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MULTIPLE CHANNEL QUEUES IN HEAVY TRAFFIC. I

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1. Introduction and summary

The queueing systems considered in this paper consist of r independent arrival channels and s independent service channels, where as usual the arrival and service channels are independent. Arriving customers form a single queue and are served in the order of their arrival without defections. We shall treat two distinct modes of operation for the service channels. In the standard system a waiting customer is assigned to the first available service channel and the servers (servers $\equiv$ service channels) are shut off when they are idle. Thus the classical $GI/G/s$ system is a special case of our standard system. In the modified system a waiting customer is assigned to the service channel that can complete his service first and the servers are not shut off when they are idle. While the modified system is of some interest in its own right, we introduce it primarily as an analytical tool. Let λ_i denote the arrival rate (reciprocal of the mean interarrival time) in the ith arrival channel and μ_j the service rate (reciprocal of the mean service time) in the jth service channel. Then $\lambda = \sum_{i=1}^{r}\lambda_i$ is the total arrival rate to the system and $\mu = \sum_{j=1}^{s}\mu_j$ is the maximum service rate of the system. As a measure of congestion we define the traffic intensity $\rho = \lambda/\mu$.

We shall restrict our attention to systems in which $\rho \geqq 1$. Under this condition the systems are of course unstable (a proof of this fact is an easy by-product of our results). Our principal objective will be to obtain functional central limit theorems (invariance principles) for the stochastic processes characterizing these systems after appropriately scaling and translating the processes.

There has been a growing literature on queues in heavy traffic beginning with Kingman ((1961), (1962)); see Kingman (1965) for a summary of his work. We use the term heavy traffic in a broader sense than Kingman. While he considered queueing systems with traffic intensity less than but close to

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one, we now let heavy traffic include both single queueing systems with $\rho \geqq 1$ and sequences of queueing systems with $\rho_n \to \rho \geqq 1$. We shall only mention a few of these papers which are most relevant to the present work and refer the reader to Whitt (1968) for a comprehensive discussion of this literature. There have been essentially two avenues of attack in heavy traffic theory. The first begins with the sequence of waiting times of successive customers which is easily related to a sequence of partial sums in the case of a single server. This method was used by Kingman ((1961), (1962)), Prohorov (1963), and Whitt (1968). Unfortunately, this method has drawbacks in a general multiple channel queue. The second method begins with the queue length process which is (not as) easily related to renewal processes characterizing the arrival and departure processes. The second method has been championed by Borovkov (1965) and used to a lesser extent by Whitt (1968). Most of the work in heavy traffic has dealt with the single server case, the principal exceptions being Borovkov (1965) and Presman (1965).

The natural setting for the limit theorems obtained in this paper is the weak convergence of probability measures on the function space $D[0,1]$ ($\equiv D$). Since an excellent treatment of this subject was recently published by Billingsley (1968), we shall only make a few remarks here about our terminology and notation. The stochastic processes characterizing the queueing system give rise to sequences of random functions in D, the space of all right-continuous functions on $[0,1]$ having left limits and endowed with the Skorohod metric, d; in Billingsley this metric is denoted by d_0. With d, D is a complete, separable metric space. Let $\mathscr{D}$ be the class of Borel sets of D. Then if P_n and P are probability measures on $\mathscr{D}$ which satisfy

$$\lim_{n \to \infty} \int_D f\,dP_n = \int_D f\,dP$$

for every bounded, continuous, real-valued function f on D, we shall say that P_n *converges weakly* to P as $n \to \infty$ and write $P_n \Rightarrow P$. A *random function* X is a measurable mapping from some probability space $(\Omega, \mathscr{B}, \mathscr{P})$ into D having *distribution* $P = \mathscr{P}X^{-1}$ on $(D, \mathscr{D})$. We say a sequence of random functions $\{X_n\}$ converges weakly to the random function X, and write $X_n \Rightarrow X$ if the distribution P_n of X_n converges weakly to the distribution P of X. A sequence of random functions $\{X_n\}$ converges to X in probability if X_n and X are defined on a common domain and for all $\varepsilon > 0$, $P\{d(X_n, X) \geqq \varepsilon\} \to 0$. When X is a constant function (not random), convergence in probability is equivalent to weak convergence. In such cases we shall write $d(X_n, X) \Rightarrow 0$ or $X_n \Rightarrow X$. If X_n and Y_n have a common domain, we also write $d(X_n, Y_n) \Rightarrow 0$ when for all $\varepsilon > 0$, $P\{d(X_n, Y_n) > \varepsilon\} \to 0$. We shall also use the uniform metric ρ which is defined by $\rho(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|$ for $x, y \in D$.

Our analysis begins in Section 2 with a study of the modified system. While this modified system is different from our standard system, in heavy traffic it is essentially the same and has the virtue of being easy to analyze. This idea, due to Borovkov (1965), is the key to our analysis. Let $Q'(t)$ denote the number of customers in the modified system at time t and let $\{Q_n'\}$ be the corresponding sequence of random functions in D:

$$Q_n' \equiv [Q'(nt) - (\lambda - \mu)nt]/cn^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1,$$

where c is a constant to be specified later. (The letter c will be used to denote a generic constant.) For $\rho = 1$ $(\lambda = \mu)$ we show that $Q_n' \Rightarrow f(\xi)$. where ξ is a standard Wiener process and $f: D \to D$ is a continuous function which acts as an impenetrable barrier at the origin. (Borovkov showed this for a single value of t.) Throughout the paper Wiener processes will be denoted by ξ with and without subscripts or superscripts. The process $f(\xi)$ has the same distribution as $|\xi|$, the one-dimensional Bessel process. For $\rho > 1$, we show that $Q_n' \Rightarrow \xi$. In Section 3 we obtain comparable results for the standard system when $\rho \geqq 1$.

In Section 4 we study the departure process, $D(t)\,[D'(t)]$, defined to be the total number of customers which depart from the standard [modified] system in the interval $(0, t]$. Let $\{D_n\}$ $[\{D_n'\}]$ be the corresponding sequence of random functions in D:

$$D_n \equiv [D(nt) - (\lambda \wedge \mu)nt]/cn^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

We use $a \wedge b$ for $\min(a, b)$ and $a \vee b$ for $\max(a, b)$. For $\rho > 1$, $D_n \Rightarrow \xi$. For $\rho = 1$, D_n converges weakly to a functional of two independent Wiener processes. For $\rho < 1$, we shall show in a forthcoming paper that $D_n \Rightarrow \xi$.

In Sections 5, 6, and 7 we restrict our attention to the case $\rho = 1$. Section 5 deals with the weak limits of a sequence of random functions induced by the process $Q^i(t)$ which is defined to be the number of customers in the system at time t which will be processed through the ith service channel. We show that the appropriate sequence of random functions converges to $(\mu_i/\mu)|\xi|$. Section 6 treats the virtual waiting time, the load at the ith service channel, and the total load in the system, when the servers are identical. All these processes when appropriately scaled converge weakly to $|\xi|$. Section 7 is concerned with embedded sequences obtained by looking at the processes above only at arrival points or only at departure points. We give a general method for obtaining weak convergence theorems for sequences of random functions induced by such embedded sequences. As a particular example we treat the sequence of waiting times of successive customers. Again we find that the appropriate sequence of random functions converges to $|\xi|$.

In Section 8 we discuss limit theorems for the time of the nth arrival and the time of the nth departure. Finally, in Section 9 we discuss limit theorems for the busy period. Since such theorems may be obtained by applying an appropriate functional, this last section illustrates one of the major advantages of the weak convergence.

There are a number of interesting problems, for which we have further results, that have been omitted in an attempt to keep the length and complexity of the paper within reasonable bounds. We mention these here in passing and leave the details to further publications. Our entire analysis has dealt with a single queueing system with $\rho \geqq 1$. By taking a sequence of systems with $\rho_n \to 1$ other limit processes can be obtained (cf. [2], [11] and [12]). The weak convergence analysis was carried out in the space $D[0, 1]$. For some problems, particularly first-passage times, it is more natural to work in $D[0, \infty)$. We have used $D[0, 1]$ because its properties are well known, but we could have done all our work in $D[0, \infty)$. With an appropriate metric, $D[0, \infty)$ is a complete separable metric space in which all our theorems hold. The arrival processes at each of the arrival channels can be made up of quite general dependent sequences. Some of this analysis for the single server case was done by Whitt (1968). Having limit theorems for the departure process, we can construct networks of systems such as ours and analyze those as we do a single system in this paper. Furthermore, it is possible to handle certain barriers on our processes which restrict them from above. One such barrier might correspond to a finite waiting room. Finally, almost identical arguments yield corresponding limit theorems for the stochastic processes arising in dams.

2. The modified queueing system

We follow Borovkov (1965) and begin by introducing a modified multiple channel queueing system. The modified system differs from the standard system in two respects. First, the servers are not shut off when they become idle. With each server (and not with each customer, as is usually done) we associate a sequence of potential service times (random variables). If a server faces continued demand for service, then the actual service times of his successive customers are just these potential service times; but if there is no demand during any potential service time, then that potential service time is ignored and there is no actual service and no departure. After a server has begun working in the absence of demand, then the next demand will in general occur in the middle of some potential service time. Let the remaining portion of that potential service time be that next customer's actual service time.

The second difference in the modified system is that customers are served by the server who can complete the service first, which is not necessarily the first idle server. This means that customers will depart in the order they arrived. Moreover, every completion of a potential service time will generate an actual departure as long as there is a customer demanding service somewhere in the system. This property allows us to work directly with the net potential output process obtained by superimposing the potential outputs from the separate servers. This modified server system is of interest in its own right. For us, it is a device.

Assume now that customers arrive one at a time in each of r channels and then immediately join a single queue in front of the s servers. Equivalently, each customer immediately upon arrival can be assigned to one of s separate queues in front of the s servers. In this case, we look ahead and assign the customer to the server who would eventually serve him. We are given as initial data $r+s$ independent sequences of non-negative, independent, identically distributed random variables with finite variance: $\{u_n^i, n \geqq 1\}$ $(i = 1, \cdots, r)$ and $\{v_n^j, n \geqq 1\}$ $(j = 1, \cdots, s)$ all defined on a common probability space $(\Omega, \mathscr{F}, \mathscr{P})$. The variable u_n^i represents the interarrival time between the $(n-1)$th and nth customers in the ith arrival channel and the variable v_n^j represents the nth potential service time of the jth server. Assume that the system is initially empty, although our limit theorems do not depend on this condition.

We now define renewal processes associated with each channel. Let

$$A^i(t) = \begin{cases} \max\{k: u_1^i + \cdots + u_k^i \leqq t\}, & u_1^i \leqq t \\ 0, & u_1^i > t \end{cases}$$

for all $t \geqq 0$, $1 \leqq i \leqq r$, and

$$S^j(t) = \begin{cases} \max\{k: v_1^j + \cdots + v_k^j \leqq t\}, & v_1^j \leqq t \\ 0, & v_1^j > t \end{cases}$$

for all $t \geqq 0$, $1 \leqq j \leqq s$.

