Optimal Battery Purchasing and Charging at an Electric Vehicle Battery Swap Station

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A battery swap station (BSS) is a facility where electric vehicle owners can quickly exchange their depleted battery (DB) for a fully-charged one. In order for battery swap to be economically sound, the BSS operator must make a long-term decision on the number of charging bays in the facility, a medium-term decision on the number of batteries in the system, and short-term decisions on when and how many batteries to recharge. In this paper, we formulate a two-stage decision model to find an optimal battery purchasing and charging policy that best trades off battery investment cost and operating cost (including charging cost and cost of customer waiting). In the first stage, an optimal number of batteries is identified. Then the second stage involves solving a Markov decision process (MDP) that models charging operations at a BSS facing non-stationary stochastic demand for battery swap and non-stationary prices for charging DBs. Because solving the MDP can be computationally challenging for large systems, we propose a fluid-based optimization problem which is more amenable to analysis. We characterize the fluid-based optimal battery purchasing and charging policy. We derive an explicit upper bound for the optimal amount of battery fluid. We present numerical studies to illustrate that (i) when the demand and the price function are unsynchronized, the BSS can achieve a lower operating cost with a smaller number of batteries; (ii) each additional battery can help reduce the operating cost, but the marginal gain is decreasing.

Key words: battery swap, electric vehicles, Markov decision process, queueing models, fluid models, inventory control.
1. Introduction

Today more and more people are turning to electric vehicles (EVs) as a more environmentally friendly and economically viable driving option. Unlike conventional internal combustion engine vehicles, EVs do not have tailpipe emissions, thereby reducing negative impacts to the environment. In addition, EVs often can be refueled using energy produced from domestic sources, including coal, nuclear, natural gas, and renewable energy. EV sales are climbing, yet there are still concerns centering around the range and charging time of the batteries. Charging times are decreasing, due to the emergence of specialized fast-charging facilities such as Tesla’s supercharger stations, but not as quickly as consumers would like. In addition, there can be long lines at those stations, especially during holiday seasons. Those charging sites are likely to become even more congested as the adoption of EVs continues to increase. In the meantime, options from the past are showing up, such as battery swap. This energy supply mode has been applied in many pilot cities of China. BAIC BJEV, the leading new energy electro-mobile producer, has built 106 new battery swap stations and put additional 5,000 battery-swapping vehicles into operation in 2017, and plans to build over 3,000 swapping stations in 100 cities nationwide by 2020.

Battery swap has several advantages over plug-in charging. First, rapid charging can decrease the life expectancy of the battery by adding stress. Swapping would allow to charge batteries over time to reduce these stresses and hence prolong their life expectancy. Second, a BSS can use alternative energy on-site to recharge batteries; see Larcher and Tarascon (2015). Solutions like this work well for developing nations without the infrastructure to generate consistent power. In addition, a BSS can use grid electricity when it is off-peak, cheapest, or when the more environmental energy generation is available. Third, battery swap provides a more rapid way of refueling the EV and can enable EVs to travel essentially nonstop on long road trips.

Where BSSs may especially thrive is in companies with fleet vehicles. For example, Both China and South Korea use electric buses and swap the batteries to keep cities
moving; see Zheng et al. (2014) and Kim et al. (2015). Another example is Tesla Semi, a battery-powered semi-trailer truck. It is widely speculated that a commercial application of these electric trucks may rely on battery swap since the charging times for these trucks are much longer and companies that buy the Semi can choose to lease the batteries separately. According to third-party analysis, recharging a semi to around 80 percent (to cover a further 320 km, or 200 miles) takes about 90 minutes. Since companies make money by keeping the vehicles on the road, reducing a car or bus’s downtime with a battery swap station may help boost productivity and profits.

