	8%		89%	6%		86%	4%
E[Q]	5.78E+0	0.00E+0	1.89E+0	4.06E+0	0.00E+0	1.89E+0	2.88E+0	0.00E+0	1.89E+0
rel. err.		100%	67%		100%	53%		100%	34%
Var(Q)	1.04E+2	0.00E+0	1.59E+1	5.67E+1	0.00E+0	1.59E+1	3.22E+1	0.00E+0	1.59E+1
rel. err.		100%	85%		100%	72%		100%	51%
E[V]	6.16E-2	0.00E+0	1.89E-2	4.37E-2	0.00E+0	1.89E-2	3.14E-2	0.00E+0	1.89E-2
rel. err.		100%	69%		100%	57%		100%	40%
Var(V)	1.13E-2	0.00E+0	1.78E-3	6.24E-3	0.00E+0	1.78E-3	3.60E-3	0.00E+0	1.78E-3
rel. err.		100%	84%		100%	72%		100%	51%
PoD	4.49E-1	0.00E+0	6.73E-1	4.06E-1	0.00E+0	6.73E-1	3.64E-1	0.00E+0	6.73E-1
rel. err.		100%	50%		100%	66%		100%	85%
PoA	6.09E-3	NaN	1.88E-3	1.07E-2	NaN	4.70E-3	1.51E-2	NaN	9.38E-3
rel. err.		NaN	69%		NaN	56%		NaN	38%

	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	DGA	TGA	Exact	DGA	TGA	Exact	DGA	TGA
E[X]	9.50E+1	9.50E+1	same	9.38E+1	9.50E+1	same	9.28E+1	9.50E+1	same
rel. err.		0%			1%			2%	
Var(X)	9.50E+1	9.50E+1	same	7.78E+1	9.50E+1	same	6.64E+1	9.50E+1	same
rel. err.		0%			22%			43%	
E[B]	9.31E+1	9.50E+1	9.31E+1	9.26E+1	9.50E+1	9.31E+1	9.21E+1	9.50E+1	9.31E+1
rel. err.		2%	0%		3%	1%		3%	1%
Var(B)	5.16E+1	9.50E+1	5.31E+1	5.16E+1	9.50E+1	5.31E+1	5.12E+1	9.50E+1	5.31E+1
rel. err.		84%	3%		84%	3%		86%	4%
E[Q]	1.92E+0	0.00E+0	1.89E+0	1.21E+0	0.00E+0	1.89E+0	7.23E-1	0.00E+0	1.89E+0
rel. err.		100%	1%		100%	57%		100%	161%
Var(Q)	1.69E+1	0.00E+0	1.59E+1	8.28E+0	0.00E+0	1.59E+1	3.82E+0	0.00E+0	1.59E+1
rel. err.		100%	6%		100%	92%		100%	315%
E[V]	2.13E-2	0.00E+0	1.89E-2	1.38E-2	0.00E+0	1.89E-2	8.60E-3	0.00E+0	1.89E-2
rel. err.		100%	11%		100%	37%		100%	120%
Var(V)	1.93E-3	0.00E+0	1.78E-3	9.79E-4	0.00E+0	1.78E-3	4.76E-4	0.00E+0	1.78E-3
rel. err.		100%	8%		100%	81%		100%	273%
PoD	3.17E-1	0.00E+0	6.73E-1	2.68E-1	0.00E+0	6.73E-1	2.21E-1	0.00E+0	6.73E-1
rel. err.		100%	112%		100%	151%		100%	205%
PoA	2.02E-2	NaN	1.87E-2	2.54E-2	NaN	3.70E-2	3.04E-2	NaN	7.26E-2
rel. err.		NaN	7%		NaN	46%		NaN	139%

