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Necessary conditions in limit theorems for cumulative processes

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Abstract

We show that sufficient conditions in terms of moments for cumulative processes (additive functionals of regenerative processes) to satisfy the central limit theorem and the weak law of large numbers established in Glynn and Whitt (Stochastic Process. Appl. 47 (1993) 299–314) are also necessary, as previously conjectured. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $X \equiv (X_n\colon n \geqslant 0)$ be an irreducible positive-recurrent discrete-time Markov chain (DTMC) taking values in a finite or countably infinite state space S. Given a real-valued function $f: S \to \mathbb{R}$, our primary concern in this paper is on the identification of necessary and sufficient (N&S) moment-type conditions under which there exist (deterministic finite) constants γ and σ such that

$$n^{-1/2}\left(\sum_{j=0}^{n-1} f(X_j) - n\gamma\right) \Rightarrow \sigma N(0,1) \tag{1.1}$$

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as $n \to \infty$, where $N(0,1)$ denotes a standard (mean 0, variance 1) normal random variable and $\Rightarrow$ denotes convergence in distribution. The above central limit theorem (CLT) arises in many different application settings, providing approximations for: cumulative-reward distributions, confidence intervals for statistical estimators, and efficiency criteria for simulation algorithms. Our main result shows that the classic sufficient condition for the DTMC CLT given in Chung (1967, Section 16), originally due to Doeblin (1937, 1938)—see Lindvall (1991), is in fact necessary.

Following Glynn and Whitt (1993), we study this problem in the more general context of a positive-recurrent (classically) regenerative stochastic process. In addition to the above class of DTMC's, this family of stochastic processes includes continuous-time Markov chains (CTMC's), one-dimensional recurrent diffusions and certain Harris-recurrent Markov processes on a general state space. Thus, the results we develop here automatically cover these other classes of processes, as well as DTMC's.

Our results in this paper solve open problems in Glynn and Whitt (1993). In our previous paper, we found N&S conditions for both ordinary and functional strong laws of large numbers (SLLNs) and laws of the iterated logarithms (LILs) for additive functionals of regenerative processes (also known as cumulative processes). We also found N&S conditions for a functional weak law of large numbers (FWLLN) and a functional central limit theorem (FCLT), but we only obtained sufficient conditions for the ordinary weak law of large numbers (WLLN) and the ordinary CLT. We conjectured that the established sufficient conditions for the WLLN and CLT were actually N&S. In this paper, we prove that those earlier conjectures are indeed correct.

The key to obtaining such N&S conditions is to use the regenerative structure to reduce the problem to one involving random sums of independent and identically distributed (iid) random variables. For the sufficiency with random sums, we can apply the CLT for random sums of iid random variables, e.g., see Gut (1988, p. 15): Suppose that $(Z_n\colon n \geqslant 1)$ is a sequence of iid real-valued random variables and $N \equiv (N(t)\colon t \geqslant 0)$ is a stochastic process satisfying

$$t^{-1}N(t) \Rightarrow \lambda \quad \text{as } t \to \infty, \quad 0 < \lambda < \infty, \tag{1.2}$$

where λ is a constant. If, in addition, $EZ_n = 0$ and $EZ_n^2 = \beta^2 < \infty$, then

$$t^{-1/2} \sum_{i=1}^{N(t)} Z_i \Rightarrow \lambda^{1/2}\beta N(0,1) \quad \text{as } t \to \infty. \tag{1.3}$$

Our key step is to show the necessity of these moment conditions. We do so when N is a renewal process. In our context, with N being a renewal process, we show that the limit in (1.3) holds if and only if

$$t^{-1/2} \sum_{i=1}^{\lfloor \lambda t \rfloor} Z_i \Rightarrow \lambda^{1/2}\beta N(0,1) \quad \text{as } t \to \infty, \tag{1.4}$$

which in turn is known to hold if and only if $EZ_n = 0$ and $EZ_n^2 = \beta^2 < \infty$; see p. 181 of Gnedenko and Kolmogorov (1968). It remains an open problem to determine whether limits (1.2) and (1.3) imply (1.4) more generally, when N is *not* a renewal process. We will exploit the renewal-process structure (or, equivalently, the regenerative structure) in our proof.

