On Gaussian Markov Processes and Polya Processes

Kerry Fendick[a], Ward Whitt[b]

[a]*Communications System Branch, Johns Hopkins Applied Physics Laboratory, Laurel, MD 20723, USA*
[b]*Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027-6699, USA*

Abstract

In previous work we characterized Gaussian Markov processes with stationary increments and showed that they arise as asymptotic approximations for stochastic point processes with a random rate such as Polya processes, which can be useful to model over-dispersion and path-dependent behavior in service system arrival processes. Here we provide additional insight into these stochastic processes.

Keywords: Gaussian Markov processes, path-dependent stochastic processes, over-dispersion, Polya point process, queues, heavy-traffic

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1. Introduction

The net input processes of queueing systems - queueing networks as well as individual queues - are often approximated by Brownian motion (BM), leading to approximations of the content (workload or queue length) by reflected Brownian motion; e.g., see [1, 2]. The BM model tends to be relatively tractable because it is a Gaussian Markov process with stationary and independent increments. However, the property of independent increments fails to capture positive correlations among increments of the arrival process over nonoverlapping intervals, often referred to as over-dispersion, which are often found in measurements; e.g., see [3, 4, 5].

The interest in modelling net input processes without independent increments led us to investigate Gaussian Markov processes without independent increments and their applications to queues in [6, 7, 8, 9]. Significant contributions for applications are: (i) a characterization of multidimensional Gaussian Markov processes with stationary (but not independent) increments, called ψ-GMPs (with ψ pronounced "SI" being a mnemonic "for stationary increments"), (ii) expressions for the transient distribution of the workload in a queue with a ψ-GMP input; see Theorems 5 and 6 of [6] and §5 of [7] and (ii) heavy-traffic limits for queueing models with input modelled as generalized Polya processes (GPPs) from [10]; see §3-§5 of [7]. The paper [7] considers one-dimensional GPPs with stationary increments; the paper [8] considers GPPs without stationary increments; the paper [9] considers multidimensional GPPs with stationary increments and their applications to queueing networks.

The purpose of this paper is to provide new insight into GMPs and GPPs. In section §2 we obtain a new representation for a multivariate ψ-GMP with parameter matrices (A, B); we show that it can be represented as a sum of two independent processes, one a BM with covariance matrix A and the other a constant t times a normal random vector with covariance matrix B. In §3 we show that the set of all univariate ψ-GPPs coincides with the set of all Polya processes, which in turn can be characterized as Poisson processes with a gamma distributed random rate; see Chapter 4 of [11]. In §4 we obtain a new version of the one-dimensional FCLTs in §3-§4 of [7], where the limit is expressed directly in the (one-dimensional version of the) ψ-GMP in §2 here. In §5 we also obtain a new supporting multivariate FCLT, yielding all possible multivariate ψ-GMPs as limits.

2. Stationary-Increment Gaussian Markov Processes

In this section we give a new simple representation for a stationary-increment Gaussian Markov process ($\psi - GMP$), which was studied in [6]. We first review the definition and some of its properties.

We work with column vectors, so that if U and V are two k-dimensional random vectors in $\mathbb{R}^k$, then the $k \times k$ covariance matrix is $Cov[U,V] \equiv E[VU^t] - E[V]E[U^t]$. Let D^k be the k-fold product of the space $D \equiv D[0,\infty)$ with the usual Skorohod topology and the product topology, as in [2]. (The limits will have continuous sample paths, so the topology will correspond to the topology of uniform convergence over bounded intervals, but with the usual sigma-field; see §11.5.3 of [2].)

Definition 1. *A process X in D^k for $k \geq 1$ is a ψ-GMP with parameter matrices (A,B) and drift vector ω in $\mathbb{R}^k$ if X is a Gaussian process with $E[X(t)] = \omega t$ and*

$$Cov\left[X(s), X(t)\right] = s\left(A + Bt\right), \quad 0 \leq s \leq t < \infty,$$

where A and B are (strictly) positive definite, symmetric matrices of $k \times k$ real scalars.