These processes represent the total number of arrivals or the total number of potential service times in the appropriate channel in the time interval $(0, t]$. Because of the service discipline in this modified system, it is particularly easy to express the queue length process, $Q'(t)$, in terms of these basic renewal processes. Throughout this paper all queue length processes count the customers being served as well as those waiting. We also place no upper bound on the number of waiting customers. For each $\omega \in \Omega$ and $t \geqq 0$, we have

$$Q'(t) = X(t) - \inf\{X(s), 0 \leqq s \leqq t\}, \tag{2.1}$$

where

$$A(t) = A^1(t) + \cdots + A^r(t),$$

$$S(t) = S^1(t) + \cdots + S^s(t),$$

and

$$X(t) = A(t) - S(t).$$

Limit theorems for $Q'(t)$ in heavy traffic follow immediately from (2.1). As we indicated in Section 1, we shall let $\lambda_i = 1/Eu_1^i$, $\mu_j = 1/Ev_1^j$, $\lambda = \sum_{i=1}^r \lambda^i$, $\mu = \sum_{j=1}^s \mu_j$, and $\rho = \lambda/\mu$. Observe that the usual traffic intensity for the $GI/G/s$ queue is just a special case of ρ. Furthermore, we let $\alpha_i^2 = \lambda_i^3 \sigma^2[\mu_1^i]$, $\sigma_j^2 = \mu_j^3 \sigma^2[v_1^j]$, and $\gamma^2 = \sum_{i=1}^r \alpha_i^2 + \sum_{j=1}^s \sigma_j^2$. Now let A_n^i $(i = 1, \cdots, r)$, $S_n^j (j = 1, \cdots, s)$, X_n, and Q_n' be random functions in $D[0,1]$ defined by

$$A_n^i \equiv [A^i(nt) - \lambda_i nt]/\alpha_i n^{\frac{1}{2}},$$

$$S_n^j \equiv [S^j(nt) - \mu_j nt]/\sigma_j n^{\frac{1}{2}},$$

$$X_n \equiv [X(nt) - (\lambda - \mu)nt]/\gamma n^{\frac{1}{2}},$$

$$Q_n' \equiv [Q'(nt) - (\lambda - \mu)nt]/\gamma n^{\frac{1}{2}},$$

where for each random function, $t \in [0,1]$. The following lemma is easily proved by appealing to the continuous mapping theorem. As we indicated in the introduction, ξ is a Wiener process.

Lemma 2.1. $X_n \Rightarrow \xi$.

Proof. Let D^{r+s} be the product of $r + s$ copies of D and let

$$\{\xi^i : i = 1, 2, \cdots, r + s\}$$

be $r + s$ independent Wiener processes. Then form $(\xi^1, \cdots, \xi^{r+s})$, an element of D^{r+s} with the product measure. Since the random functions

$$\{A_n^i, S_n^j : i = 1, \cdots, r; j = 1, \cdots, s\}$$

are independent, we have

$$(A_n^1, \cdots, A_n^r, S_n^1, \cdots, S_n^s) \Rightarrow (\xi^1, \cdots, \xi^{r+s}) \tag{2.2}$$

if and only if $A_n^i \Rightarrow \xi^i$ and $S_n^j \Rightarrow \xi^{r+j}$. But this latter fact follows from the functional central limit theorem for renewal processes (cf. [1], Theorem 17.3 with Theorem 16.1). Furthermore, (2.2) is equivalent to

$$(\alpha_1 A_n^1, \cdots, \alpha^r A_n^r, \sigma_1 S_n^1, \cdots, \sigma_s S_n^s) \Rightarrow (\alpha_1 \xi^1, \cdots, \alpha_r \xi^r, \sigma_1 \xi^{r+1}, \cdots, \sigma_s \xi^{r+s})$$

by the continuous mapping theorem ([1], Theorem 5.1).

Now apply the continuous mapping theorem once more with the function $g: D^{r+s} \to D$, defined for any $(x_1, \cdots, x_{r+s}) \in D^{r+s}$ by

$$g(x_1, \cdots, x_{r+s}) = x_1 + \cdots + x_r - x_{r+1} - \cdots - x_{r+s}.$$

Since $\alpha_1 \xi^1 + \cdots + \alpha_r \xi^r - \sigma_1 \xi^{r+1} - \cdots - \sigma_s \xi^{r+s}$ has the same distribution as $(\alpha_1^2 + \cdots + \alpha_r^2 + \sigma_1^2 + \cdots + \sigma_s^2)^{\frac{1}{2}} \xi = \gamma \xi$, for some new independent Wiener process ξ, we obtain the desired result: $X_n \Rightarrow \xi$.

To proceed now to our limit theorem for Q_n', we introduce the continuous mapping $f: D \to D$ which corresponds to an impenetrable barrier at the origin. For $x \in D$, f is defined by $f(x)(t) = x(t) - \inf_{0 \leqq s \leqq t} x(s)$, $0 \leqq t \leqq 1$. The limit theorem for Q_n' when $\rho = 1$ is given next.

Theorem 2.1. If $\rho = 1$, then $Q_n' \Rightarrow f(\xi)$ for all initial queue lengths. The random function $f(\xi)$ has the same distribution as $|\xi|$.

Proof. Assume first that $Q'(0) = 0$. Since $\rho = 1$, the translation terms in both X_n and Q_n' are zero. Therefore, from (2.1) we have $Q_n' = f(X_n)$, where $f: D \to D$ is defined above. Since f is continuous, we may apply the continuous mapping theorem and Lemma 2.1 to obtain $Q_n' \Rightarrow f(\xi)$. The initial conditions are easily handled with [1], Theorem 4.1.

Since ξ is a Wiener process without drift, $f(\xi)$ has the same distribution as the reflecting Brownian motion or the one-dimensional Bessel process (cf. [5], pages 40–42, 59).

For a single time point $t > 0$, we obtain

Corollary 2.1. If $\rho = 1$, then

$$\lim_{t \to \infty} P\left\{\frac{Q'(t)}{\gamma t^{\frac{1}{2}}} \leqq x\right\} = \begin{cases} (2/\pi)^{\frac{1}{2}} \int_0^x \exp\{-y^2/2\} dy, & x \geqq 0 \\ 0, & x < 0. \end{cases}$$

Proof. Apply the continuous mapping theorem again with the projection $\pi_1: D \to R$, defined for any $x \in D$ by $\pi_1(x) = x(1)$. Also use weak convergence of the random elements Q_s' where s goes to infinity in a continuous manner (cf. [1], page 16).

The corresponding results for $\rho > 1$ are given in the next theorem and corollary.

Theorem 2.2. If $\rho > 1$, then $Q_n' \Rightarrow \xi$.

Proof. Using Lemma 2.1 and Theorem 4.1 of [1] it suffices to show that $d(X_n, Q_n') \Rightarrow 0$. Since $d(X_n, Q_n') \leqq \rho(X_n, Q_n') = -\inf_{0 \leqq s \leqq n}\{X(s)/\gamma n^{\frac{1}{2}}\}$, we have only to show that for each $\varepsilon > 0$

$$P[-\inf_{0 \leqq s \leqq n} \{X(s)/\gamma n^{\frac{1}{2}}\} \leqq \varepsilon] \to 1, \quad \text{as } n \to \infty. \tag{2.3}$$

We begin by recalling that $X(t)/t \xrightarrow{a.s.} (\lambda - \mu) > 0$ by the strong law of large numbers for renewal processes (cf. [4]). Thus, for any $\delta > 0$, there exists a $t_0 > 0$ such that

$$P\left[\sup_{t_0 \leqq t} \left| \frac{X(t)}{t} - (\lambda - \mu)\right| \leqq \varepsilon\right] \geqq 1 - \delta.$$

Hence, the

$$P[\sup_{t_0 \leqq s \leqq n} \{-X(s)/\gamma n^{\frac{1}{2}}\} \leqq \varepsilon] \geqq P[\sup_{t_0 \leqq s} \{-X(s)\} \leqq 0] \geqq 1-\delta$$

for $n \geqq [t_0]+1$. On the other hand,

$$\sup_{0 \leqq s \leqq t_0} \{-X(s)/\gamma n^{\frac{1}{2}}\} \leqq \sup_{0 \leqq s \leqq t_0} \frac{A(s)+S(s)}{\gamma n^{\frac{1}{2}}} \leqq \frac{A(t_0)+S(t_0)}{\gamma n^{\frac{1}{2}}}$$

which converges in probability to 0. Since

$$\sup_{0 \leqq s \leqq t_0} \{-X(s)/\gamma n^{\frac{1}{2}}\} \quad \text{and} \quad \sup_{t_0 \leqq s \leqq n} \{-X(s)/\gamma n^{\frac{1}{2}}\}$$

both converge to 0 in probability, we have (2.3).

The proof of the result for a single time point $t > 0$ is the same as in Corollary 2.1.

Corollary 2.2. If $\rho > 1$, then

$$\lim_{t \to \infty} P\left\{\frac{Q(t) - (\lambda-\mu)t}{\gamma t^{\frac{1}{2}}} \leqq x\right\} = (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\{-y^2/2\} dy.$$

3. The standard queueing system

We now investigate the standard multiple channel queueing system in which customers are served in the order of their arrival by the first idle server. The classical $GI/G/s$ queue is the special case arising when there is only one arrival channel and s identical servers.

The central idea, due to Borovkov ([2], Section 5), is to define the standard system in terms of the same basic sequences of random variables already used for the modified system. We then show that the two queue length processes differ very little in heavy traffic. In fact, by applying Theorem 4.1 of [1], we show that the two corresponding sequences of random functions in $D[0,1]$ converge to the same limit. We obtain stronger results than Borovkov with simpler arguments by applying weak convergence theory.

In order to define the standard system, we must generate the actual service times from the given sequences of potential service times. For each server, we let the actual service times be a subsequence of the potential service times, chosen so that this subsequence is also i.i.d. If there is still demand for service after a server has just served a customer, then let the next actual service time be just the next random variable in the basic sequence of potential service times. If there has been no demand before receiving a customer at time t, let the next actual service time be the first unused random variable occurring after time t in the basic sequence of potential service times; that is, let the index of the potential service time which is to be the actual service time of the next customer be $1+\max\{k, S^j(t)\}$, where k is the index of the potential service time which was the last actual service time. It is easy to see that this selection procedure provides a subsequence of i.i.d. random variables.

The standard system is now defined. As before, customers arrive one at a time in each of the r channels and then immediately join a single queue in front of the s servers. They are then served in the order of their arrival, but by the first idle server. Their service times have been specified above.