2. The results

Let $X \equiv (X(t)\colon t \geqslant 0)$ be an S-valued stochastic process. (Note that a discrete-time sequence $(X_n\colon n \geqslant 0)$ can be embedded in continuous time by setting $X(t) = X_{\lfloor t \rfloor}$ for $t \geqslant 0$.) Given a nonnegative nondecreasing sequence $(T(n)\colon n \geqslant 0)$ with $T(n) \to \infty$ a.s., set $T(-1) = 0$, $\tau_i \equiv T(i) - T(i-1)$, and

$$W_i(t) \equiv \begin{cases} X(T(i-1)+t), & 0 \leqslant t < \tau_i, \\ \Delta, & t \geqslant \tau_i \end{cases} \tag{2.1}$$

for $t \geqslant 0$, where $\Delta \notin S$. We require that X be classically regenerative with respect to $(T(n)\colon n \geqslant 0)$. In other words, we demand that $((W_i(t)\colon t \geqslant 0)\colon i \geqslant 1)$ be a sequence of iid random elements independent of $(W_0(t)\colon t \geqslant 0)$. We assume throughout this paper that X be positive recurrent in the sense that

$$E\tau_i \equiv \lambda^{-1} < \infty. \tag{2.2}$$

Let $f : S \to \mathbb{R}$ be a measurable function for which

$$\int_0^t |f(X(s))| \, \mathrm{d}s < \infty \quad \text{a.s. for } t \geqslant 0, \tag{2.3}$$

so that the corresponding cumulative process

$$C(t) \equiv \int_0^t f(X(s)) \, \mathrm{d}s, \quad t \geqslant 0, \tag{2.4}$$

is well-defined for $t \geqslant 0$. In this context, our primary goal is to develop a N&S moment condition equivalent to the following:

A. There exist constants γ and σ such that

$$t^{-1/2}(C(t) - \gamma t) \Rightarrow \sigma N(0,1) \quad \text{as } t \to \infty. \tag{2.5}$$

The key is to reduce the analysis of $C(t)$ to that of a corresponding random sum of iid random variables. That is accomplished through the representation

$$C(t) - \gamma t = \sum_{i=0}^{N(t)} Z_i(\gamma) + R(t,\gamma), \tag{2.6}$$

where

$$N(t) \equiv \max\{n \geqslant -1\colon T(n) \leqslant t\},$$

$$Z_i(\gamma) \equiv \int_{T(i-1)}^{T(i)} [f(X(s)) - \gamma] \, \mathrm{d}s,$$

$$R(t,\gamma) \equiv \int_{T(N(t))}^{t} [f(X(s)) - \gamma] \, \mathrm{d}s. \tag{2.7}$$

The first step is to control the "remainder term" $R(t,\gamma)$. The following is a consequence of Proposition 9 of Glynn and Whitt (1993).

Proposition 1. *For any constant γ, the family of random variables $\{R(t,\gamma)\colon t \geqslant 0\}$ is tight.*

Note that Proposition 1 is valid universally, and requires no moment conditions whatsoever (other than the blanket hypothesis that $E\tau_1 < \infty$). In view of Proposition 1, condition A is clearly equivalent to condition B below:

B. There exist constants γ and σ such that

$$t^{-1/2}\sum_{i=1}^{N(t)} Z_i(\gamma) \Rightarrow \sigma N(0,1) \quad \text{as } t \to \infty. \tag{2.8}$$

Because of the classical regenerative structure of X, the sequence $(Z_i(\gamma)\colon i \geqslant 1)$ is iid. Thus, B reduces the CLT problem to one involving a random sum of iid random variables, as indicated in Section 1.