In Definition 1 we have changed the sign of the matrix B from [6], but kept the meaning unchanged. We also do not consider all cases studied in [6]. The definition of a ψ-GMP in [6] relaxes the positive definite property for the matrix B, but to do so may require X be defined on a bounded interval $[0,T]$ for $T < \infty$. The cases we consider always have positively correlated increments by Proposition 2 of [6].

By Theorem 4 of [9], the k-dimensional ψ-GMP with drift satisfies the following non-ergodic law of large numbers (LLN). We thus say that the ψ-GMP is *path-dependent.*

Proposition 2.1. (*Theorem 4 of [9]*) *If X is a ψ-GMP in D^k with parameter matrices (A,B) and drift vector ω, then*

$$n^{-1}X(n) \Rightarrow N(\omega, B) \quad in \quad \mathbb{R}^k \quad as \quad n \to \infty, \tag{1}$$

where $N(m,\Sigma)$ denotes a normal or Gaussian random vector with mean vector m and covariance matrix Σ.

By Theorem 3 of [6], X has a representation as a solution to the linear stochastic differential equation (SDE), which is a well known way to generate GMPs; see §5.6 of [12]. In particular, the ψ-GMP can be expressed as

$$X(t) = \omega t + Y(t), \quad t \geq 0, \tag{2}$$

where $Y(t)$ is characterized as the solution of the stochastic differential equation

$$dY(t) = B(A + Bt)^{-1} Y(t)\, dt + \sqrt{A}\, dW(t), \quad t \geq 0, \tag{3}$$

with $Y(0) \equiv 0$, W is standard k-dimensional Brownian motion (BM) or Wiener process (with mean 0 and covariance matrix the identity matrix I).

It follows that a ψ-GMP has almost surely continuous sample paths. When A and B have the assumed properties, $A^{1/2}$ exists, and $(A + Bt)^{-1}$ always exists because $A + Bt$ is positive definite for all $t \geq 0$. If we relax the positive definite assumption for B and assume that $B = 0$, then X is a multivariate Brownian motion with drift ω and $Cov\,[X(s), X(t)] = sA$ for $0 \leq s \leq t < \infty$.

We now show that there is a simple alternative representation for a ψ-GMP.

Theorem 2.1. (*alternative representation for a ψ-GMP*). *An equivalent representative for the ψ-GMP in* (2) *and* (3) *above with parameter matrices* (A, B) *and drift vector* ω *is*

$$X(t) = N(\omega, B)t + \sqrt{A}W(t), \quad t \geq 0, \tag{4}$$

where W is again standard multivariate BM that is independent of a normal random vector $N(\omega, B)$ with mean vector ω and covariance matrix B.

Proof. Clearly the two representations are both for Gaussian processes with drift. To show equivalence, it suffices to show that the mean vectors and covariance matrices coincide. Clearly the processes have the same mean vectors. When (4) holds,

$$\begin{aligned} E[X(t)X(t)^t] &= E[E[X(t)X(t)^t | N(0, B)] \\ &= At + E[N(0, B)N(0, B)^t]t^2 = At + Bt^2 + \omega\omega^t t^2, \end{aligned} \tag{5}$$

so that

$$Var[X(t)] = Cov[X(t), X(t)] = At + Bt^2, \quad t \geq 0. \tag{6}$$

It follows from (4) that $X(t-s) \stackrel{\mathrm{d}}{=} X(t) - X(s)$ and $Cov[X(s), X(t)] = Cov[X(t), X(s)]$. Therefore,

$$\begin{aligned} Var[X(t-s)] &= Var[X(t) - X(s)] \qquad (7)\\ &= Var[X(t)] + Var[X(s)] - 2Cov[X(s), X(t)], \quad 0 \le s \le t, \end{aligned}$$

so that we conclude that

$$Cov[X(s), X(t)] = s(A + Bt), \quad 0 \le s \le t. \quad \blacksquare \qquad (8)$$