In this paper, we consider a stochastic model for the BSS as illustrated in Figure 1. Exogenous demand for battery swap comes from vehicles arriving stochastically to the BSS. That demand is fulfilled by exchanging a depleted battery (DB) for a fully-charged battery (FB), but the EV must wait if the FB is not available because an EV arriving at a BSS with a DB may not have sufficient energy to reach another charging facility. Using inventory terminology, we say that the demand is then backlogged. (We assume that the EV waits until the next FB becomes available.) The BSS mainly does two jobs: (i) It provides battery swap service for EVs, which is an uncontrollable operation. (ii) It recharges DBs so as to produce FBs for future use; in doing so it can dynamically control the number of DBs to be charged at the same time (or the FB production rate). Two types of capacity resources constrain the BSS’s capability of producing FBs. The number of charging bays restricts the number of DBs that can be charged simultaneously, whereas the number of batteries in the system limits the utilization of the charging bays. These two resources together determine the effective charging capacity of the BSS. Here we regard the decision on the number of charging bays as part of long-term planning. In the present study we do not consider long-term strategies; hence we take the capacity as given in our model.

In order for a BSS to run efficiently, the system operator must resolve several issues. First, the system manager must decide when to perform the charging since both demand for battery swap and electricity price can be time-varying. It clearly would be beneficial
for the BSS to recharge DBs at full capacity when the electricity price is low and to use the FBs in stock to satisfy the demand when the electricity price becomes high. On the other hand, high demand for battery swap produces a large number of DBs that can be used for recharging, and thus recharging DBs at those times can increase the utilization of the batteries in system. Then a dilemma arises when high-price intervals coincide with high-demand periods, in which case building up a large amount of FB inventory would be an inevitable outcome if the BSS wants to avoid service degradation and meanwhile keep its charging cost down. Second, the operator needs to determine an appropriate number of batteries in circulation. With too few batteries, the BSS is faced with either large charging cost or large cost of waiting. This is especially true when electricity prices are aligned with demand rates, e.g., high at the same times, thereby making it impossible for the BSS to build enough FB inventory before demand surges. On the other hand, having too many batteries in the system is costly. These complexities suggest that the optimization can help manage the BSS.

In this paper, we formulate a two-stage decision model to help BSS developer optimize battery purchasing and charging decisions. At the first stage an optimal number of batteries is identified. Then a Markov decision process (MDP) is solved to determine
the optimal charging policy using the number of batteries determined at the first stage. Noting that real-world instances tend to have large state and action spaces, which poses computational challenges, we introduce a fluid-based formulation that can serve as an approximation of the original decision model.

We are by no means the first to consider the battery purchasing and recharging problem for a BSS. A problem that concurrently optimizes the number of batteries and the charging decisions has been formulated and carefully studied by Schneider et al. (2017) under an MDP framework. In contrast to their study, we propose a fluid-based optimization model which is especially appropriate for large-scale implementations of battery swap in the future. Our fluid-based formulation not only results in algorithmic simplifications but produces insights that can be absent under an MDP framework. For example, using the fluid model we are able to quantify the joint effect of demand patterns and electricity prices on battery investment decisions in a rather explicit fashion.

Although we are primarily motivated by BSS operations, our modeling framework and solution approaches may be applied to other systems that share common features with a BSS. Examples include a bike-sharing system that pays some of its members (referred to as bike Angels) to redistribute the bikes themselves so as to resolve the “rebalancing problem”, in which riders overload a system’s most popular takeoff points and destinations, rendering docks useless; see Chung et al. (2018). Specifically, consider an idealized model with two bike locations $A$ and $B$ connected by a one-way street with direction from $A$ and $B$. One can draw an analogy between these two models by regarding each bike as a battery and each pick-up at location $A$ as an EV arrival. Then each reverse trip (from $B$ to $A$) corresponds to a charge completion for the BSS. Moreover, the transit time from $A$ to $B$ and that from $B$ to $A$ can be thought of as the battery-swapping and battery charging times, respectively. But the forms of controls used are slightly different. A bike-sharing system influences its number of reverse commuters indirectly by dynamically adjusting its rewards whereas a BSS can decide the number of working chargers directly. Due to a striking resemblance between these two types of systems, we
believe that the recipes provided here can be easily adapted to suit the bike rebalancing problem.