In preparation for our main theorem and the arguments in later sections, we give two lemmas which clarify the relationship between the function spaces C and D. As is demonstrated in [1], analysis is much easier in C, but many processes of interest are not continuous and must be regarded as elements of D. The standard procedure has been to consider linearly-interpolated versions of such processes, which will be in C, but it recently has become clear that such devices are unnecessary in most cases; the analysis in C may often be used for processes in D. We state a lemma communicated orally to us by T. Liggett and B. Rosén. (For a proof, see [12], page 46.)

Lemma 3.1. (Liggett and Rosén). Let $\{X_n\}$ be a sequence of random functions in (D,d), $\{Y_n\}$ a sequence of random functions in (C,ρ), and X a random function in (C,ρ). If $d(X_n, Y_n) \Rightarrow 0$, then $X_n \Rightarrow X$ in (D,d) if and only if $Y_n \Rightarrow X$ in (C,ρ).

As an easy consequence of Lemma 3.1, the functional central limit theorems for random functions induced in D by sequences of partial sums or renewal processes are equivalent to the corresponding theorems for the linearly-interpolated random functions in C.

In the same spirit, we prove a lemma which gives us C-tightness for sequences of random functions in D. Knowing that a sequence of random functions converges weakly in D, we shall want to use the resulting tightness for other arguments. In a sense, we want a converse to Theorem 15.5 of [1]. The characterization of tightness in C is much easier to use than the characterization in D (cf. [1], pages 55, 125).

The main condition for C-tightness is expressed in terms of the modulus of continuity, $w(\delta): C \to R$, defined for any $x \in C$ by

$$w_x(\delta) = \sup_{\substack{0 \leqq s,t \leqq 1 \\ |s-t| < \delta}} |x(t) - x(s)|.$$

Lemma 3.2. Let $\{X_n\}$ be a sequence of random functions in (D,d), and X a random function such that $P\{X \in C\} = 1$. If $X_n \Rightarrow X$, then $\{X_n\}$ is C-tight: for all positive ε and η, there exists a δ $(0 < \delta < 1)$ and an integer n_0 such that

$$P\{w_X(\delta) \geqq \varepsilon\} \leqq \eta$$

for $n \geqq n_0$.

Proof. The modulus of continuity is a measurable mapping of D into R which is continuous almost everywhere with respect to X. Theorem 5.1 of [1] implies that

$$w_{X_n}(\delta) \Rightarrow w_X(\delta),$$

but $w_X(\delta) \Rightarrow 0$ as $\delta \downarrow 0$.

Before returning to our main argument, we state one more lemma. It gives various sufficient conditions on a sequence of random variables $\{X_n\}$ in order to have $\max_{1 \leqq k \leqq n}\{|X_k|/n^{\frac{1}{2}}\} \Rightarrow 0$. Such conditions are well known, but we include them because the result is so frequently used in weak convergence arguments. The statements are somewhat stronger than we need in this paper, but the generality is important for extending the weak convergence theorems to include dependent sequences. The almost-sure convergence will not be used later. (For further discussion, see [12], pages 41–43, 71.)

Lemma 3.3. Let $\{X_n\}$ be a sequence of random variables.

a) If $\{X_n\}$ are identically distributed (not necessarily independent) with finite variance, then $\max_{1 \leqq k \leqq n}\{|X_k|/n^{\frac{1}{2}}\} \to 0$ a.s.

b) If there exists a random variable X with finite variance such that for all $n \geqq 1$ and all $a > 0$, $P\{|X_n| > a\} \leqq P\{|X| > a\}$, then $\max_{1 \leqq k \leqq n}\{|X_k|/n^{\frac{1}{2}}\} \to 0$ a.s.

c) If $Y_n \Rightarrow Y$ in D with $P\{Y \in C\} = 1$, where $Y_n \equiv S_{[nt]}/n^{\frac{1}{2}}$, $0 \leqq t \leqq 1$, and $S_k = X_1 + \cdots + X_k - kv$ for some constant v, then $\max_{1 \leqq k \leqq n}\{|X_k|/n^{\frac{1}{2}}\} \Rightarrow 0$.

We now return to the standard multiple channel queueing system. Let $Q(t)$ denote the number of customers either waiting or being served at time t. Define the corresponding random function in D by

$$Q_n \equiv [Q(nt) - (\lambda - \mu)nt]/\gamma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

Our main result is

Theorem 3.1. If $\rho = 1$, then $Q_n \Rightarrow f(\xi)$; if $\rho > 1$, then $Q_n \Rightarrow \xi$.

Proof. We shall show that $d(Q_n, Q_n') \Rightarrow 0$. Theorems 2.1 and 2.2 together with Theorem 4.1 of [1] then give us the desired result.

We now follow Borovkov ((1965), Section 5) quite closely. We first determine the time between the occurrence of a random variable in the basic potential service time sequence and the occurrence of that same random variable as an actual service time in the corresponding service channel. We call this time the *shift*; let $\theta_j(t)$ denote the shift in channel j at time t. The shift can be expressed in terms of the idle periods and the unused potential service times. If the jth server is serving a customer at time t, $I_1^j, \cdots, I_k^j$ are the idle periods in service channel j up to time t, and $\{v_{n_k}^j\}$ is the subsequence consisting of the unused potential service times, then

$$\theta_j(t) = (v^j_{n_1} + \cdots + v^j_{n_l}) - (I^j_1 + \cdots + I^j_k),$$

where

$$v^j_{n_1} + \cdots + v^j_{n_l} \quad \geqq I^j_1 + \cdots + I^j_k,$$

but

$$v^j_{n_1} + \cdots + v^j_{n_{l-1}} < I^j_1 + \cdots + I^j_k.$$

In other words, if the server is actually serving someone at time t, then $\theta_j(t)$ is the residual lifetime of the renewal process generated by the subsequence $\{v^j_{n_k}\}$, evaluated at time $I^j_1 + \cdots + I^j_k$. If the server is idle at time t, then we shall define $\theta_j(t) = \theta_j(t'')$, where t'' is the last time before t the server was busy. The following diagram helps make these ideas clear.

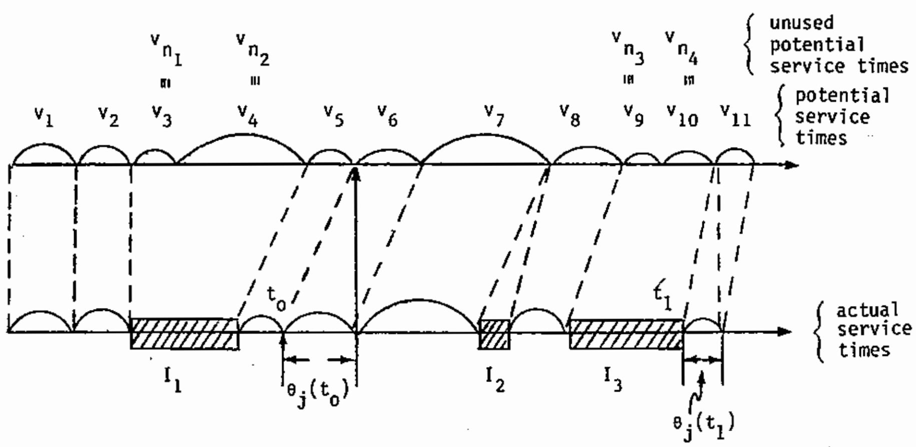

Figure 1

Now we establish bounds on the difference between the two queue length processes, $Q'(t)$ and $Q(t)$. We shall show, for all $\omega \in \Omega$ and $t \geqq 0$, that there exists a $t_0 \leqq t$ such that

$$Q(t) \leqq Q'(t) + s + \sum_{j=1}^{s} [S^j(t_0 + \theta_j(t_0)) - S^j(t_0)] \tag{3.1}$$

and that

$$Q'(t) \leqq Q(t) + \sum_{j=1}^{s} [S^j(t + \theta_j(t)) - S^j(t)]. \tag{3.2}$$

We first verify (3.1). If $Q(t) < s$, then the inequality holds trivially. Suppose then that $Q(t) \geqq s$. Let

$$t_0 = \sup\{\tau \leqq t: Q(\tau) < s\}.$$

We clearly have

$$Q(t) = Q(t_0) + \sum_{i=1}^{r} [A^i(t) - A^i(t_0)] - \sum_{j=1}^{s} [S^j(t + \theta_j(t)) - S^j(t_0 + \theta_j(t_0))]$$

$$\leqq s + \sum_{i=1}^{r} [A^i(t) - A^i(t_0)] - \sum_{j=1}^{s} [S^j(t + \theta_j(t) - S^j(t_0 + \theta_j(t_0))],$$

while

$$Q'(t) \geqq Q'(t_0) + \sum_{i=1}^{r} [A^i(t) - A^i(t_0)] - \sum_{j=1}^{s} [S^j(t)) - S^j(t_0)]$$

$$\geqq \sum_{i=1}^{r} [A^i(t) - A^i(t_0)] - \sum_{j=1}^{s} [S^j(t + \theta_j(t)) - S^j(t_0)].$$

These two inequalities imply (3.1).

We now verify (3.2). If $Q'(t) = 0$, then the inequality holds trivially. Suppose then that $Q'(t) > 0$. Let $t_1 = \sup\{\tau \leqq t : Q'(\tau) = 0\}$.

We clearly have

$$Q'(t) = \sum_{i=1}^{r} [A^i(t) - A^i(t_1)] - \sum_{j=1}^{s} [S^j(t) - S^j(t_1)],$$

while

$$Q(t) \geqq \sum_{i=1}^{r} [A^i(t) - A^i(t_1)] - \sum_{j=1}^{s} [S^j(t + \theta_j(t)) - S^j(t_1)],$$

which together imply (3.2). Therefore,

$$d(Q_n, Q_n') \leqq \rho(Q_n, Q_n') = \sup_{0 \leqq t \leqq 1} \frac{|Q(nt) - Q'(nt)|}{\gamma n^{\frac{1}{2}}}$$

$$\leqq \sup_{0 \leqq t \leqq 1} \frac{s + \sum_{j=1}^{s} [S^j(nt + \theta_j(nt)) - S^j(nt)]}{\gamma n^{\frac{1}{2}}}.$$

In order to complete the proof, it thus suffices to show for all j, $1 \leqq j \leqq s$, that

$$\sup_{0 \leqq t \leqq 1} \frac{S^j(nt + \theta_j(nt)) - S^j(nt)}{n^{\frac{1}{2}}} \Rightarrow 0. \tag{3.3}$$

Now let us obtain a handle on the shift. Observe that

$$\sup_{0 \leqq t \leqq 1} \theta_j(nt) \leqq \max_{k \leqq \tilde{S}^j(n)+1} v_{n_k}$$

where $\tilde{S}^j(t)$ is the renewal process generated by the subsequence, $\{v_{n_k}\}$, consisting of the unused potential service times. Note that the subsequence $\{v_{n_k}\}$ is i.i.d. and independent of the idle times. Since $\tilde{S}_j(nt) \leqq \tilde{S}_j(n)$, $0 \leqq t \leqq 1$, and $\tilde{S}^j(n)/n \Rightarrow \mu_j$, in order to have $\sup_{0 \leqq t \leqq 1}\{\theta_j(nt)/n^{\frac{1}{2}}\} \Rightarrow 0$, it suffices to show that $\max_{1 \leqq k \leqq n}\{v^j_{n_k}/n^{\frac{1}{2}}\} \Rightarrow 0$, which holds by Lemma 3.3.