3. Polya Point Processes

In our previous work [7, 8], we viewed Polya processes (PPs) as special cases of stationary generalized Polya processes ($\psi - GPPs$), drawing on the paper by Cha [10] that introduced GPPs. In this approach, a GPP with parameter triple $(\kappa(t), \gamma, \beta)$ is defined as a Markov point process with intensity function (defined in terms of the internal histories $\mathcal{H}_t$; e.g., see §1.8 of [13]) by

$$\lambda^*(t) \equiv \lambda^*(t|\mathcal{H}_t) \equiv \lim_{h \downarrow 0} \frac{E[N(t+h) - N(t)|\mathcal{H}_t]}{h} \equiv (\gamma N(t-) + \beta)\kappa(t), \qquad (9)$$

where $N(0) = 0$, γ and β are positive constants, $\kappa(t)$ is a positive integrable real-valued function and $\equiv$ denotes equality by definition. As observed by [10], the special case of (9) with

$$\kappa(t) = \frac{1}{\gamma t + 1}, \quad t \ge 0, \qquad (10)$$

is a Polya point process. Theorem 1 of [7] shows that the GPP with κ in (10) is stationary. We refer to one such ψ-GPP as an γ, β ψ-GPP. We now show that the set of all $\psi - GPP$'s coincides with the set of PPs.

Theorem 3.1. (ψ*-GPP representation theorem) A GPP has stationary increments if and only if it is a Polya process as defined by* (9) *and* (10)*.*

Proof. We apply our results for GPPs, in particular, Theorem 1 and Corollary 3 of [8], which constructs a $\psi - GPP$ in terms of its instantaneous mean function. Corollary 3 to Theorem 1 of [8] and the following Remark 2 there show that any $\psi - GPP$ has a constant instantaneous mean function, defined as

$$\lambda(t) \equiv \lim_{h \downarrow 0} \frac{E[N(t+h) - N(t)]}{h} \tag{11}$$

where $N(0) = 0$. Note that the instantaneous mean function $\lambda(t)$ differs from the intensity $\lambda^*(t)$ in (9) by not conditioning on the internal histories $\mathcal{H}_t$. Suppose that a GPP with parameter triple $(\kappa(t), \gamma, \beta)$ has instantaneous mean function $\lambda(t) = c$, and so is a $\psi - GPP$. Apply (2.10) in Theorem 1 of [8] to show that the associated function $\kappa(t)$ in that GPP is

$$\kappa(t) = \frac{c}{\beta + \gamma ct}. \tag{12}$$

Thus the associated intensity function is

$$\lambda^*(t) \equiv \lambda^*(t|\mathcal{H}_t) \equiv (\gamma N(t-) + \beta)\kappa(t) = \frac{c\gamma N(t-) + c\beta}{\gamma ct + \beta}. \tag{13}$$

Now divide the numerator and denominator by β to obtain an expression for a new GPP with parameter triple $(\hat{\kappa}(t), \hat{\gamma}, \hat{\beta})$, where

$$\hat{\kappa}(t) \equiv 1/(\hat{\gamma}t + 1), \quad \hat{\gamma} \equiv c\gamma/\beta \quad \text{and} \quad \hat{\beta} \equiv c. \tag{14}$$

However, this $\psi - GPP$ is of the form of a PP as defined in (9) and (10) above, following (2) of [7], drawing on Theorem 1 there. ■

Remark 3.1. (*alternate proof based on the MPP representation*) An alternative proof of Theorem 3.1 above can be based on Grandell [11], which in turn is based on the seminal book Lundberg [14], where a Polya process is defined as a mixed Poisson process (MPP) with a gamma structure distribution; i.e., if $\Pi(t)$ is a standard Poisson process with rate 1, then the Polya process is defined as

$$N(t) \equiv \Pi(t\Lambda), \quad t \geq 0, \tag{15}$$

where Λ is a random variable independent of Π with a gamma distribution, i.e., with probability density function (pdf)

$$f(x; \delta, \nu) \equiv \frac{\nu e^{-\nu x}(\nu x)^{\delta-1}}{\Gamma(\delta)}, \quad x \geq 0, \tag{16}$$

where Γ is the gamma function with $\Gamma(n) = (n-1)!$ for n integer, while ν is called *the rate* and δ is called the *shape*. With (16), the mean of the gamma distribution with pdf in (16) is δ/ν and the variance is δ/ν^2. With this notation, the intensity function given in Example 4.1 of [11] is

$$\lambda^*(t) = \frac{N(t-) + \delta}{t + \nu}. \tag{17}$$

In fact, three equivalent variants of this definition are given on pages 62-66 of [11].