We make three contributions in this paper:

1. We introduce a two-stage approach for determining an optimal number of batteries (to be purchased) and the corresponding charging policy. We formulate the second-stage problem as an infinite-horizon average-cost MDP in a time-periodic context (where the demand and electricity price are taken as jointly periodic functions of time) to trade off the charging and waiting costs using the number of batteries optimally determined at the first stage. The time-periodic features - both demand and cost of production - is relatively nonstandard. The only previous example we know of is Aviv and Federgruen (1997). Under the time-periodic assumptions, we show that the MDP produces policies that retain the time-periodic structure.

2. We propose a fluid-based formulation that serves as a deterministic, continuous-time approximation of the original decision problem. Using Pontryagin’s maximum principle, we explicitly characterize the optimal solution. Leveraging the fluid-model analysis, we obtain useful managerial insights for optimizing the operations of the BSS under non-stationary demand and electricity price: (i) Each additional battery can help lower the charging and the cost of waiting combined, but the marginal gain is decreasing in the number of batteries in system. (ii) When the demand and electricity price are unsynchronized, namely, the high-demand period coinciding with the low-price period, the BSS can achieve a lower operating cost with a smaller number of batteries.

3. We illustrate through extensive numerical examples the effect of key system parameters on the solution to the battery purchasing and charging problem. We identify the key factors that one should focus on in order to improve the performance of a BSS.

The remainder of the paper is structured as follows. In §2, we review related literature. In §3, we introduce our two-stage decision model. In §4, we present the corresponding fluid model and provide important analytical results. In §5 we provide extensive
numerical experiments and offer managerial insights. In §6 we draw conclusions. We put technical proofs in the appendix.

2. Literature Review

Our battery swap problem is similar to some inventory control problems, especially the research on a closed-loop supply-chain inventory system in which failed items (DBs) are returned and replaced by functioning ones (FBs), and the returned items are then repaired (recharged) and put back into the inventory. Early work on supply chains with repairable items dates back to Sherbrooke (1968) and Graves (1985), where the repair capacity is assumed to be infinite. Extensions of these models with limited repair capacity are sometimes framed as a closed queueing network; see, e.g., Gross et al. (1983), Diaz and Fu (1997). These papers assume the repair cost (if any) to be constant and the demand to be time-stationary and mainly focus on steady-state analysis, while we take both the charging cost and the demand (for battery swap) to be time-varying. More recently, Georgiadis et al. (2006) formulate a dynamic model for capacity planning of closed-loop supply chains with remanufacturing. There is a large body of literature on this topic, as can be seen from Muckstadt (2004) and Guide and Wassenhove (2006).

Our paper is also related to a fluid approximation of a time-varying stochastic system, which is essentially a deterministic process that approximates the evolution of the stochastic model, subject to some scaling or limiting criteria. While the reference list presented here is by no means exhaustive, it should give an indication of the many research studies making use of this technique. The fluid model analysis is mostly seen in queueing literature. An early work on fluid approximation for time-varying queues is Mandelbaum et al. (1998) where a time-varying fluid approximation is developed for a Markovian queue and supported by a functional laws of large numbers and a functional central limit theorem. A fluid model for a many-server queue with time-periodic arrivals, services and abandonments is developed by Puhalskii (2013). Whitt (2006) relies on deterministic fluid models to derive staffing solutions for a call center with uncertain arrival rate and employee absenteeism. More recently, Li et al. (2017) model a bike-
sharing system as a multiclass closed queueing network and derive its fluid and diffusion limits. See also Hampshire and Massey (2010) for a comprehensive review of optimal control of time-varying fluid queues. The use of fluid model analysis also appears in the revenue management literature. For instance, Maglaras and Meissner (2006) perform a unified analysis of the pricing and capacity control problem in the context of multi-product revenue management and develop a deterministic fluid formulation that gives rise to a closed-form characterization of the optimal control which in turn leads to useful fluid heuristics. For recent development in using fluid-based analysis to study network revenue management problems, see Dai et al. (2018).