We complete the proof by verifying (3.3). We use the tightness associated with the sequence of random functions $\{S_n^j\}$. Since $S_n^j \Rightarrow \xi$, we have C-tightness for $\{S_n^j\}$ in D by virtue of Lemma 3.2. Thus, for all positive ε and η, there exists a δ $(0 < \delta < 1)$ and an integer n_0 such that

$$P\left\{\sup_{\substack{0 \leq s,t \leq 1 \\ |s-t| < \delta}} \left| \frac{S^j(nt) - S^j(ns) - \mu_j n(t-s)}{\sigma_j n^{\frac{1}{2}}} \right| \geq \varepsilon \right\} \leq \eta$$

for $n \geq n_0$. This in turn implies, for all positive ε and η, there exists a δ $(0 < \delta < 1)$ and an integer n_0 such that

$$P\left\{\sup_{\substack{0 \leq t \leq 1 \\ 0 \leq s < \delta}} \left| \frac{S^j(n[t+s]) - S^j(nt) - \mu_j ns}{\sigma_j n^{\frac{1}{2}}} \right| \geq \varepsilon \right\} \leq \eta$$

for $n \geq n_0$. Instead of the constant s, we have in (3.3) $\theta_j(nt)/n$, but $\sup_{0 \leq t \leq 1} \theta_j(nt)/n^{\frac{1}{2}} \Rightarrow 0$. Thus, for any positive η and δ, there exists an integer n_1 such that

$$P\{\sup_{0 \leq t \leq 1} \theta_j(nt) > n\delta\} < \eta$$

for $n > n_1$. Furthermore, the translation term in the tightness characterization goes to 0:

$$\sup_{0 \leq t \leq 1} \frac{\mu_j n[\theta_j(nt)/n]}{\sigma_j n^{\frac{1}{2}}} = \sup_{0 \leq t \leq 1} \frac{\mu_j \theta_j(nt)}{\sigma_j n^{\frac{1}{2}}} \Rightarrow 0.$$

Consequently, the C-tightness of $\{S_n^j\}$ implies (3.3) and the proof is complete.

Corollaries 2.1 and 2.2 also hold for $Q(t)$.

4. The departure process

Let the departure processes for the standard and modified queueing systems be denoted by $\{D(t), t \geq 0\}$ and $\{D'(t), t \geq 0\}$ respectively. In this section we seek weak convergence theorems for these processes when $\rho \geq 1$. As usual we assume $Q(0) = Q'(0) = 0$, but other initial conditions can be handled by Theorem 4.1 of [1].

From the definitions of the departure processes, $D(t) = A(t) - Q(t)$ and $D'(t) = A(t) - Q'(t)$. From the definition of $Q'(t)$, we have

$$\begin{aligned} D'(t) &= A(t) - \{X(t) - \inf_{0 \leq s \leq t} X(s)\} \\ &= S(t) + \inf_{0 \leq s \leq t} [A(s) - S(s)]. \end{aligned}$$

Now define the random function D_n' by

$$D_n' \equiv [D'(nt) - (\lambda \wedge \mu)nt]/n^{\frac{1}{2}}, \qquad 0 \leq t \leq 1,$$

and the continuous mapping $g:D[0,1]\times D[0,1]\to D[0,1]$ by

$$g(x,y)(t) = y(t) + \inf_{0\leqq s\leqq t}[x(s)-y(s)], \qquad 0\leqq t\leqq 1.$$

The process D_n is defined exactly like D_n' with $D(nt)$ replacing $D'(nt)$.

First we use the old trick of showing that D_n and D_n' are essentially the same if $\rho \geqq 1$.

Lemma 4.1. If $\rho\geqq 1$, then $d(D_n, D_n') \Rightarrow 0$.

Proof. Since $D_n - D_n' = Q_n' - Q_n$, the result follows from the proof of Theorem 3.1.

Now since $(\alpha A_n, \sigma S_n) \Rightarrow (\alpha\xi_1, \sigma\xi_2)$, we can apply the continuous mapping theorem and Lemma 4.1 to obtain

Theorem 4.1. If $\rho = 1$, then $D_n' \Rightarrow g(\alpha\xi_1,\sigma\xi_2)$ and $D_n \Rightarrow g(\alpha\xi_1,\sigma\xi_2)$.

On the other hand, for $\rho > 1$ we use the fact that $d(X_n, Q_n') \Rightarrow 0$ from the proof of Theorem 2.2, which gives us $d(D_n', S_n) \Rightarrow 0$. Thus we have

Theorem 4.2. If $\rho > 1$, then $D_n' \Rightarrow \sigma\xi$ and $D_n \Rightarrow \sigma\xi$.

This completes the limit theorems for the departure processes in heavy traffic.

For $\rho < 1$, we shall show in a sequel to this paper that $D_n \Rightarrow \alpha\xi$ and that $D_n' \Rightarrow \alpha\xi$.

The other open problem for the departure processes is to obtain the distribution of $g(\alpha\xi_1,\sigma\xi_2)$ in a more useful form. For the $M/M/1$ queue with $\rho = 1$, we have been able to calculate a limit theorem for $D(t)$ using the combinatorial ideas of Champernowne (1956) (cf. [9], page 11). Using the invariance principal idea implicit for all functional central limit theorems, this gives us the distribution of $g(\alpha\xi_1,\sigma\xi_2)(t)$ when $\alpha = \sigma$. However, the process seems to have different properties when $\alpha < \sigma$, $\alpha = \sigma$, and $\alpha > \sigma$. A simpler characterization of $g(\alpha\xi_1,\sigma\xi_2)$ as a process is still needed. We remark however, that for a fixed $t > 0$, $g(\alpha\xi_1,\sigma\xi_2)(t)$ is distributed as

$$\inf_{0\leqq s\leqq t}\{\sigma\xi_2(t-s)+\alpha\xi_1(s)\}.$$

5. The queue length process at the *i*th service channel

Let $Q^i(t)$ be the number of customers in the standard system at time t which will be processed through the ith service channel and let $\{Q_n^i\}$ be the corresponding sequence of random functions in D where

$$Q_n^i \equiv Q^i(nt)/\gamma n^{\frac{1}{2}}, \qquad 0\leqq t\leqq 1.$$

We shall show that $Q_n^i \Rightarrow (\mu_i/\mu)f(\xi)$ when $\rho = 1$.

Our analysis begins with the process $L^i(t)$ which we define to be the work load (future service time required for all customers in the system) at time t which will be processed through the ith service channel. Define the corresponding sequence of random functions $\{L_n^i\}$ in D by

$$L_n^i = L^i(nt)/\gamma n^{\frac{1}{2}}, \qquad 0 \leq t \leq 1.$$

The first lemma is an immediate consequence of the queue discipline.

Lemma 5.1. If $\rho \geqq 1$, then $\rho(L_n^i, L_n^j) \Rightarrow 0$, $i,j = 1,\cdots,s$.

Proof. Since a waiting customer goes to the first available server, $L^i(nt)$ and $L^j(nt)$ can differ at most by a potential service time. Hence, we have

$$\rho(L_n^i, L_n^j) \leqq \left(\max_{1 \leq k \leq S^i(n+\theta_i(n))} v_k^i/\gamma n^{\frac{1}{2}} \right) \vee \left(\max_{1 \leq k \leq S^j(n+\theta_j(n))} v_k^j/\gamma n^{\frac{1}{2}} \right).$$

Since $\theta_j(n)/n \Rightarrow 0$, $S^j(n)/n \Rightarrow \mu_j$, and $\max_{1 \leq k \leq cn} v_k^j/\gamma n^{\frac{1}{2}} \Rightarrow 0$ for any $c > 0$ and all j $(j = 1,\cdots,s)$, we have $\rho(L_n^i, L_n^j) \Rightarrow 0$.

Lemma 5.2. If $\rho = 1$, then $\rho(\mu_i^{-1}Q_n^i, \mu_j^{-1}Q_n^j) \Rightarrow 0$, $i,j = 1,\cdots,s$.

Proof. First we relate $Q^i(t)$ to $L^i(t)$. Let $B^i(t)$ be the total number of customers which arrive in $(0,t]$ and are processed through the ith service channel. Then it is easy to see that

$$L^i(t) = \sum_{k=B^i(t)-[Q^i(t)-1]^+ +1}^{B^i(t)} v_k^i + r^i(t), \tag{5.1}$$

where $r^i(t)$ is the residual service time of the customer being served at time t, and $\{v_k^i\}$ are the actual service times in the ith service channel. Since

$$\sup_{0 \leq t \leq 1} r^i(nt) \leqq \sup_{1 \leq k \leq B^i(n)} v_k^i,$$

and

$$\frac{B^i(n)}{n} \leqq \frac{A(n)}{n} \Rightarrow \lambda,$$

the maximal residual service time will be killed by the factor $n^{-\frac{1}{2}}$, so we omit it from here on. We thus can write

$$L^i(nt)/\gamma n^{\frac{1}{2}} = (1/\gamma n^{\frac{1}{2}}) \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1}) + (1/\gamma n^{\frac{1}{2}})[Q^i(nt)-1]^+ \mu_i^{-1}.$$

After dropping some terms of order $n^{-\frac{1}{2}}$, this leads via the triangle inequality to

$$\rho(\mu_i^{-1}Q_n^i, \mu_j^{-1}Q_n^j) \leqq \rho(L_n^i, L_n^j) + \sup_{0 \leq t \leq 1} \Bigg\{ \Bigg| (1/\gamma n^{\frac{1}{2}}) \sum_{k=B^i(nt)-[Q^i(nt)-1]^+ +1}^{B^i(nt)} (v_k^i - \mu_i^{-1})$$
$$- (1/\gamma n^{\frac{1}{2}}) \sum_{k=B^j(nt)-[Q^j(nt)-1]^+ +1}^{B^j(nt)} (v_k^j - \mu_j^{-1}) \Bigg| \Bigg\}.$$