Note that (17) above coincides with (9) in §3 for new parameters. To see this, first divide the numerator and denominator of (17) by ν. then let $(\gamma, \beta) \equiv (1/\nu, \delta/\nu)$.

We can also obtain the stationary increments property from the established stationarity of the MPP representation, drawing on McFadden's theorem, Theorem 6.2 on p. 110 of [11] or Nawrotzki's theorem, Theorem 6.3 on p. 113 of [11]. ■

4. A Connecting Limit Theorem

In this section we establish an alternative version of Theorem 4 of [7], the FCLT for a one-dimensional PP that yields a ψ-GMP limit. Now we establish a version that directly yields the one-dimensional version of the new representation of the $\psi - GMP$ in §2.

4.1. The Two Gamma Parameters

Let $\Pi(t)$ be a standard Poisson process with rate 1 and Λ be an independent random variable with a gamma distribution as in (16).

We match the first two moments of the Polya process (PP) in Theorem 1 of [7] to a gamma mixture of Poisson processes to obtain the first two moments of the gamma distribution that serves as the mixing distribution applied to the parameter λ in the mixed Poisson process (MPP) representation of the Polya process in (15).

Proposition 4.1. *The mixing random variable* Λ *in the MPP representation of the PP in* (15) *with mean* βt *and variance* $\beta t(1+\gamma t)$ *has first two moments*

$$E[\Lambda] = \beta \quad \textit{and} \quad E[\Lambda^2] = \beta(\gamma + \beta) \tag{18}$$

and thus variance $Var[\Lambda] = \beta\gamma$.

Proof. For the mean, we have the equality

$$\beta t = E[N(t)] = \int_0^\infty (\lambda t) dP_\Lambda(\lambda) = E[\Lambda]t, \tag{19}$$

from which we deduce that the mean of the random Poisson parameter Λ must be $E[\Lambda] = \beta$ as in (18).

For higher moments, we use the well known property that the k^{th} moment of a mixture is the mixture of the underlying k^{th} moments. To apply this property with higher moments, it is convenient to work with factorial moments. Recall that the r^{th} factorial moment of a nonnegative-integer-valued random variable Y is $Y_{(r)} = E[Y(Y-1)\ldots(Y-r+1)]$. It is easy to see that the k^{th} factorial moment of a mixture is the mixture of the underlying k^{th} factorial moments.

In particular, for a Poisson process with parameter λ, the second factorial moment is

$$\begin{aligned} E[(N(t))_2] \;&\equiv\; E[N(t)(N(t)-1)] = E[N(t)^2] - E[N(t)] \\ &= \lambda t + (\lambda t)^2 - \lambda t = (\lambda t)^2. \end{aligned} \tag{20}$$

To obtain the second factorial moment of the PP, we use Theorem 1 of [7], which concludes for the Polya process that

$$Var(N(t)) = \beta t(1 + \gamma t) = \beta t + \beta\gamma t^2. \tag{21}$$

As a consequence, the second factorial moment of the Polya process is

$$\beta(\beta+\gamma)t^2 = E[(N(t)_2)] = \int_0^\infty (\lambda t)^2 dP_\Lambda(\lambda) = E[\Lambda^2]t^2. \tag{22}$$

Hence, the second moment of the mixing distribution must be $E[\Lambda^2] = \beta(\beta+\gamma)$ as in (18), so that the variance is

$$Var(\Lambda) = E[\Lambda^2] - E[\Lambda]^2 = \beta(\beta+\gamma) - \beta^2 = \beta\gamma. \quad \blacksquare \tag{23}$$

Proposition 4.1 above is consistent with the analysis in Remark 3.1 above. Given that the PP has mean βt and variance $\beta\gamma t$, the underlying gamma distribution in the MPP representation must be of the form (16) with $(\gamma, \beta) \equiv (1/\nu, \delta/\nu)$. We will let $\Lambda(\beta, \beta\gamma)$ denote a random variable with mean β and variance $\beta\gamma$ as arises in the MPP representation.