Since battery swap is in the early planning stages, research on optimizing the operation of a BSS remains limited. Widrick et al. (2018) formulates and analyzes a finite-horizon, discrete-time, non-stationary MDP. They include in their model an ability of a BSS to discharge energy back to the power grid, but they do not consider cost of waiting due to backlogged demand. In contrast, we do not consider discharging operations and take system congestion as an important component in the operating cost. In addition, unlike the assumptions made in the present paper, they regard the charging times to be fairly short, being equal to the length of the period between two successive decision epochs. Schneider et al. (2017) also formulates and analyzes an MDP, which aims to decide the number of batteries to purchase and charge over time while minimizing the total cost comprised of initial investment, charging cost and penalty for congestion. They formulate the problem over an infinite horizon and embed the opportunity cost of unused batteries into the initial investment. Moreover, they model the electricity price as a random process and assume the charging times to be fixed. In comparison to their study, we take the time-varying electricity prices to be given. This tends to be justified in applications because estimates of electricity prices can be obtained in the day-ahead electricity market. Finally, instead of assuming constant charging times, we take charging times to be random. In particular, we assume all charging times are geometrically distributed, which entails the memoryless property. Under our MDP framework, it is
possible to model non-Markovian charging times, but if we did so, then we would lose much analytical tractability.

Similar to what we do, Zhang et al. (2012) consider an adequacy model and use Monte Carlo simulations to determine the adequate number of batteries circulating in the system, but they do not take into account the charging or discharging operations in the BSS. Mak et al. (2013) analyze a BSS location problem using a robust optimization approach. Tan et al. (2017) model a BSS as a mixed queueing network and analyzes the system capacity parameters as the number of batteries asymptotically approaches infinity. Using the same queueing-theoretic approach, Sun et al. (2017) formulate the charging operation problem as a stationary constrained MDP to minimize the charging cost while ensuring a certain quality of service.

3. The Two-Stage Decision Model

We model the state of the batteries at a fundamental level where each battery is either fully-charged or depleted. Since the consumption of an FB automatically creates a DB, the total number of batteries (i.e., the sum of FBs and DBs) is kept a constant. Throughout the rest of the section, we will use $K$ and $B$ to denote the numbers of charging bays and batteries purchased, respectively. We choose to describe the second-stage problem first which we formulate as an MDP. We present the first-stage problem thereafter.

3.1. The Second-Stage Problem

Under the MDP framework, we define the state of the BSS system as the FB inventory level. Our action taken at each decision epoch is to decide the number of DBs to be put in the charging bays and start charging. The objective of the MDP is to find a policy for charging batteries that can best trade off the charging cost and the cost of waiting. We adopt a discrete-time formulation where time is discretized into small slots of length $\delta$, indexed by $k = 1, 2, \ldots$. The non-stationarity characteristics of the MDP include demand for battery swap and electricity price. To capture the fact that demand and electricity price depend on the time of day and the day of the week, we allow the parameters in both state transition probability and one-slot cost function to be time-varying. Specifically
we assume that the number of EV arrivals over the $k$-th slot $[k\delta, (k+1)\delta]$, denoted by $\xi_k$, to be a Bernoulli random variable with parameter $\lambda_k \delta$. Here we assume $\{\lambda_k; k \geq 0\}$ to be a periodic sequence with cycle length $\tau$, i.e., $\lambda_k = \lambda_{k + \tau}$ for any $k$. With slight abuse of notation, we write $\lambda_k$ in place of $\lambda_k \delta$ and refer to $\lambda_k$ as either the demand rate or probability of arrival in slot $k$. We model the system such that the amount of time for a DB to receive a full charge is geometrically distributed with parameter $\mu \delta$. Again, with slight abuse of notation, we write $\mu$ in place of $\mu \delta$ and refer to $\mu$ as either the charging rate or the probability that a battery finishes charging during the slot.

We remark that while the geometric charging times may not capture the exact charging-time distribution in practice (states of charge often show a sharp increase in initial charging period followed by a saturation phase), it is close to reality since it captures (i) the mean charging time and (ii) the independence between different batteries. The assumption is mostly motivated by mathematical convenience. Indeed, the memoryless property of the geometric distribution allows us to gain much analytical tractability which is not possible for other probability distributions.