Table 19: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $(\lambda, \rho) = (110, 1.10)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	2.00E+2	2.00E+2	0%	1.40E+2	1.40E+2	0%	1.20E+2	1.20E+2	0%
Var(X)	1.10E+3	1.10E+3	0%	4.23E+2	4.40E+2	4%	2.08E+2	2.20E+2	6%
E[Q]	1.00E+2	1.00E+2	0%	4.04E+1	4.02E+1	1%	2.08E+1	2.06E+1	1%
Var(Q)	1.09E+3	1.10E+3	0%	4.13E+2	4.18E+2	1%	1.87E+2	1.88E+2	0%
E[V]	9.58E-1	9.53E-1	1%	3.90E-1	3.83E-1	2%	2.03E-1	1.96E-1	3%
Var(V)	9.96E-2	9.98E-2	0%	3.77E-2	3.80E-2	1%	1.72E-2	1.71E-2	1%
PoD	9.99E-1	9.99E-1	0%	9.80E-1	9.72E-1	1%	9.29E-1	9.11E-1	2%
PoA	9.09E-2	9.05E-2	1%	9.19E-2	9.03E-2	2%	9.46E-2	9.16E-2	3%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.10E+2	1.10E+2	0%	1.04E+2	1.05E+2	1%	1.01E+2	1.02E+2	2%
Var(X)	1.10E+2	1.10E+2	0%	6.53E+1	5.50E+1	16%	4.41E+1	2.75E+1	38%
E[Q]	1.09E+1	1.09E+1	0%	5.77E+0	6.10E+0	6%	3.06E+0	3.57E+0	17%
Var(Q)	8.26E+1	8.08E+1	2%	3.57E+1	3.45E+1	4%	1.51E+1	1.49E+1	1%
E[V]	1.08E-1	1.04E-1	4%	5.88E-2	5.82E-2	1%	3.23E-2	3.41E-2	5%
Var(V)	7.72E-3	7.35E-3	5%	3.43E-3	3.14E-3	9%	1.52E-3	1.36E-3	11%
PoD	8.42E-1	8.30E-1	1%	7.31E-1	7.50E-1	3%	6.12E-1	6.83E-1	12%
PoA	9.92E-2	9.59E-2	3%	1.05E-1	1.05E-1	0%	1.11E-1	1.18E-1	6%

Table 21: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $(\lambda, \rho) = (102, 1.02)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.29E+2	1.20E+2	7%	1.13E+2	1.08E+2	5%	1.06E+2	1.04E+2	2%
Var(X)	6.70E+2	1.02E+3	52%	2.82E+2	4.08E+2	45%	1.62E+2	2.04E+2	26%
E[Q]	2.97E+1	2.52E+1	15%	1.48E+1	1.27E+1	15%	8.75E+0	7.91E+0	10%
Var(Q)	6.03E+2	6.19E+2	3%	2.14E+2	2.07E+2	3%	9.84E+1	9.34E+1	5%
E[V]	2.98E-1	2.49E-1	16%	1.51E-1	1.26E-1	17%	9.01E-2	7.84E-2	13%
Var(V)	5.95E-2	6.07E-2	2%	2.14E-2	2.03E-2	5%	9.96E-3	9.16E-3	8%
PoD	8.72E-1	7.34E-1	16%	7.74E-1	6.54E-1	16%	6.86E-1	6.10E-1	11%
PoA	2.91E-2	2.43E-2	16%	3.64E-2	3.03E-2	17%	4.29E-2	3.74E-2	13%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.02E+2	1.02E+2	0%	9.91E+1	1.01E+2	2%	9.71E+1	1.00E+2	3%
Var(X)	1.02E+2	1.02E+2	0%	7.11E+1	5.10E+1	28%	5.45E+1	2.55E+1	53%
E[Q]	5.09E+0	5.10E+0	0%	2.91E+0	3.37E+0	16%	1.62E+0	2.27E+0	40%
Var(Q)	4.46E+1	4.31E+1	3%	1.98E+1	2.03E+1	3%	8.54E+0	9.71E+0	14%
E[V]	5.34E-2	5.06E-2	5%	3.13E-2	3.34E-2	7%	1.82E-2	2.25E-2	24%
Var(V)	4.61E-3	4.23E-3	8%	2.11E-3	1.99E-3	6%	9.58E-4	9.54E-4	0%
PoD	5.92E-1	5.78E-1	2%	4.96E-1	5.56E-1	12%	4.06E-1	5.40E-1	33%
PoA	4.99E-2	4.74E-2	5%	5.70E-2	6.11E-2	7%	6.37E-2	7.97E-2	25%