4.2. FCLT for a PP based on the MPP Representation

Let

$$X_n(t) \equiv n^{-1/2}(X^n(t) - \beta n t), \quad t \geq 0, \tag{24}$$

be the scaled GPP in (9) of [7] based on the superposition of n i.i.d. GPPs, each with parameter triple $(\kappa(t), \beta, \gamma)$ for $\kappa(t) = 1/(\gamma t + 1)$ as in (10). (We have changed the notation from A in [7] to X to avoid confusion with the ψ-GMP parameters in Definition (1) here.) Recall that X^n is an $(n\beta, \gamma)$ $\psi - GPP$ by Proposition 3 of [7]. Let $\mathcal{D}$ be the function space $D[0, \infty)$ with the usual Skorohod J_1 topology, as in [2]. The following derives an alternative representation for the limiting $\psi - GMP$ in Theorem 4 of [7].

Theorem 4.1. (*FCLT with new representation for the GMP*) *Under the assumptions above* (*the same as in Theorem 4 of* [7]),

$$X_n \Rightarrow X \quad as \quad n \to \infty, \tag{25}$$

where X_n *is defined in* (24) *and*

$$X(t) \quad = \quad N(0, \beta\gamma)t + \sqrt{\beta} W(t) \quad t \geq 0, \tag{26}$$

with W *being a standard Brownian motion or Wiener process that is independent of a Gaussian random variable* $N(0, \beta\gamma)$ *with mean* 0 *and variance* $\beta\gamma$.

Proof. By §3, we have the representation

$$X^n(t) = \Pi(t\Lambda(n\beta, n\beta\gamma)), \quad t \geq 0, \tag{27}$$

where $\Lambda(m, v)$ is a gamma random variable with mean m and variance v that is independent of a rate-1 Poisson process $\Pi \equiv \{\Pi(t) : t \geq 0\}$. We apply the well known FCLT for the Poisson process, yielding

$$n^{-1/2}(\Pi(nt) - nt) \Rightarrow W(t) \quad \text{in} \quad \mathcal{D} \quad \text{as} \quad n \to \infty \tag{28}$$

and the well known CLT and LLN for the gamma distribution, e.g., see §II.2 and §VI.3 of [15], yielding

$$\frac{\Lambda(n\beta, n\beta\gamma) - n\beta}{\sqrt{n}} \Rightarrow N(0, \beta\gamma) \quad \text{in} \quad \mathbb{R} \tag{29}$$

and

$$\frac{\Lambda(n\beta, n\beta\gamma)}{n} \Rightarrow \beta \quad \text{in} \quad \mathbb{R} \tag{30}$$

as $n \to \infty$.

Because the gamma random variable is independent of the Poisson process and the LLN has a deterministic limit, we can apply Theorems 11.4.4 and 11.4.5 of [2] to obtain the joint convergence

$$\begin{aligned}&\left(n^{-1/2}(\Pi(nt) - nt), n^{-1/2}(\Lambda(n\beta, n\beta\gamma) - n\beta), n^{-1}\Lambda(n\beta, n\beta\gamma)\right)\\ &\quad \Rightarrow (W(t), N(0, \beta\gamma), \beta) \quad \text{in} \quad \mathcal{D} \times \mathbb{R}^2 \quad \text{as} \quad n \to \infty.\end{aligned} \tag{31}$$