The sequence of events in each time slot is as follows. At the beginning of slot $k$, batteries that receive a full charge become available and backlogged demand (if any) for battery swap is fulfilled immediately from on-hand FBs. Then the system operator observes the system state and electricity price $p_k$, and decides how many batteries to start charging. Here we assume $\{p_k; k \geq 0\}$ to be a periodic sequence as well with the cycle length equal to $\tau$, i.e., $p_k = p_{k + \tau}$ for all $k$. Thus $\{\lambda_k; k \geq 0\}$ and $\{p_k; k \geq 0\}$ are jointly periodic with a common cycle length $\tau$. At the end of the slot, the demand $\xi_k$ is realized, and finally the charging and waiting costs are incurred. We now mathematically characterize the MDP using the notation introduced above.

1. The state of the system in slot $k$, $x_k \in X \equiv \{-\infty, \ldots, -1, 0, 1, \ldots, B\}$ represents the inventory level of FBs at the beginning of period $k$. Here, a positive value indicates the existence of FBs in system and a negative value is understood to be backlogged demand.
Thus, the number of DBs at time slot \( k \) equals \( B - x_+^k \), where \( x^+ \) denotes the positive part of \( x \), i.e., \( x^+ = \max(x, 0) \).

2. The action taken at the beginning of the \( k \)-th slot, \( a_k \in \mathcal{A}_k(x_k) \equiv \{0, 1, \ldots, \min(B - x^+_k, K)\} \), is the number of DBs placed in the charging bays. A decision rule, \( \pi_k : \mathcal{X} \rightarrow \mathcal{A}_k(x_k) \), is a function mapping from the state space \( \mathcal{X} \) to the action space \( \mathcal{A}_k(x_k) \), which indicates how the system operator selects an action \( a_k \in \mathcal{A}_k(x_k) \) at a decision epoch \( k \) when the system state is \( x_k \in \mathcal{X} \). Because the decision rules depend on the current system state only rather than the entire history, we are essentially restricting ourselves to Markovian decision rules. We use \( \pi \equiv (\pi_1(x_1), \pi_2(x_2), \ldots) \) to denote a policy specifying the decision at all decision epochs. Since we consider a periodic system, we anticipate that both the state and the optimal control policy should exhibit time-periodic structure as well, namely, \( x_k = x_{k+\tau} \) and \( \pi_k(x_k) = \pi_{k+\tau}(x_{k+\tau}) \), which will be rigorously shown in Theorem 1. Denote by \( \Pi \) the set of the deterministic periodic-stationary policies.

3. We denote by \( q_k(j|x_k, a_k) \) the transition probability that the system state reaches \( j \) at time \( k+1 \) from \( x_k \) when action \( a_k \) is taken. Let \( \eta_k \equiv \eta_k(a_k) \) denote the number of batteries that become fully-charged at the beginning of the \( (k+1) \)-th slot. From our distributional assumption on the charging times, it can be verified easily that \( \eta_k \) is a binomial random variable with parameters \( (a_k, \mu) \). The transition equation for the state is then given by

\[
x_{k+1} = x_k - \xi_k + \eta_k.
\] (1)

It is readily checked from (1)

\[
q_k(j|x_k, a_k) = \begin{cases} 
\lambda_k(1 - \mu)^{a_k} & \text{if } j = x_k - 1, \\
\lambda_k \frac{a_k}{j - x_k + 1}(1 - \mu)^{a_k - j + x_k - 1} + (1 - \lambda_k) \frac{a_k}{j - x_k} \mu^{j - x_k} (1 - \mu)^{a_k - j + x_k} & \text{if } x_k \leq j < x_k + a_k, \\
(1 - \lambda_k) \mu^{a_k} & \text{if } j = x_k + a_k.
\end{cases}
\] (2)

4. The one-slot cost when action \( a_k \) is taken in state \( x_k \) at time \( k \) that leads to transition to \( x_{k+1} \) at time \( k+1 \) is the cost of system incurred over the \( k \)-th slot, given by

\[
c_k \equiv c_k(x_k, a_k) = p_k a_k + c x_k^-,
\] (3)
where we have written $p_k$ and $c$ in place of $p_k\delta$ and $c\delta$, respectively, similar to what we did with $\lambda_k\delta$ and $\mu\delta$, and $x^-$ denotes the negative part of $x$, i.e., $x^- \equiv \max(-x, 0)$. Here $x_k^-$ represents the backlog, namely, the number of EVs waiting for battery swap, $p_k$ is the charging cost per battery per unit time, $c$ is the cost of waiting per EV in queue per unit time.