Table 22: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $(\lambda, \rho) = (101, 1.01)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.23E+2	1.10E+2	10%	1.10E+2	1.04E+2	6%	1.05E+2	1.02E+2	3%
Var(X)	5.78E+2	1.01E+3	75%	2.60E+2	4.04E+2	55%	1.55E+2	2.02E+2	30%
E[Q]	2.40E+1	1.83E+1	24%	1.26E+1	1.02E+1	19%	7.63E+0	6.72E+0	12%
Var(Q)	4.96E+2	4.78E+2	4%	1.85E+2	1.71E+2	8%	8.68E+1	8.05E+1	7%
E[V]	2.43E-1	1.82E-1	25%	1.29E-1	1.01E-1	22%	7.92E-2	6.69E-2	15%
Var(V)	4.96E-2	4.73E-2	5%	1.87E-2	1.69E-2	9%	8.90E-3	7.97E-3	10%
PoD	8.23E-1	6.23E-1	24%	7.27E-1	5.79E-1	20%	6.42E-1	5.56E-1	13%
PoA	2.37E-2	1.78E-2	25%	3.12E-2	2.45E-2	22%	3.78E-2	3.20E-2	15%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.01E+2	1.01E+2	0%	9.84E+1	1.00E+2	2%	9.66E+1	1.00E+2	4%
Var(X)	1.01E+2	1.01E+2	0%	7.21E+1	5.05E+1	30%	5.61E+1	2.52E+1	55%
E[Q]	4.52E+0	4.53E+0	0%	2.61E+0	3.09E+0	18%	1.47E+0	2.13E+0	45%
Var(Q)	3.99E+1	3.85E+1	4%	1.79E+1	1.86E+1	4%	7.77E+0	9.09E+0	17%
E[V]	4.78E-2	4.51E-2	6%	2.84E-2	3.08E-2	8%	1.66E-2	2.12E-2	27%
Var(V)	4.18E-3	3.82E-3	9%	1.94E-3	1.85E-3	5%	8.85E-4	9.03E-4	2%
PoD	5.53E-1	5.40E-1	2%	4.63E-1	5.28E-1	14%	3.79E-1	5.20E-1	37%
PoA	4.47E-2	4.23E-2	5%	5.17E-2	5.64E-2	9%	5.83E-2	7.51E-2	29%

Table 23: $M(\lambda^{-1})/M(1)/100 + M(\theta^{-1})$ with $(\lambda, \rho) = (100.1, 1.001)$ and $0.1 \leq \theta \leq 4$