Recall that the strong law for renewal (counting) processes establishes that $t^{-1}N(t) \to \lambda$ a.s. as $t \to \infty$, where $\lambda = 1/E\tau_1$. This suggests the approximation

$$t^{-1/2}\sum_{i=1}^{N(t)} Z_i(\gamma) \approx t^{-1/2}\sum_{i=1}^{\lfloor \lambda t \rfloor} Z_i(\gamma) \tag{2.9}$$

and leads to the following condition.

C. There exist constants γ and σ such that

$$t^{-1/2}\sum_{i=1}^{\lfloor \lambda t \rfloor} Z_i(\gamma) \Rightarrow \sigma N(0,1) \quad \text{as } t \to \infty. \tag{2.10}$$

The classical theory for sums of iid random variables (involving a deterministic number of summands) shows that C is equivalent to condition D below; see p. 181 of Gnedenko and Kolmogorov (1968).

D. There exist constants γ and σ such that $EZ_1(\gamma) = 0$ and $\operatorname{var} Z_1(\gamma) = \lambda^{-1}\sigma^2$.

Condition D is, of course, the desired moment condition that we have been seeking. Consequently, we have a N&S moment condition for the CLT in A, provided that we can rigorously justify approximation (2.9). (We remark that this would be easy to do if we had initially assumed a FCLT.) The justification of this "random time change" is the key result underlying the validity of the following theorem.

Theorem 2.1. *Conditions* A, B, C, *and* D *are all equivalent.*

It turns out that Theorem 2.1 generalizes easily to nonstandard CLTs with different scalings and stable-law limits. To do the space scaling in the nonstandard CLT, we use slowly varying and regularly varying functions; see Bingham et al. (1989). A positive, Lebesgue measurable real-valued function L (on some interval (b,∞) for $b > 0$) is *slowly varying* (at infinity) if

$$\frac{L(ct)}{L(t)} \to 1 \quad \text{as } t \to \infty \quad \text{for all } c > 0. \tag{2.11}$$

A positive, Lebesgue measurable, real-valued function ψ (on some interval (b, ∞) for $b > 0$) is *regularly varying* of index α (at infinity), and we write $\psi \in \mathscr{R}(\alpha)$, if

$$\frac{\psi(ct)}{\psi(t)} \to c^{\alpha} \quad \text{as } t \to \infty \quad \text{for all } c > 0. \tag{2.12}$$

A regularly varying function of index 0 is slowly varying. Any regularly varying function ψ of index α can be represented as $\psi(x) = x^{\alpha}L(x)$ for some slowly varying function L.

Theorem 2.2. *The following four conditions are equivalent*:

$\tilde{\mathrm{A}}$. *There exists a constant γ, a proper random variable A, and a regularly varying function ψ of index $v < 0$ for which*

$$\psi(t)(C(t) - \gamma t) \Rightarrow A \quad \text{as } t \to \infty. \tag{2.13}$$

$\tilde{\mathrm{B}}$. *There exists a constant γ, a proper random variable A, and a regularly varying function ψ of index $v < 0$ for which*

$$\psi(t) \sum_{i=1}^{N(t)} Z_i(\gamma) \Rightarrow A \quad \text{as } t \to \infty. \tag{2.14}$$

$\tilde{\mathrm{C}}$. *There exists a constant γ, a proper random variable A, and a regularly varying function ψ of index $v < 0$ for which*

$$\psi(t) \sum_{i=1}^{\lfloor \lambda t \rfloor} Z_i(\gamma) \Rightarrow A \quad \text{as } t \to \infty. \tag{2.15}$$

$\tilde{\mathrm{D}}$. *There exists a constant γ, a proper random variable A and a regularly varying function ψ of index $v < 0$ for which*

$$\psi(t) \sum_{i=1}^{\lfloor t \rfloor} Z_i(\gamma) \Rightarrow \lambda^{v} A. \tag{2.16}$$