Apply the continuous mapping theorem with the function $\phi : \mathcal{D}\times\mathbb{R}\times\mathbb{R} \to \mathcal{D}$ taking $(x(t), y, z)$ into $x(zt) + yt$. (Use Theorem 3.4.3 of [2] and the fact that W has continuous paths w.p1.) By direct application of this function, we get

$$\begin{aligned}&n^{-1/2}(\Pi(t\Lambda(n\beta, n\beta\gamma)) - t\Lambda(n\beta, n\beta\gamma)) + n^{-1/2}(t\Lambda(n\beta, n\beta\gamma) - nt\beta)\\ &= n^{-1/2}(\Pi(t\Lambda(n\beta, n\beta\gamma)) - nt\beta) = A_n(t), \quad t \geq 0.\\ &\quad \Rightarrow W(\beta t) + tN(0, \beta\gamma)\end{aligned} \tag{32}$$

as claimed. ■

By a minor modification of the same argument and the argument used in Corollary 3 of [7], we obtain the alternative representation of the limit with a drift term. As assumed in (14) of [7], assume that

$$\mu_n \to 1 \quad \text{and} \quad \sqrt{n}(\mu_n - 1) \to \mu \quad \text{as} \quad n \to \infty. \tag{33}$$

Let the modified scaled process be defined as in (15) of [7], i.e.,

$$X_n^d(t) \equiv n^{-1/2}(X^n(\mu_n t - \beta nt), \quad t \geq 0. \tag{34}$$

Corollary 4.1. *If* (33) *holds in addition to the assumptions of Theorem 4.1, then*

$$X_n^d(t) \Rightarrow W(\beta t) + N(\beta\mu, \beta\gamma)t \quad \text{in} \quad \mathcal{D} \quad \text{as} \quad n \to \infty. \tag{35}$$

for X_n^d *defined in* (34).

Corollary 4.1 yields the one-dimensional version of the representation for the $\psi - GMP$ with drift in §2.

Remark 4.1. (*non-Poisson mixture processes*) Extensions of Theorem 4.1 to non-Poisson arrival processes that satisfy a FCLT with Brownian limit follow by the same argument; e.g. see §4.4 of [2] for examples.

5. A Multivariate Extension

We now show how to obtain the general k-dimension ψ-GMP from §2 as the limit in a FCLT involving a mixed multivariate counting process. Suppose that the stochastic process of interest can be given as a mixed representation $X^n(t) \equiv \tilde{\Pi}(t\Lambda(n\mathbf{1}, nB))$, where $\mathbf{1}$ is the vector $(1, 1, \ldots, 1)^t$ in $\mathbb{R}^k$ and $\tilde{\Pi}$ is a k-dimensional process with

$$\tilde{\Pi}(t\nu) \equiv (\tilde{\Pi}_1(t\nu_1), \ldots, \tilde{\Pi}_k(t\nu_k)) \tag{36}$$

for a k-dimensional vector $\nu \equiv (\nu_1, \ldots, \nu_k)$ with $\tilde{\Pi}$ satisfying the FCLT in $\mathcal{D}^k$

$$n^{-1/2}(\tilde{\Pi}(nt\mathbf{1}) - nt\mathbf{1}) \Rightarrow \sqrt{A}W(t) \quad \text{in} \quad \mathcal{D}^k \quad \text{as} \quad n \to \infty, \tag{37}$$

where $W(t) \equiv (W_1(t), \ldots, W_k(t))$ is k-dimensional BM and A is a symmetric positive-definite real matrix, while $\{\Lambda(n\mathbf{1}, nB) : n \geq 1\}$ is a sequence of nonnegative random vectors in $\mathbb{R}^k$ with mean vectors $n\mathbf{1}$ and $k \times k$ covariance matrices nB which satisfies a CLT and LLN in $\mathbb{R}^k$, i.e.,

$$\begin{aligned} n^{-1}\Lambda(n\mathbf{1}, nB) &\Rightarrow \mathbf{1} \qquad \text{and} \\ n^{-1/2}(\Lambda(n\mathbf{1}, nB) - n\mathbf{1}) &\Rightarrow \quad N(0, B) \quad \text{in} \quad \mathbb{R}^k \end{aligned} \tag{38}$$