The first term on the right converges in probability to 0 by Lemma 5.1. Thus it will suffice to show that

$$\sup_{0\leq t\leq 1}\left|(1/\gamma n^{\frac{1}{2}})\sum_{k=B^i(nt)-[Q^i(nt)-1]^++1}^{B^i(nt)}(v_k^i-\mu_i^{-1})\right| \Rightarrow 0.$$

This we shall do by exploiting the C-tightness of the partial sums of $X_k^i = v_k^i - \mu_i^{-1}$ $(k = 1,2,\cdots)$ which holds by virtue of Donsker's theorem (cf. [1], page 137) and Lemma 3.1. Now define events $A_{n\varepsilon}$, B_n, $C_{n\delta}$, and $D_{n\delta\varepsilon}$ as follows:

$$A_{n\varepsilon} = \left\{\omega:\ \sup_{0\leq t\leq 1}\left|(1/\gamma n^{\frac{1}{2}})\sum_{k=B^i(nt)-Q^i(nt)-1]^++1}^{B^i(nt)}(v_k^i-\mu_i^{-1})\right|\geqq\varepsilon\right\},$$

$$B_n = \left\{\omega:\ \sup_{0\leq t\leq 1} B^i(nt)\geqq 2\lambda n\right\},$$

$$C_{n\delta} = \left\{\omega:\ \sup_{0\leq t\leq 1}[Q^i(nt)-1]^+\geqq n\delta\right\},$$

and

$$D_{n\delta\varepsilon} = \left\{\omega:\ \sup_{0\leq t\leq 2\lambda}\left|\frac{S^i_{[nt]}-S^i_{[ns]}}{\gamma n^{\frac{1}{2}}}\right|\geqq\varepsilon\right\},$$

where $S_k^i = X_1^i + \cdots + X_k^i$, $X_k^i = v_k^i - \mu_i^{-1}$, and $S_0^i = 0$.

Observe that $A_{n\varepsilon} \subseteq B_n \cup C_{n\delta} \cup D_{n\delta\varepsilon}$ for all n, δ and ε, so that

$$P(A_{n\varepsilon}) \leqq P(B_n) + P(C_{n\delta}) + P(D_{n\delta\varepsilon}).$$

We now show that, for any positive ε and η, there exists an n_0 such that $P(A_{n\varepsilon}) < \eta$ for $n \geqq n_0$. Since $\sup_{0\leq t\leq 1}B^i(nt) = B^i(n) \leqq A(n)$, and $A(n)/n \Rightarrow \lambda$, there is an n_1 such that $P(B_n) < \eta/3$ for $n \geqq n_1$. Using the tightness of the random functions induced by the sequence of partial sums $\{S_k^i\}$, we know that there exists a δ $(0 < \delta < 1)$ and n_2 such that $P(D_{n\delta\varepsilon}) < \eta/3$ for $n \geqq n_2$. Since $Q_n \Rightarrow f(\xi)$ by Theorem 3.1, we may apply the continuous mapping theorem to assert that $\sup_{0\leq t\leq 1} Q(nt)/\gamma n^{\frac{1}{2}}$ converges to a non-degenerate limit. Hence, $\sup_{0\leq t\leq 1}Q(nt)/n \Rightarrow 0$. Therefore, for any fixed $\delta > 0$ (that needed above), there exists an n_3 such that $P(C_{n\delta}) < \eta/3$. Finally, if $n_0 = n_1 \vee n_2 \vee n_3$, $P(A_{n\varepsilon}) < \eta$ for $n \geqq n_0$.

Lemma 5.3. If $\rho = 1$, then $\rho[(\mu_j/\mu)Q_n, Q_n^j] \Rightarrow 0$, $j = 1,\cdots,s$.

Proof. It suffices to show that $\rho[Q_n,(\mu/\mu_j)Q_n^j] \Rightarrow 0$, but

$$\rho\,[\,Q_n,(\mu/\mu_j)Q_n^j] = \rho\left[Q_n^1 + \cdots + Q_n^s, \left(\frac{\mu_1}{\mu}\right)Q_n^j + \cdots + \left(\frac{\mu_s}{\mu_j}\right)Q_n^j\right]$$

$$\leqq \sum_{i=1}^{s}\rho\left[Q_n^i,\left(\frac{\mu_i}{\mu_j}\right)Q_n^j\right] \Rightarrow 0$$

by Lemma 5.2.

Finally, we have our desired result.

Theorem 5.1. If $\rho = 1$, then $Q_n^i \Rightarrow (\mu_i/\mu)f(\xi)$, $i = 1, \cdots, s$.

Proof. Apply Theorem 3.1, Lemma 5.3, and Theorem 4.1 of [1].

6. The load and waiting time processes

In Section 5 we introduced the load at the ith service channel, $L^i(t)$. The total load for the entire system, $L(t)$, is obviously just $L^1(t) + \cdots + L^s(t)$. In a single-server queue $L^i(t) \equiv L(t)$ is just the virtual waiting time, $W(t)$, the time a potential customer arriving at time t would have to wait before reaching the server. Here $W(t) = \min_{1 \leq j \leq s}\{L^j(t)\}$. In this section we obtain functional central limit theorems for $L^i(t)$, $L(t)$, and $W(t)$ when $\rho = 1$ and all servers are identical.

The total load can be expressed as

$$L(t) = \sum_{i=A(t)-[Q(t)-s]^+ + 1}^{A(t)} v_i + \sum_{j=1}^{s} r_j(t),$$

where the $\{v_i\}$ are actual service times and $r_j(t)$ is the residual service time of the customer being served by the jth server at time t. Clearly the $[Q(t) - s]^+$ complete service times are independent of $[Q(t) - s]^+$, but if we allow different servers, as we have so far, then the $\{v_i\}$ are in general neither independent nor identically distributed. If we look at $L^i(t)$ [see (5.1)] instead, then the $[Q^i(t) - 1]^+$ complete service times are i.i.d., but these service times are not independent of $[Q^i(t) - 1]^+$. Therefore, we restrict our attention to the case in which all servers are identical; then, the $[Q(t) - s]^+$ complete service times in $L(t)$ are i.i.d. and independent of $[Q(t) - s]^+$. As we have seen before, the residual service times will be killed when we normalize by $n^{-\frac{1}{2}}$, hence we shall ignore them.

In the usual way, define the sequence of random functions $\{L_n\}$ induced in D by $L(t)$ by

$$L_n \equiv L(nt)/\gamma n^{\frac{1}{2}}, \qquad 0 \leq t \leq 1.$$

Theorem 6.1. If the service channels are identical and $\rho = 1$, then $L_n \Rightarrow (s/\mu)f(\xi)$.

We shall break up the proof of this theorem into a number of lemmas. First we show convergence at a single time point.

Lemma 6.1. If the service channels are identical and $\rho = 1$, then $L(nt)/\gamma n^{\frac{1}{2}} \Rightarrow s\mu^{-1}f(\xi)(t)$ for all $t \in [0,1]$.

Proof. If $Q(nt) > s$, then we can write

$$\frac{L(nt)}{\gamma n^{\frac{1}{2}}} = \frac{Q(nt)}{\gamma n^{\frac{1}{2}}} \cdot \frac{[Q(nt)-s]^+}{Q(nt)} \cdot \frac{L(nt)}{[Q(nt)-s]^+}.$$

From Theorem 3.1 we know that $Q(nt)/\gamma n^{\frac{1}{2}} \Rightarrow f(\xi)(t)$ through an application of the continuous mapping theorem with the projection $\pi_t: D \to R$, defined for any $x \in D$ by $\pi_t(x) = x(t)$. For $t = 0$, $L(0)/\gamma n^{\frac{1}{2}} = 0 = s\mu^{-1}f(\xi)(0)$ for all n. Thus, it suffices to consider from here on only $t > 0$. Since $f(\xi)(t)$ has no atom at zero, $Q(nt) \Rightarrow +\infty$ which implies that $[Q(nt)-s]^+/Q(nt) \Rightarrow 1$. Furthermore. the weak law of large numbers implies that $L(nt)/[Q(nt)-s]^+ \Rightarrow s\mu^{-1}$. Using Theorem 4.4 of [1], we obtain

$$\left(Q(nt)/\gamma n^{\frac{1}{2}},\ \frac{[Q(nt)-s]^+}{Q(nt)},\ \frac{L(nt)}{[Q(nt)-s]^+}\right) \Rightarrow (f(\xi)(t), 1, s\mu^{-1}).$$

Finally, an application of the continuous mapping theorem with the function $h: R^3 \to R^1$, defined for any $(x, y, z) \in R^3$ by $h(x, y, z) = xyz$, yields the result.

We now turn to the convergence of the finite-dimensional distributions of L_n.

Lemma 6.2. If the service channels are identical and $\rho = 1$, then $[L(nt_1)/\gamma n^{\frac{1}{2}}, \cdots, L(nt_k)/\gamma n^{\frac{1}{2}}] \Rightarrow [s\mu^{-1}f(\xi)(t_1), \cdots, s\mu^{-1}f(\xi)(t_k)]$ for all

$$0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_k \leqq 1.$$

Proof. From Theorem 3.1 we have $[Q(nt_1)/\gamma n^{\frac{1}{2}}, \cdots, Q(nt_k)/\gamma n^{\frac{1}{2}}] \Rightarrow [f(\xi)(t_1), \cdots, f(\xi)(t_k)]$, using the projection $\pi_{t_1,\cdots,t_k}: D \to R^k$, defined for any $x \in D$ by $\pi_{t_1,\cdots,t_k}(x) = [x(t_1), \cdots, x(t_k)]$. From Lemma 6.1, $L(nt)/[Q(nt)-s]^+ \Rightarrow s\mu^{-1}$ and $[Q(nt)-s]^+/Q(nt) \Rightarrow 1$ for any $t > 0$, which implies that

$$\{L(nt_1)/[Q(nt_1)-s]^+, \cdots, L(nt_k)/[Q(nt_k)-s]^+\} \Rightarrow (s\mu^{-1}, \cdots, s\mu^{-t})$$

and $\{[Q(nt_1)-s]^+/Q(nt_1), \cdots, [Q(nt_k)-s]^+/Q(nt_k)\} \Rightarrow (1, \cdots, 1)$ for $t_1 > 0$. Since $L(0)/\gamma n^{\frac{1}{2}} = 0 = f(\xi)(0)$ for all n, the case $t_1 = 0$ can be handled separately without any difficulty. Theorem 4.4 of [1] again implies convergence of the appropriate joint distributions in R^{3k}. The proof is completed by applying the continuous mapping theorem with the function $h: R^{3k} \Rightarrow R^k$, defined for any $(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) \in R^{3k}$ by $h(x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, \cdots, z_k) = (x_1y_1z_1, \cdots, z_ky_kz_k)$.