The equivalence of $\tilde{\mathrm{C}}$ and $\tilde{\mathrm{D}}$ in Theorem 2.2 follows immediately from (2.12). When A is a stable law, condition $\tilde{\mathrm{D}}$ can be reformulated in terms of a N&S condition involving the tails of the r.v. $Z_1(\gamma)$. To describe this classic condition, let $S_{\alpha}(\sigma, \beta, \mu)$ be a stable random variable with index α, scale parameter σ, skewness parameter β, and shift parameter μ; see Samorodnitsky and Taqqu (1994). Let $F_{\gamma}(x) \equiv P(Z_1(\gamma) \leqslant x)$, $G_{\gamma}(x) \equiv P(|Z_1(\gamma)| \leqslant x)$, $F_{\gamma}^{c}(x) = 1 - F_{\gamma}(x)$ and $G_{\gamma}^{c}(x) \equiv 1 - G_{\gamma}(x)$. The following result is classical; see Gnedenko and Kolmogorov (1968), Feller (1971) and p. 50 of Samorodnitsky and Taqqu (1994).

Proposition 2. *The following are equivalent*:

(i) *Condition $\tilde{\mathrm{D}}$ holds for $v = -1/\alpha, 0 < \alpha < 2$ and $\alpha \neq 1$, with $\lambda^{v}A$ having the distribution of the stable law $S_{\alpha}(1, \beta, 0)$*;

(ii) *There exists a constant* γ *such that both* $G_\gamma^c \in \mathscr{R}(-\alpha)$ *for* $0 < \alpha < 2$ *with* $\alpha \neq 1$ *and*

$$\lim_{x\to\infty} F_\gamma^c(x)/G_\gamma^c(x) = \frac{1+\beta}{2}. \tag{2.17}$$

If the conditions of Proposition 2(ii) hold, then we can identify the centering constant γ: For $0 < \alpha < 1$, we can let $\gamma=0$; for $1 < \alpha < 2$, γ must be chosen so that $EZ_1(\gamma)=0$.

We may want to insist that the multiplicative scaling is done by a simple power. Then we want to exploit classical results for a distribution to be in the normal domain of attraction of a stable law.

Proposition 3. *The following are equivalent*:

(i) *Condition* $\tilde{\mathrm{D}}$ *holds with* $\psi(t)=ct^{-1/\alpha}$ *for some constants c and* α *with* $0 < \alpha < 2$, $\alpha \neq 1$, *and* $\lambda^{-1/\alpha}A$ *having the distribution of the stable law* $S_\alpha(1,\beta,0)$;

(ii) *There exist constants* γ *and* C *such that*

$$\lim_{x\to\infty} x^\alpha G_\gamma^c(x) = C \tag{2.18}$$

and

$$\lim_{x\to\infty} F_\gamma^c(x)/G_\gamma^c(x) = (1+\beta)/2. \tag{2.19}$$

In the setting of Proposition 3, the centering constant again can be $\gamma=0$ for $0 < \alpha < 1$ and such that $EZ_1(\gamma) = 0$ for $1 < \alpha < 2$. The constant c in the multiplicative scaling term $\psi(t) = ct^{-1/\alpha}$ should be

$$c = (C/C_\alpha)^{-1/\alpha}, \tag{2.20}$$

where C is from (2.18) and

$$C_\alpha = (1-\alpha)/\Gamma(2-\alpha)\cos(\pi\alpha/2) \tag{2.21}$$

where Γ is the gamma function.

We can apply Theorem 2.2 to obtain a criterion for a cumulative process to obey a WLLN.

Corollary 1. *A WLLN holds for the cumulative process, i.e., condition* $\tilde{\mathrm{A}}$ *holds with* $A = 0$ *and* $\psi(t) = t^{-1}$, *if and only if*

$$tP(|Y_1| > t) \to 0 \quad \text{as } t \to \infty \tag{2.22}$$

and

$$E(Y_1; |Y_1| \leqslant t) \to \gamma \quad \text{as } t \to \infty, \tag{2.23}$$

where

$$Y_i \equiv \int_{T(i-1)}^{T(i)} f(X(s))\,\mathrm{d}s. \tag{2.24}$$

Proof. See p. 235 of Feller (1971). □

3. Proof

We now prove Theorem 2.2. We will actually establish the following stronger result.