as $n \to \infty$, where A and B are $k \times k$ positive definite symmetric real matrices. Then let

$$\begin{aligned} X^n(t) &\equiv \tilde{\Pi}(t\Lambda(n\mathbf{1}, nB)) \\ &= (\tilde{\Pi}_1(t\Lambda_1(n\mathbf{1}, nB)), \ldots, \tilde{\Pi}_k(t\Lambda_k(n\mathbf{1}, nB))), \quad t \geq 0, \end{aligned} \tag{39}$$

and

$$\begin{aligned} X_n &\equiv n^{-1/2}(X^n(t) - nt\mathbf{1}) \\ &= n^{-1/2}(\tilde{\Pi}_1(t\Lambda_1(n\mathbf{1}, nB)) - nt, \ldots, \tilde{\Pi}_k(t\Lambda_k(n\mathbf{1}, nB))) - nt), \quad t \geq 0. \end{aligned} \tag{40}$$

For example, it is natural for $\tilde{\Pi}$ to be a k-dimensional strictly stationary stochastic point process that possesses appropriate mixing properties to justify the FCLT, as in §4.4 of [2] and references there. (Here mixing refers to the dependence assumptions as opposed to a random rate.) It is also natural for $\Lambda(\mathbf{1}, B)$ to have an infinitely divisible multivariate distribution, such as the multivariate gamma distributions in [16] and references there.

Then, by a vector version of essentially the same argument used to prove Theorem 4.1, we obtain the following multivariate extension and generalization of Theorem 4.1. (Recall that the product space $\mathcal{D}^k$ is endowed with the product topology, so that the one-dimensional argument applies in each coordinate.)

Theorem 5.1. (*multivariate FCLT*) *Under the conditions above,*

$$X_n(t) \equiv n^{-1/2}(X^n(t\mathbf{1}) - \mathbf{1}nt) \Rightarrow \sqrt{A}W(t) + N(0,B)t \quad as \quad n \to \infty \quad (41)$$

for X_n *in* (40) *with* X^n *in* (39)*, where the limit has the structure of* (4) *in Theorem* 2.1*.*

By this argument we get a ψ-GMP limit with parameter matrices (A, B), just as in Definition 1. In summary, we obtain the matrix A from the FCLT for $\tilde{\Pi}$ in (37) and we obtain the matrix B from the CLT for $\Lambda(n\mathbf{1}, nB)$ in (38). In contrast, if we assume that (37) holds with the limit process having independent marginals and try to obtain the full parameter pair (A, B) from a version of (38), we see that we can only obtain matrices A that are diagonal matrices. In such a random-time representation, non-diagonal matrices A must come from the FCLT for $\tilde{\Pi}$. In particular, the multivariate Polya process constructed by Zocher [17, 18], which is based on a multivariate Poisson process with i.i.d. marginal one-dimensional processes, cannot be used to yield all ψ-GMPs.

Remark 5.1. (*connection to* [9]) The restriction to diagonal matrices for A is consistent with Lemma 1 of [9] in our limits for queueing networks, but the linear maps M used in [9] to create the generalized Polya superposition processes (GPSPs) apply to the entire process. The construction of an GPSP leads to $\tilde{\Pi}$ with dependent coordinate processes in the above representation, so that $\sqrt{A}$ is not diagonal. Nevertheless, not all ψ-GMPs can be obtained as limits of a ψ-GPSP. That is, the limit in [9] can be regarded as a special case of Theorem 5.1.

To elaborate, we now prove that the framework here is more general than the ψ-GPSP framework in [9]. For a ψ-GPSP, $A = MUM^t$ and $B = MVM^t$ where U and V are diagonal matrices and the matrix M is common to A and B. Any positive definite matrix has a decomposition of the form MUM^t, where M is a matrix of its eigenvectors, while U is a diagonal matrix of its eigenvalues. The decomposition is unique up to an ordering of the eigenvectors. Therefore, the vectors $A = MUM^t$ and $B = MVM^t$ have the same eigenvectors, which implies that they must commute. That need not be the case for general positive definite matrices.

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