	$\theta = 0.1$			$\theta = 0.25$			$\theta = 0.5$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.18E+2	1.01E+2	14%	1.08E+2	1.00E+2	7%	1.03E+2	1.00E+2	3%
Var(X)	5.00E+2	1.00E+3	100%	2.41E+2	4.00E+2	66%	1.49E+2	2.00E+2	34%
E[Q]	1.96E+1	1.31E+1	33%	1.08E+1	8.18E+0	24%	6.70E+0	5.74E+0	14%
Var(Q)	4.08E+2	3.54E+2	13%	1.60E+2	1.40E+2	13%	7.68E+1	6.93E+1	10%
E[V]	2.00E-1	1.31E-1	34%	1.12E-1	8.18E-2	27%	7.01E-2	5.74E-2	18%
Var(V)	4.12E-2	3.54E-2	14%	1.63E-2	1.40E-2	15%	7.97E-3	6.93E-3	13%
PoD	7.72E-1	5.13E-1	34%	6.81E-1	5.08E-1	25%	6.01E-1	5.06E-1	16%
PoA	1.96E-2	1.29E-2	34%	2.70E-2	1.98E-2	27%	3.35E-2	2.75E-2	18%
	$\theta = 1$			$\theta = 2$			$\theta = 4$		
Perf.	Exact	TGA	rel. err.	Exact	TGA	rel. err.	Exact	TGA	rel. err.
E[X]	1.00E+2	1.00E+2	0%	9.77E+1	1.00E+2	2%	9.61E+1	1.00E+2	4%
Var(X)	1.00E+2	1.00E+2	0%	7.30E+1	5.00E+1	31%	5.76E+1	2.50E+1	57%
E[Q]	4.04E+0	4.04E+0	0%	2.36E+0	2.84E+0	20%	1.34E+0	2.00E+0	49%
Var(Q)	3.59E+1	3.45E+1	4%	1.62E+1	1.72E+1	6%	7.10E+0	8.55E+0	20%
E[V]	4.30E-2	4.04E-2	6%	2.59E-2	2.85E-2	10%	1.53E-2	2.00E-2	31%
Var(V)	3.81E-3	3.45E-3	9%	1.78E-3	1.72E-3	4%	8.19E-4	8.56E-4	5%
PoD	5.17E-1	5.04E-1	3%	4.33E-1	5.03E-1	16%	3.54E-1	5.02E-1	42%
PoA	4.03E-2	3.80E-2	6%	4.72E-2	5.22E-2	11%	5.36E-2	7.11E-2	33%

Table 24: Comparison with the engineering approximation in [35] and simulation for the $M(102^{-1})/GI(1, c_s^2)/100/200 + E_2$ models

Serv. Dist.	$D, c_s^2 = 0$				$E_2, c_s^2 = 0.5$			
Perf.	Sim	Eng. Approx. (W05)	DGA	TGAG	Sim	Eng. Approx. (W05)	DGA	TGAG
$\mathbb{P}(W = 0)$	1.80E-1	2.50E-1	1.88E-1	same	2.17E-1	2.50E-1	2.49E-1	same
rel. err.	±1.30E-3	28%	4%		±2.10E-3	13%	15%	
PoA	3.09E-2	3.81E-2	3.53E-2	same	3.51E-2	3.81E-2	4.37E-2	same
rel. err.	±1.70E-4	19%	14%		±2.90E-4	8%	25%	
E[Q]	1.11E+1	1.14E+1	1.07E+1	1.21E+1	1.15E+1	1.14E+1	1.07E+1	1.31E+1
rel. err.	±4.20E-2	3%	3%	9%	±7.50E-2	1%	7%	14%
Var(Q)	8.93E+1	1.22E+2	1.54E+2	1.08E+2	1.12E+2	1.22E+2	2.55E+2	1.60E+2
rel. err.	±4.00E-1	27%	72%	21%	±7.10E-1	8%	127%	42%
E[N]	1.10E+2	1.10E+2	1.11E+2	same	1.10E+2	1.10E+2	1.11E+2	same
rel. err.	±4.90E-2	0%	1%		±9.20E-2	0%	1%	
Serv. Dist.	$D, c_s^2 = 0$				$E_2, c_s^2 = 0.5$			
Perf.	Sim	Eng. Approx. (W05)	DGA	TGAG	Sim	Eng. Approx. (W05)	DGA	TGAG
$\mathbb{P}(W = 0)$	2.46E-1	2.50E-1	2.65E-1	same	2.33E-1	2.50E-1	2.65E-1	same
rel. err.	±2.00E-3	2%	8%		±2.10E-3	7%	14%	
PoA	3.78E-2	3.81E-2	4.69E-2	same	3.70E-2	3.81E-2	4.69E-2	same
rel. err.	±3.20E-4	1%	24%		±2.70E-4	3%	27%	
E[Q]	1.18E+1	1.14E+1	1.07E+1	1.35E+1	1.17E+1	1.14E+1	1.07E+1	1.35E+1
rel. err.	±7.50E-2	3%	9%	15%	±6.30E-2	3%	9%	15%
Var(Q)	1.29E+2	1.22E+2	2.97E+2	1.80E+2	1.23E+2	1.22E+2	2.97E+2	1.80E+2
rel. err.	±9.40E-1	6%	130%	39%	±7.20E-1	1%	141%	46%
E[N]	1.10E+2	1.10E+2	1.11E+2	same	1.10E+2	1.10E+2	1.11E+2	same
rel. err.	±9.10E-2	0%	1%		±7.20E-1	0%	1%	