Theorem 3.1. *If any one of conditions* $\tilde{\mathrm{A}}$, $\tilde{\mathrm{B}}$, $\tilde{\mathrm{C}}$ *or* $\tilde{\mathrm{D}}$ *hold, then*

$$\psi(t)\left(C(t),\sum_{i=1}^{N(t)} Z_i(\gamma),\sum_{i=1}^{\lfloor \lambda t\rfloor} Z_i(\gamma)\right) \Rightarrow (A,A,A) \quad \text{in } \mathbb{R}^3 \tag{3.1}$$

and all four of $\tilde{\mathrm{A}}$, $\tilde{\mathrm{B}}$, $\tilde{\mathrm{C}}$, $\tilde{\mathrm{D}}$ *hold.*

First, Proposition 1 implies that $\tilde{A}$ and $\tilde{B}$ are equivalent and that either one implies that

$$\psi(t)\left(C(t),\sum_{i=1}^{N(t)} Z_i(\gamma)\right) \Rightarrow (A,A). \tag{3.2}$$

As noted above, (2.12) implies that $\tilde{\mathrm{C}}$ and $\tilde{\mathrm{D}}$ are equivalent. Next, the CLT limit in $\tilde{\mathrm{C}}$ directly implies the FCLT generalization, as was shown by Skorokhod (1957). The standard random-time-change argument then shows that $\tilde{\mathrm{C}}$ implies $\tilde{\mathrm{B}}$ and (3.1) holds; e.g., see Sections 7.4 and 13.3 of Whitt (2002). It thus suffices to show that $\tilde{\mathrm{A}}$ implies $\tilde{\mathrm{C}}$. Consequently, in order to establish Theorem 3.1, it suffices to show that

$$\psi(t)\left(C(t)-\sum_{i=1}^{\lfloor \lambda t\rfloor} Z_i(\gamma)\right) \Rightarrow 0, \tag{3.3}$$

assuming that $\tilde{\mathrm{A}}$ holds. Instead of (3.3) we will find a deterministic function $(a_t\colon t \geqslant 0)$ such that $a_t \downarrow 0$ and

$$\psi(t)\left(C(t)-\sum_{i=1}^{\lfloor \lambda t(1-a_t)\rfloor} Z_i(\gamma)\right) \Rightarrow 0. \tag{3.4}$$

However, given $\tilde{\mathrm{A}}$, (3.4) implies (3.3). To see that, note that from $\tilde{\mathrm{A}}$ and (3.4) we obtain

$$\psi(t)\sum_{i=1}^{\lfloor \lambda t(1-a_t)\rfloor} Z_i(\gamma) \Rightarrow A \tag{3.5}$$

and

$$\psi(t)\sum_{i=1}^{\lfloor \lambda t\rfloor} Z_i(\gamma) \Rightarrow A. \tag{3.6}$$

However, given (3.6),

$$\psi(t)\left(\sum_{i=1}^{\lfloor\lambda t\rfloor} Z_i(\gamma) - \sum_{i=1}^{\lfloor\lambda t(1-a_t)\rfloor} Z_i(\gamma)\right) = \psi(t) \sum_{i=\lfloor\lambda t(1-a_t)\rfloor+1}^{i=\lfloor\lambda t\rfloor} Z_i(\gamma)$$

$$\stackrel{\mathrm{d}}{=} \psi(t) \sum_{i=1}^{\lfloor\lambda t a_t\rfloor} Z_i(\gamma) \Rightarrow 0, \tag{3.7}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. By the triangle inequality, (3.3) follows from (3.4) and (3.7). Hence we work to establish (3.4).

We establish (3.4) with the aid of three lemmas, the first of which is elementary.