The remaining step in the proof of Theorem 6.1 is to show that $\{L_n\}$ is tight; (cf. [1], Theorem 15.1). From Theorem 15.5 of [1], it suffices to show that, for any positive ε and η, there exists a δ $(0<\delta<1)$ and an n_0 such that $P\{w_{L_n}(\delta) \geqq \varepsilon\} \leqq \eta$ for $n \geqq n_0$.

We shall use an argument similar to that used in Lemma 5.2.

Lemma 6.3. The sequence $\{L_n\}$ is tight.

Proof. Begin by using the representation

$$L(nt) = \sum_{k=A(nt)-[Q(nt)-s]^{+}+1}^{A(nt)} (v_k - s\mu^{-1}) + [Q(nt) - s]^{+} s\mu^{-1}.$$

Thus

$$\sup_{\substack{0 \leqq \tau, t \leqq 1 \\ |\tau - t| \leqq \delta}} \left| \frac{L(nt) - L(n\tau)}{\gamma n^{\frac{1}{2}}} \right| \leqq 2 \sup_{0 \leqq t \leqq 1} \left| (1/\gamma n^{\frac{1}{2}}) \sum_{k=A(nt)-[Q(nt)-s]^{+}+1}^{A(nt)} (v_k - s\mu^{-1}) \right|$$

$$+ \sup_{\substack{0 \leqq \tau, t \leqq 1 \\ |\tau - t| \leqq \delta}} s\mu^{-1} \left| \frac{Q(nt) - Q(n\tau)}{\gamma n^{\frac{1}{2}}} \right| + \frac{2s\mu^{-1}}{\gamma n^{\frac{1}{2}}}.$$

The last term on the right goes to zero and we ignore it. Since $Q_n \Rightarrow f(\xi)$ with $f(\xi) \in C$, Lemma 3.2 gives C-tightness for $\{Q_n\}$; that is, for any positive ε and η, there exists a δ_1 $(0 < \delta_1 < 1)$ and an n_1 such that

$$P\left\{ \sup_{\substack{0 \leqq \tau, t \leqq 1 \\ |\tau - t| \leqq \delta}} \left| \frac{Q(nt) - Q(n\tau)}{\gamma n^{\frac{1}{2}}} \right| \geqq \varepsilon/2 \right\} \leqq \eta/2$$

for $n \geqq n_1$. Furthermore,

$$P\left\{ 2 \sup_{0 \leqq t \leqq 1} \left| (1/\gamma n^{\frac{1}{2}}) \sum_{k=A(nt)-[Q(nt)-s]^{+}+1}^{A(nt)} (v_k - s\mu^{-1}) \right| \geqq \varepsilon/2 \right\} \leqq \eta/2$$

for $n \geqq n_2$, for sufficiently large n_2, by an argument exactly like that used to prove Lemma 5.2. Thus $P\{w_{L_n}(\delta) \geqq \varepsilon\} \leqq \eta$ by taking $\delta = \delta_1$ and $n_0 = n_1 \vee n_2$.

Combining Lemmas 6.2 and 6.3 yields Theorem 6.1. From here it is easy to obtain the f.c.l.t.'s for $L^i(t)$ and $W(t)$. Define the random function W_n by

$$W_n \equiv W(nt)/\gamma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

Theorem 6.2. If the service channels are identical and $\rho = 1$, then $L_n^i \Rightarrow \mu^{-1} f(\xi)$ and $W_n \Rightarrow \mu^{-1} f(\xi)$.

Proof. From Lemma 5.1 we know that $\rho(L_n^i, L_n^j) \Rightarrow 0$ for $i, j = 1, \cdots, s$. Since

$$d(sL_n^i, L_n) \leqq \sum_{j=1}^{s} d(L_n^i, L_n^j) \leqq \sum_{j=1}^{s} \rho(L_n^i, L_n^j),$$

$d(sL_n^i, L_n) \Rightarrow 0$ and the first result follows. On the other hand,

$$d(W_n, L_n^i) \leqq \max_{1 \leqq j \leqq s} d(L_n^j, L_n^i) \Rightarrow 0,$$

and the second result follows.

7. Embedded sequences

A standard method of analysis in queueing theory is to study certain embedded sequences which are obtained by looking at the continuous-time queueing processes only at arrival points or only at departure points (cf. [9], page 38). Usually this is done to provide a more tractable process than the original one, in particular, a Markov process. For example, in the $GI/G/1$ queue the waiting time of the nth customer, W_n', is often studied because $\{W_n'\}$ is a Markov process while $W(t)$ is not. Although our analysis makes no use of such Markov properties, we shall indicate in this section how f.c.l.t.'s can be obtained for embedded sequences. We illustrate the idea by considering the waiting time of the nth customer, W_n', in the standard queueing system.

We begin by introducing the notion of an inverse random time change. The notion of a random time change is discussed in Chapter 17 of [1]. As in [1], page 144, we let D_0 consist of those functions ϕ in D that are non-decreasing and satisfy $0 \leqq \phi(t) \leqq 1$. It is easy to show that D_0 is a closed subset of D. Therefore, D_0 is a complete separable metric space with the Skorohod metric d of D. The function ϕ represents a transformation of the time interval $[0,1]$.

Assume that $\{X_n\}$ is any sequence of random functions in D and $\{\Phi_n\}$ is any sequence of random functions in D_0, with X_n and Φ_n defined on a common domain for each n. Assume that the prospective limits X and Φ are also defined on a common domain. Billingsley ([1], page 145 and Theorem 4.4) has shown

Lemma 7.1. (Billingsley) If $X_n \Rightarrow X$ with $P\{X \in C\} = 1$ and $d(\Phi_n, \phi) \Rightarrow 0$, where ϕ is a constant function in $C \cap D_0$, then $X_n \circ \Phi_n \Rightarrow X \circ \phi$, where

$$X_n \circ \Phi_n \equiv X_n(\Phi_n(t)), \quad 0 \leqq t \leqq 1,$$

and

$$X \circ \phi \equiv X(\phi(t)), \quad 0 \leqq t \leqq 1.$$

We shall find a converse of sorts to Lemma 7.1. We want to show under appropriate conditions that $X_n \Rightarrow X$ if $X_n \circ \Phi_n \Rightarrow X \circ \phi$. For this purpose, we introduce the inverse random time change Φ^{-1}, defined for any random function $\Phi \in D_0$ by

$$\Phi^{-1}(\tau) = \begin{cases} \inf[t \geqq 0 \colon \Phi(t) \geqq \tau,\ 0 \leqq t \leqq 1], & \text{if the set is non-empty,} \\ 1, & \text{otherwise,} \end{cases}$$

for $0 \leqq \tau \leqq 1$. For each ω, this function will be left-continuous with right limits. For each τ, $\Phi^{-1}(\tau)$ is a random variable so Φ^{-1} is a legitimate random function in D_0^L (cf. [1], page 128), where D_0^L is just D_0 with left-continuity instead of right-continuity for all functions. We could define a right-continuous version $\tilde{\Phi}^{-1} \in D_0$ by

$$\tilde{\Phi}^{-1}(\tau) = \lim_{s \downarrow \tau} \Phi^{-1}(s), \qquad 0 \leqq \tau \leqq 1,$$

but it is not necessary; we can use $\Phi^{-1} \in D_0^L$.

Before we state our results, we remark that we shall require the limiting constant function ϕ to be strictly increasing with $\phi(0) = 0$ and $\phi(1) = 1$. This enables us to say that ϕ^{-1} is of the same form and $\phi(\phi^{-1}(t)) = t$ for all $t \in [0,1]$. Our first tool is

Lemma 7.2. If $d(\Phi_n, \phi) \Rightarrow 0$ in D_0 with $\phi \in C \cap D_0$ and ϕ strictly increasing, then $\rho(\Phi_n^{-1}, \phi^{-1}) \Rightarrow 0$.

Proof. Since $\phi \in C$, $d(\Phi_n, \phi) \Rightarrow 0$ implies $\rho(\Phi_n, \phi) \Rightarrow 0$. By looking at the graph of ϕ, it is easy to see that, for any sample point ω, if $\rho(\Phi_n, \phi) < \varepsilon$, then $|\Phi_n^{-1}(\tau) - \phi^{-1}(\tau)| \leqq |\phi^{-1}([\tau + \varepsilon] \wedge 1) - \phi^{-1}([\tau - \varepsilon] \vee 0)|$ for any $\tau \in [0,1]$, so that

$$\rho(\Phi_n^{-1}, \phi^{-1}) \leqq \sup_{0 \leqq \tau \leqq 1} |\phi^{-1}([\tau + \varepsilon] \wedge 1) - \phi^{-1}([\tau - \varepsilon] \vee 0)|. \tag{7.1}$$

Since ϕ^{-1} is uniformly continuous with ϕ as above, the right side of (7.1) converges to 0 as $\varepsilon \to 0$. Hence, $\rho(\Phi_n^{-1}, \phi^{-1}) \Rightarrow 0$ as claimed.

Our main result is

Theorem 7.1. Let $X_n \in D$, $\Phi_n \in D_0$, and ϕ be a strictly increasing constant function in C with $\phi(0) = 0$ and $\phi(1) = 1$. If $d(\Phi_n, \phi) \Rightarrow 0$, $X_n \circ \Phi_n \Rightarrow X \circ \phi$, $\{X_n\}$ is C-tight, and $P\{X \in C\} = 1$, then $X_n \Rightarrow X$.

Proof. Let $I: [0,1] \to [0,1]$ be the identity function: $I(t) = t$, $0 \leqq t \leqq 1$. Clearly ϕ^{-1} is of the same form as ϕ and $\phi \circ \phi^{-1} = I$. Using Billingsley's argument ([1], page 145) and Lemma 7.2, we have

$$X_n \circ \Phi_n \circ \Phi_n^{-1} \Rightarrow X \circ \phi \circ \phi^{-1} = X,$$

and

$$\Phi_n \circ \Phi_n^{-1} \Rightarrow \phi \circ \phi^{-1} = I.$$

To complete the proof, we show that $\rho(X_n \circ \Phi_n \circ \Phi_n^{-1}, X_n) \Rightarrow 0$ and apply Theorem 4.1 of [1]. For all n, δ and ε,

$$\{\omega: \rho(X_n \circ \Phi_n \circ \Phi_n^{-1}, X_n) \geqq \varepsilon\} \subseteq$$

$$\{\omega: \sup_{\substack{0 \leqq s,t \leqq 1 \\ |s-t| < \delta}} |X_n \circ \Phi_n \circ \Phi_n^{-1}(t) - X_n \circ \Phi_n \circ \Phi_n^{-1}(s)| \geqq \varepsilon\}$$

$$\bigcup \{\omega: \sup_{0 \leqq t \leqq 1} |\Phi_n \circ \Phi_n^{-1}(t) - t| > \delta\}.$$

Since $I \in C$, $\rho(\Phi_n \circ \Phi_n^{-1}, I) \Rightarrow 0$. By assumption, $\{X_n\}$ is C-tight. Hence, $\rho(X_n \circ \Phi_n \circ \Phi_n^{-1}, X_n) \Rightarrow 0$.