Table 25: Comparison with the engineering approximation in [35] and simulation for the $M(102^{-1})/GI(1, c_s^2)/100/200 + LN(1, 1)$ models

Serv. Dist.	$E_2, c_s^2 = 0.5$				$M, c_s^2 = 1$			
Perf.	Sim	Eng. Approx. (W05)	DGA	TGAG	Sim	Eng. Approx. (W05)	DGA	TGAG
$\mathbb{P}(W = 0)$	2.11E-1	2.47E-1	1.76E-1	same	2.42E-1	2.47E-1	1.95E-1	same
rel. err.	±1.30E-3	15%	17%		±2.60E-3	2%	20%	
PoA	3.48E-2	3.79E-2	5.13E-2	same	3.76E-2	3.79E-2	5.51E-2	same
rel. err.	±2.10E-4	8%	47%		±3.20E-4	1%	47%	
E[Q]	1.14E+1	1.10E+1	1.29E+1	1.43E+1	1.14E+1	1.10E+1	1.29E+1	1.46E+1
rel. err.	±3.90E-2	3%	13%	25%	±7.10E-2	4%	13%	27%
Var(Q)	1.03E+2	1.07E+2	1.99E+2	1.43E+2	1.16E+2	1.07E+2	2.31E+2	1.61E+2
rel. err.	±3.90E-1	4%	94%	40%	±4.60E-1	8%	100%	39%
E[N]	1.10E+2	1.09E+2	1.13E+2	1.13E+2	1.10E+2	1.09E+2	1.13E+2	same
rel. err.	±5.30E-2	1%	3%		±9.20E-2	0%	3%	
Serv. Dist.	$LN(1, 1)$				$LN(1, 4)$			
Perf.	Sim	Eng. Approx. (W05)	DGA	TGAG	Sim	Eng. Approx. (W05)	DGA	TGAG
$\mathbb{P}(W = 0)$	2.29E-1	2.47E-1	1.95E-1	same	2.11E-1	2.47E-1	2.41E-1	same
rel. err.	±1.50E-3	7%	15%		±1.30E-3	15%	14%	
PoA	3.66E-2	3.79E-2	5.51E-2	same	3.48E-2	3.79E-2	6.66E-2	same
rel. err.	±2.40E-4	3%	51%		±2.10E-4	8%	91%	
E[Q]	1.14E+1	1.10E+1	1.29E+1	1.46E+1	1.14E+1	1.10E+1	1.29E+1	1.55E+1
rel. err.	±5.10E-2	4%	13%	27%	±3.90E-2	3%	13%	36%
Var(Q)	1.11E+2	1.07E+2	2.31E+2	1.61E+2	1.03E+2	1.07E+2	3.40E+2	2.16E+2
rel. err.	±4.30E-1	3%	109%	45%	±3.90E-1	4%	231%	111%
E[N]	1.10E+2	1.09E+2	1.13E+2	same	1.10E+2	1.09E+2	1.13E+2	same
rel. err.	±6.20E-2	1%	3%		±5.30E-2	1%	3%	