Lemma 3.2. *For any deterministic function c_t such that $c_t \downarrow 0$,*

$$P\left(\left|\frac{T(\lfloor\lambda t(1-2\lambda c_t)\rfloor)}{\lfloor\lambda t(1-2\lambda c_t)\rfloor} - \lambda^{-1}\right| > c_t\right) \to 0 \quad as\ t \to \infty. \tag{3.8}$$

Proof. The regenerative structure and the moment condition (2.2) imply the SLLN $n^{-1}T(n) \to \lambda^{-1}$ w.p.1, which implies that

$$\sup_{2^{-1}\leqslant s\leqslant 1}\left|\frac{T(\lfloor\lambda ts\rfloor)}{\lfloor\lambda ts\rfloor} - \lambda^{-1}\right| \to 0 \quad \text{w.p.1} \quad \text{as } t \to \infty. \tag{3.9}$$

That in turn implies (3.8). □

Let $f_c(x) = f(x) - \gamma$.

Lemma 3.3. *Let c_t be a deterministic function such that $c_t \downarrow 0$ and let $d_t \equiv \max\{c_t, t^{-1}\}$. Then, for any $\varepsilon > 0$,*

$$P\left(\psi(t)\left|\int_0^t f_c(X(s))\,\mathrm{d}s - \sum_{i=1}^{\lfloor\lambda t(1-2\lambda c_t)\rfloor} Z_i\right| > \varepsilon\right)$$

$$= \int_{B_t} P\left(\psi(t)\left|\int_0^{t-u} f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\right) P(T(\lfloor\lambda t(1-2\lambda c_t)\rfloor) \in \mathrm{d}u) + \mathrm{o}(1)$$

$$\leqslant \sup_{u\in B_t}\left\{P\left(\psi(t)\left|\int_0^{t-u} f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\right)\right\} + \mathrm{o}(1), \tag{3.10}$$

where B_t is the interval

$$B_t \equiv [t(1-3\lambda c_t + \mathrm{o}(d_t)), t(1-\lambda c_t + \mathrm{o}(d_t))]. \tag{3.11}$$

Proof. Let

$$C_t \equiv \left\{ \psi(t) \left| \int_0^t f_c(X(s))\,\mathrm{d}s - \sum_{i=1}^{\lfloor \lambda t(1-2\lambda c_t)\rfloor} Z_i \right| > \varepsilon \right\} \tag{3.12}$$

and

$$D_t \equiv \left\{ \left| \frac{T(\lfloor \lambda t(1-2\lambda c_t)\rfloor)}{\lfloor \lambda t(1-2\lambda c_t)\rfloor} - \lambda^{-1} \right| > c_t \right\}. \tag{3.13}$$

Then

$$P(C_t) = P(C_t D_t^c) + P(C_t D_t), \tag{3.14}$$

where

$$P(C_t D_t) \leqslant P(D_t) = \mathrm{o}(1) \quad \text{as } t \to \infty \tag{3.15}$$

by Lemma 3.2. Also,

$$\begin{aligned} P(C_t D_t^c) &= P(C_t; \lfloor \lambda t(1-2\lambda c_t)\rfloor(\lambda^{-1} - c_t) \\ &\leqslant T(\lfloor \lambda t(1-2\lambda c_t)\rfloor) \leqslant \lfloor \lambda t(1-2\lambda c_t)\rfloor(\lambda^{-1} + c_t)) \\ &= P(C_t; t - 3\lambda c_t t + \mathrm{o}(td_t) \leqslant T(\lfloor \lambda t(1-2\lambda c_t)\rfloor) \\ &\leqslant t - \lambda c_t t + \mathrm{o}(td_t)) \end{aligned} \tag{3.16}$$

for t large enough. Hence,

$$P(C_t D_t^c) = \int_{B_t} P\left(\psi(t) \left| \int_0^t f_c(X(s))\,\mathrm{d}s - \int_0^u f_c(X(s))\,\mathrm{d}s \right| > \varepsilon;\ T(\lfloor \lambda t(1-2\lambda c_t)\rfloor \in \mathrm{d}u \right) \tag{3.17}$$

for B_t in (3.11), which equals the first expression on the right-hand side in (3.10) because of the regenerative structure ($T(\lfloor t \rfloor)$ is independent of $\int_{T(\lfloor t\rfloor)}^{\cdots} f_c(X(s))\,\mathrm{d}s$). The subsequent inequality in (3.10) is elementary. □

Lemma 3.4. *If* Ã *holds, then the family*

$$\left\{ \psi(1+t) \int_0^t f_c(X(s))\,\mathrm{d}s\colon t \geqslant 0 \right\} \tag{3.18}$$

is tight.