We now return to our queues. Define the random functions Y_n induced in D by the sequence of waiting times of successive customers, $\{W_n\}$, as

$$Y_n \equiv W'_{[nt]}/\gamma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

In the single-channel queue a random time change was used to obtain weak convergence theorems for W_n from known results for Y_n, (cf. [12], page 125). We now go the other way. Define a random time change Φ_n based on the arrival process $A(t)$ as follows.

$$\Phi_n \equiv \Phi_n(t) = \frac{A(nt)}{n\lambda} \wedge 1, \qquad 0 \leqq t \leqq 1.$$

Since $A_n \Rightarrow \xi$, $\Phi_n \Rightarrow I$. Thus, by Theorem 7.1, in order to demonstrate the convergence of $\{Y_n\}$, it suffices to show that $\{Y_n\}$ is C-tight and $Y_n \circ \Phi_n \Rightarrow Y \circ I = Y$ for some Y such that $P\{Y \in C\} = 1$. Now observe that if $A(n) \leqq \lambda n$, then $Y_{\lambda n} \circ \Phi_n = \lambda^{-\frac{1}{2}} Z_n$, where

$$Z_n \equiv W'_{A(nt)}/\gamma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1,$$

and

$$Y_{\lambda n} \equiv W'_{[\lambda n t]}/\gamma(\lambda n)^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1,$$

but since $A(n)/n\lambda \Rightarrow 1$, rather than some number less than 1, we cannot follow [1], page 149 and assert that $Y_{\lambda n} \circ \Phi_n = \lambda^{-\frac{1}{2}} Z_n$ for all sufficiently large n. However, we can show that $\rho(Y_{\lambda n} \circ \Phi_n,\ \lambda^{-\frac{1}{2}} Z_n) \Rightarrow 0$. Finally, we show that $\rho(Z_n, W_n) \Rightarrow 0$ and complete the proof using Theorem 6.2. In this manner, we obtain

Theorem 7.2. If the service channels are identical and $\rho = 1$, then $Y_n \Rightarrow \mu^{-\frac{3}{2}} f(\xi)$.

Proof. By Theorem 6.2, $W_n \Rightarrow \mu^{-1} f(\xi)$. If we can show that $\rho(\lambda^{-\frac{1}{2}} W_n, Y_{\lambda n} \circ \Phi_n) \Rightarrow 0$, then by Theorem 4.1 of [1], $Y_{\lambda n} \circ \Phi_n \Rightarrow (\mu^{-\frac{3}{2}}) f(\xi)$ (recall that $\lambda = \mu$), so that Theorem 7.1 can be applied to give $Y_{\lambda n} \Rightarrow \mu^{-\frac{3}{2}} f(\xi)$, which in turn implies that $Y_n \Rightarrow \mu^{-\frac{3}{2}} f(\xi)$. The necessary C-tightness for $\{Y_{\lambda n}\}$ is obtained with the relation

$$\{W_{Y_{\lambda n}}(3\delta) \geqq \varepsilon\} \subseteq \{\rho(\Phi_n, I) > \delta\} \cup \{W_{Y_{\lambda n} \circ \Phi}(\delta) \geqq \varepsilon\}.$$

Therefore, it suffices to show that $\rho(\lambda^{-\frac{1}{2}} W_n, Y_{\lambda n} \circ \Phi_n) \Rightarrow 0$. We shall show that $\rho(W_n, Z_n) \Rightarrow 0$ and $\rho(\lambda^{-\frac{1}{2}} Z_n, Y_{\lambda n} \circ \Phi_n) \Rightarrow 0$. Define events $A_{n\varepsilon}$, $B_{n\delta\varepsilon}$, and $C_{n\delta}$ as follows.

$$A_{n\varepsilon} = \left\{\omega\colon \sup_{0 \leqq t \leqq 1} \left| \frac{W(nt) - W_{A(nt)}}{\gamma n^{\frac{1}{2}}} \right| \geqq \varepsilon \right\},$$

$$B_{n\delta\varepsilon} = \left\{\omega\colon \sup_{\substack{0 \leqq s, t \leqq 1 \\ |s-t| < \delta}} \left| \frac{W(nt) - W(ns)}{\gamma n^{\frac{1}{2}}} \right| \geqq \varepsilon \right\},$$

and

$$C_{n\delta} = \{\omega\colon \sup_{0 \leqq t \leqq 1} |nt - A(nt)| \geqq n\delta\}.$$

Observe that, for all n, δ, and ε,

$$A_{n\varepsilon} \subset B_{n\delta\varepsilon} \cup C_{n\delta}$$

so that

$$P(A_{n\varepsilon}) \leqq P(B_{n\delta\varepsilon}) + P(C_{n\delta}).$$

For any $\delta > 0$, $P(C_{n\delta}) \to 0$ since

$$\sup_{0 \leqq t \leqq 1} |nt - A(nt)| \leqq \max_{1 \leqq j \leqq r} \{ \max_{i \leqq A^j(n)+1} u_i^j \}$$

which goes to 0 in probability when divided by $n^{\frac{1}{2}}$. On the other hand, $P(B_{n\delta\varepsilon})$ can be made small by employing the C-tightness of $\{W_n\}$.

Finally, we show that $\rho(\lambda^{-\frac{1}{2}}Z_n, Y_{\lambda n} \circ \Phi_n) \Rightarrow 0$. Since

$$\rho(\lambda^{-\frac{1}{2}}Z_n, Y_{\lambda n} \circ \Phi_n) = [1/\gamma(\lambda n)^{\frac{1}{2}}] \sup_{0 \leqq t \leqq 1} | W_{A(nt)} - W_{A(nt) \wedge n\lambda} |,$$

the result follows easily from the C-tightness of $\{Z_n\}$ (in the neighborhood of $t = 1$) and the fact that $A(n)/\lambda n \Rightarrow 1$. This completes the proof of the theorem.

Clearly, other embedded sequences can be handled in the same way. We could work with other continuous-time processes, such as $Q(t)$, or the de-arture points instead of the arrival points.

8. Time of the nth departure

In this section we obtain functional central limit theorems for the time of the nth arrival and the time of the nth departure. We show that weak convergence for a sequence of random functions generated by a counting process (a renewal process or more general counting process) implies weak convergence for the corresponding sequence of random functions induced by the partial sums from the sequence of times between events in the counting process. In particular, we obtain a converse to Theorem 17.3 of [1].

Let $\{u_n\}$ be a sequence of positive random variables (not necessarily independent or identically distributed) and define the counting process

$$N(t) = \begin{cases} \max\{k : u_1 + \cdots + u_k \leqq t\}, & u_1 \leqq t \\ 0, & u_1 > t. \end{cases}$$

We shall assume the existence of positive constants ν and σ (not the same σ used in Section 2) such that $N_n \Rightarrow Y$, where Y is any random function in D and

$$N_n \equiv [N(nt) - \nu^{-1}nt]/(\nu^{-3}\sigma^2 n)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1.$$

Since $N(t)$ has unit jumps, it is easy to show that $Y \in C$. The random functions A_n^i, D_n^i, A_n, and D_n corresponding to the counting processes in each channel or for the entire system are examples of such random functions N_n. In those cases the limit Y is always the Wiener process ξ, or a function of two Wiener processes. This will usually be the case, but we allow for slightly greater generality.

Our object is to establish a f.c.l.t. for X_n, where

$$X_n \equiv S_{[nt]}/\sigma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1,$$

and $S_k = u_1 + \cdots + u_k - kv$, $S_0 = 0$.

The simplest case involves $\{u_n\}$ i.i.d. with finite variance so that $N(t)$ is an ordinary renewal process. We already know $X_n \Rightarrow \xi$ in this case. Our aim is to obtain limit theorems based on more general counting processes. For example, the net arrival stream is the superposition of r renewal processes and the net departure process is even more complicated. Our main result is

Theorem 8.1. If $N_n \Rightarrow Y$ then $X \Rightarrow -Y$.

Proof. Observe that

$$\begin{aligned} N_n &\equiv [N(nt) - v^{-1}nt]/(v^{-3}\sigma^2 n)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1, \\ &\equiv (-v^{\frac{1}{2}}) \left[\frac{nt - vN(nt)}{\sigma n^{\frac{1}{2}}}\right], \qquad 0 \leqq t \leqq 1, \\ &\equiv -v^{\frac{1}{2}} B_n, \end{aligned}$$

where

$$B_n \equiv \left[\frac{nt - vN(nt)}{\sigma n^{\frac{1}{2}}}\right], \qquad 0 \leqq t \leqq 1.$$

Since $N_n \Rightarrow Y$, $B_n \Rightarrow -v^{-\frac{1}{2}}Y$. Define the random function Z_n as

$$Z_n \equiv \sum_{i=1}^{N(nt)} (u_i - v)/\sigma n^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

The proof is completed by showing that $d(B_n, Z_n) \Rightarrow 0$ and applying the inverse time change (Theorem 7.1).

Observe that

$$d(B_n, Z_n) \leqq \rho(B_n, Z_n) \leqq 1/(\sigma n^{\frac{1}{2}}) \sup_{1 \leqq i \leqq N(n)+1} u_i = U_n, \text{ say.}$$

Since $Y \in C$, we have C-tightness for $\{N_n\}$ by Lemma 3.2. It is easy to show that this C-tightness for $\{N_n\}$ would be violated if U_n did not converge to 0 in probability. For each $n \geqq 1$, there are time points t_1 and t_2 in $[0, 1]$ such that $|t_2 - t_1| = \sigma U_n / n^{\frac{1}{2}}$ and $N(nt_2-) - N(nt_1) = 0$. Thus

$$\sup_{\substack{0 \leqq s,t \leqq 1 \\ |s-t| < \delta}} \left| \frac{N(nt) - N(ns) - n(t-s)v^{-1}}{n^{\frac{1}{2}}} \right| \geqq \frac{n^{\frac{1}{2}}\delta}{v} \wedge \frac{\sigma U_u}{v},$$

so that U_n must converge to 0 in probability. Hence $Z_n \Rightarrow -\nu^{-\frac{1}{2}}Y$ by Theorem 4.1 of [1].