Proof. We can apply Prohorov's theorem in Section 5 of Billingsley (1999). To establish relative compactness, note that for any sequence $\{t_n\colon n \geqslant 1\}$ of nonnegative numbers, there either exists a subsequence $\{t_{n_k}\colon k \geqslant 1\}$ such that either $t_{n_k} \to \infty$ or there exists a subsequence such that $t_{n_k} \to t < \infty$. First, by the assumed limit (2.13),

$$\psi(1+t_n) \int_0^{t_n} f_c(X(s))\,\mathrm{d}s \Rightarrow A \tag{3.19}$$

for any sequence $\{t_n: n \geqslant 1\}$ with $t_n \to \infty$ as $n \to \infty$. On the other hand, for any sequence $\{t_n: n \geqslant 1\}$ with $t_n \to t < \infty$ as $n \to \infty$. For all n sufficiently large,

$$\psi(1+t_n)\left|\int_0^{t_n} f_c(X(s))\,\mathrm{d}s\right| \leqslant \sup_{0\leqslant u\leqslant 1+t}\left\{\psi(1+u)\left|\int_0^u f_c(X(s)\,\mathrm{d}s\right|\right\} < \infty \quad \text{w.p.1.} \tag{3.20}$$

Since each probability measure on $\mathbb{R}$ is tight, so is the sequence on the left-hand side of (3.20) indexed by t_n with $t_n \to t < \infty$. By Prohorov's theorem, there is a subsequence of the sequence $\{t_n\}$ with $t_n \to t$ where there is convergence to a proper limit. Thus, for any sequence $\{t_n\}$, there is a subsequence where there is convergence to a proper limit, so that the family in (3.18) is relatively compact. By Prohorov's theorem, the family is tight. □

We now return to the proof of (3.4) under the assumption of $\tilde{\mathrm{A}}$. First note that $\psi(t)/\psi(1+t) \to 1$ as $t \to \infty$ by the regular variation of ψ. By Lemma 3.3, it suffices to show that, for any $\varepsilon > 0$,

$$\sup_{u\in B_t}\left\{P\left(\left|\psi(t)\int_0^{t-u} f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\right)\right\} \to 0 \quad \text{as } t \to \infty. \tag{3.21}$$

However,

$$\begin{aligned}
&\sup_{u\in B_t}\left\{P\left(\left|\psi(t)\int_0^{t-u} f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\right)\right\} \\
&\quad\leqslant \sup_{u\leqslant 3\lambda t(c_t+\mathrm{o}(d_t))}\left\{P\left(\psi(1+u)\left|\int_0^u f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\frac{\psi(1+u)}{\psi(t)}\right)\right\} \\
&\quad\leqslant \sup_{u\geqslant 0}\left\{P\left(\psi(1+u)\left|\int_0^u f_c(X(s))\,\mathrm{d}s\right| > \varepsilon\frac{\psi(1+3\lambda t(c_t+\mathrm{o}(d_t)))}{\psi(t)}\right)\right\}.
\end{aligned} \tag{3.22}$$

However, since $c_t \downarrow 0$, we can apply the regular variation property to deduce that

$$\frac{\psi(1+3\lambda t(c_t+\mathrm{o}(d_t)))}{\psi(t)} \to \infty \quad \text{as } t \to \infty. \tag{3.23}$$

Hence, the tightness established in Lemma 3.4 and (3.23) implies that the last term in (3.22) must converge to 0 as $t \to \infty$, implying (3.21).

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