Now we apply the inverse random time change. Let $\Phi_n \in D_0$ be the random change induced by $N(t)$ where

$$\Phi_n \equiv \frac{\nu N(nt)}{n} \wedge 1, \qquad 0 \leqq t \leqq 1.$$

Since $N_n \Rightarrow Y$, $\rho(\Phi_n, I) \Rightarrow 0$. Let

$$X_{\frac{n}{\nu}} \equiv S_{[\frac{nt}{\nu}]} / \sigma \left(\frac{n}{\nu}\right)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1.$$

We cannot assert that

$$\nu^{\frac{1}{2}} Z_n = X_{\frac{n}{\nu}} \circ \Phi_n,$$

but it is easy to show that $\rho(\nu^{\frac{1}{2}}Z_n, X_{n/\nu} \circ \Phi_n) \Rightarrow 0$ because

$$\rho(\nu^{\frac{1}{2}}Z_n, X_{n/\nu} \circ \Phi_n) = \nu^{\frac{1}{2}} \sup_{\tau_n \leqq t \leqq 1} \left| Z_n(t) - Z_n(\tau_n) \right|$$

where

$$\tau_n = \begin{cases} \inf\{s \geqq 0:\ N(ns) > [n/\nu], & N(n) > [n/\nu] \\ 1, & N(n) \leqq [n/\nu]. \end{cases}$$

The desired result is obtained from C-tightness of Z_n and the convergence of $\{\tau_n\}$ to 1 because $\rho(\Phi_n, I) \Rightarrow 0$.

Hence,

$$X_{\frac{n}{\nu}} \circ \Phi_n \Rightarrow -Y \equiv -Y \circ I,$$

so that by Theorem 7.1,

$$X_{\frac{n}{\nu}} \Rightarrow -Y \text{ and } X_n \Rightarrow -Y$$

as claimed.

The necessary C-tightness for $\{X_{n/\nu}\}$ is obtained by virtue of the relation

$$\{w_{X_{n/\nu}}(3\delta) \geqq \varepsilon\} \subseteq \{w_{X_{n/\nu} \circ \Phi_n}(\delta) \geqq \varepsilon\} \cup \{\rho(\Phi_n, I) > \varepsilon\}.$$

Corollary 8.1. If $N_n \Rightarrow \xi$, then $X_n \Rightarrow \xi$.

Proof. From the symmetry of the Wiener process, $-\xi \sim \xi$.

We now apply Theorem 8.1 to the standard multiple channel queueing system. Let τ_n^A be the time of the nth arrival (from all incoming channels) and let T_n^A be the corresponding random function in D where

$$T_n^A \equiv [\tau_{[nt]}^A - \lambda^{-1}nt]/(\lambda^3\alpha^2 n)^{\frac{1}{2}}, \qquad 0 \leqq t \leqq 1.$$

Recall that

$$A_n \equiv [A(nt) - \lambda nt]/\alpha n^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1,$$

so that $\nu = 1/\lambda$ and $\sigma^2 = \lambda^3\alpha^2$. Also let $\tau_n^{A^i}$ be the time of the nth arrival in the ith arrival channel and let $T_n^{A^i}$ be the associated random function in $D[0,1]$ where

$$T_n^{A^i} \equiv [\tau_{[nt]}^{A^i} - \lambda_i^{-1}nt]/(\lambda_i^3\alpha_i^2 n)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1.$$

Theorem 8.2. In the standard multiple channel queue, $T_n^A \Rightarrow \xi$ and $T_n^{A^i} \Rightarrow \xi$.

Proof. Since $A_n^i \Rightarrow \xi$ and $A_n \Rightarrow \xi$, the theorem is an immediate consequence of Theorem 8.1.

Now let τ_n^D be the time of the nth departure from the entire system and let T_n^D be the corresponding random function in $D[0,1]$. We have

$$T_n^D \equiv [\tau_{[nt]}^D - \mu^{-1}nt]/(\mu^3\sigma^2 n)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1,$$

since $D_n \equiv [D(nt) - \mu nt]/n^{\frac{1}{2}}$, $0 \leqq t \leqq 1$.

Also let D_n^j be the random function corresponding to the departure process from the jth service channel where

$$D_n^j \equiv [D^j(nt) - \mu_j(nt)]/n^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1.$$

Let $\tau_n^{D^j}$ be the time of the nth departure from the jth service channel and let $T_n^{D^j}$ be the corresponding random function in D where

$$T_n^{D^j} \equiv [\tau_{[nt]}^{D^j} - \mu_j^{-1}nt]/(\mu_j^3\sigma_j^2 n)^{\frac{1}{2}}, \quad 0 \leqq t \leqq 1.$$

Theorem 8.3. If $\rho > 1$, then $D_n^j \Rightarrow \sigma_j\xi$, $T_n^{D^j} \Rightarrow \xi$ and $T_n^D \Rightarrow \xi$, $j = 1, \cdots, s$.

Proof. In Theorem 4.2 we showed that $D_n \Rightarrow \sigma\xi$. Since $\rho > 1$, it is easy to show that $\rho(D_n^j, \sigma_j S_n^j) \Rightarrow 0$ for $j = 1, \cdots, s$. Therefore, $D_n \Rightarrow \sigma_j\xi$. The rest follows from Theorem 8.1.

Now consider the case $\rho = 1$. By Theorem 4.1, we know that $D_n \Rightarrow g(\alpha\xi_1, \sigma\xi_2)$. Hence, Theorem 8.1 gives us

Theorem 8.4. If $\rho = 1$, then $T_n^D \Rightarrow -g[(\alpha/\sigma)\xi_1, \xi_2]$.

We have not obtained the limits for D_n^j and $T_n^{D^j}$ when $\rho = 1$.

9. Related functions

Perhaps the most important feature of the weak convergence theorems is that they lead immediately to limit theorems for other related quantities of interest as a consequence of the continuous mapping theorem. In fact, Theorem 5.1 of [1] provides a limit theorem corresponding to each measurable function f on D which is continuous almost everywhere with respect to the limiting measure: if $X_n \Rightarrow X$ in D, then $f(X_n) \Rightarrow f(X)$. In this section we exhibit a few useful functions and the resulting weak convergence theorems. It is also possible to obtain limit theorems corresponding to a sequence of measurable functions $\{f_n\}$ on D which converges to another function f, so that $f_n(X_n) \Rightarrow f(X)$ if $X_n \Rightarrow X$ in D, but we shall only consider a single function here.

For any k and any set of k time points $0 \leqq t_1 \leqq \cdots \leqq t_k \leqq 1$, the projection $\pi_{t_1,\ldots,t_k}: D \to R^k$, defined for each $x \in D$ by $\pi_{t_1,\ldots,t_k}(x) = [x(t_1),\cdots,x(t_k)]$, is a measurable function on D which is continuous on C. Since the limits of all our weak convergence theorems are in C, this gives us convergence of all finite-dimensional distributions as an immediate consequence of our weak convergence theorems in D. Corollaries 2.1 and 2.2 are special cases of this result.

One is often interested in the maximum number of customers in the system during some time interval or the maximum waiting time of the first n customers. When $\rho = 1$, limit theorems for such quantities are easy to obtain using the weak convergence.

Theorem 9.1. If $\rho = 1$, then

$$(\mu^{\frac{1}{2}}/\gamma n^{\frac{1}{2}}) \max_{1 \leqq k \leqq n} \{W_k'\} \Rightarrow \sup_{0 \leqq t \leqq 1} |\xi(t)|,$$

and

$$(1/\gamma n^{\frac{1}{2}}) \sup_{0 \leqq t \leqq n} \{Q(t)\} \Rightarrow \sup_{0 \leqq t \leqq 1} |\xi(t)|,$$

where

$$P\{\sup_{0 \leqq t \leqq 1} |\xi(t)| \leqq x\} = 1 - (4/\pi) \sum_{k=1}^{\infty} [(-1)^k/(2k+1)]\exp\{-[\pi^2(2k+1)^2/8x^2]\}.$$

Proof. Use the continuous functional $h: D \to R$, defined for any $x \in D$ by $h(x) = \sup_{0 \leqq t \leqq 1} x(t)$. Since $f(\xi) \sim |\xi|$, $h[f(\xi)] \sim h[|\xi|]$. The distribution of the limit is displayed in [1], page 79.

Many queueing quantities of interest, such as the time until the system first becomes empty or the number of customers served before the system first becomes empty, can be expressed in terms of first passage time functions. Such functions are measurable on D and continuous almost everywhere with respect to Wiener measure. Hence, we can obtain limit theorems for these quantities. Let D_a be the (closed) subset of D consisting of those $x \in D$ with $x(0) = a$. Let $T_0: D_a \to R$ be the first passage time to 0; that is, for any $x \in D_a$ let $T_0(x) = \inf\{\tau \geqq 0 | x(\tau) \leqq 0\}$ where we assume the infimum of an empty set is one. We can use this function to obtain

Theorem 9.2. If $\rho = 1$, then for any $a > 0$ and any t, $0 \leqq t \leqq 1$,

$$\lim_{n\to\infty} P\{(1/n)\inf[\tau \geqq 0: Q(\tau) = 0] \leqq t \mid Q(0) = a\gamma n^{\frac{1}{2}}\}$$

$$= \lim_{n\to\infty} P\{(1/n)\inf[k \geqq 0: W_k' = 0] \leqq t \mid W(0) = a\gamma\mu^{-\frac{1}{2}}n^{\frac{1}{2}}\}$$

$$= (2/\pi t)^{\frac{1}{2}} \int_a^\infty \exp\{-y^2/2t\}dt.$$

Proof. The theorem says that $T_0(Q_n + a) \Rightarrow T_0(\xi + a)$ and $T_0(\mu^{\frac{1}{2}}Y_n + a) \Rightarrow T_0(\xi + a)$. Observe that $Q_n + a$, $\mu^{\frac{1}{2}}Y_n + a$, and $\xi + a$ are all random functions in D_a. Since $Q_n \Rightarrow f(\xi)$ and $Y_n \Rightarrow \mu^{-\frac{1}{2}}f(\xi)$, $Q_n + a \Rightarrow f(\xi) + a$ and $\mu^{\frac{1}{2}}Y_n + a \Rightarrow f(\xi) + a$. Finally, $T_0(f(\xi) + a)) \sim T_0(\xi + a)$ so that the proof is finished using Theorem 5.1 of [1]. (For the passage time distribution of the Wiener process, see [5], page 25. For further discussion, see [12], page 157.)

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MATHEMATICS IN THE ARCHAEOLOGICAL AND HISTORICAL SCIENCES

The Royal Society and the Academy of the Socialist Republic of Romania have agreed in principle to arrange an Anglo-Romanian Conference with international participation on *Mathematics in the Archaeological and Historical Sciences*, to be held in Bucharest in September 1970 (Opening Session: 16 September). Excursions to archaeological sites near Bucharest and on or near the Black Sea Littoral will be organised.

The principal themes of the conference will be as follows:

(a) Typology and Taxonomy;

(b) Chronology and Seriation;

(c) the mathematical problems common to Population Genetics, Historical Demography, and the Linkage of Manuscripts, etc.

A First Communication giving preliminary details will be sent to those interested, who are now invited to contact

either

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as soon as possible.