5. As we will be primarily interested in minimizing long-run average cost, we choose to present the infinite-horizon average-cost formulation. For simplicity, we suppose that the initial system state is zero. Let $c^\pi_k$ denote one-slot cost at time $k$ under policy $\pi$, and $C(\pi) \equiv \limsup_{T \to \infty} T^{-1} \sum_{k=1}^{T} c^\pi_k$ denote the corresponding long-run average cost. Then the infinite-horizon cost-minimization problem can be formulated as

$$\inf_{\pi \in \Pi} C(\pi) \equiv \inf_{\pi \in \Pi} \left( \limsup_{T \to \infty} T^{-1} \sum_{k=1}^{T} c^\pi_k \right). \tag{4}$$

The result below guarantees the existence of a periodic-stationary optimal policy.

**Theorem 1.** There exists a periodic-stationary optimal policy $\pi^*$ for problem (4); i.e., the decision rule $\pi^*$ at time $k = j\tau + i$ ($j \in \mathbb{Z}^+$ and $i \leq \tau$) is independent of $j$ but depends on $i$.

**Proof of Theorem 1.** The key is to show the periodic MDP can be reformulated so as to be stationary. This is done by enlarging the state and the action space. Define $\tilde{\mathcal{X}} \equiv \{(x,k); x \in \mathcal{X}, k \in \{1, \ldots, \tau\}\}$ and $\tilde{\mathcal{A}} \equiv \{(a,k); a \in \mathcal{A}_k, k \in \{1, \ldots, \tau\}\}$. Let $\tilde{c}(x,k),(a,k)) = c_k(x,a)$, and let $\tilde{q}((x,k),(a,k))$ assign probability one to $\mathcal{X} \times \{k+1\}$ for $k \neq 0$ (mod $\tau$) and to $\mathcal{X} \times \{1\}$ for $k = 0$ (mod $\tau$) with the marginal distribution of the first coordinate being $q_k(\cdot|x,a)$. Then the MDP with state space $\tilde{\mathcal{X}}$, action space $\tilde{\mathcal{A}}$, cost structure $\tilde{c}$, and transition law $\tilde{q}$ is time-stationary. Because the original action space $\mathcal{A}$ has finite elements, the new action space $\tilde{\mathcal{A}}$ is finite and hence compact. An application of Theorem 3.8 in Schä1 (1993) (see the references there for earlier related work) allows us to conclude the existence of a stationary optimal policy for the reformulated MDP. By the earlier transformation of the state and the action space, we conclude that there exists a periodic-stationary optimal policy $\pi^*$ for problem (4).

**Remark 1.** In addition to establishing the existence results, Schä1 (1993) also shows that the optimal average cost of a stationary MDP can be approximated by the value
functions of the infinite-horizon discounted model. Hence, the periodic-stationary policy \( \pi^* \) to our problem (4) can be approximated by a solution to the infinite-horizon periodic discounted model whose computational scheme is provided by Riis (1965); see also Su and Deininger (1972).

### 3.2. The First-Stage Problem

Noting that the solution to the MDP relies on the “fixed parameter” \( B \), we denote by \( V(B) \) the optimal value for the MDP. On the one hand, increasing \( B \) allows us to achieve lower objective value \( V(B) \). On the other hand, EV batteries are expensive to manufacture. We thus incorporate costs of capital and assume battery investment cost to be \( \gamma \) per unit of time. For example, if a battery costs $3,500, or $350 per year considering a 10% amortization rate, then \( \gamma \approx 1 \) if we use day as the units of time. Our first-stage problem is then formulated as follows:

\[
\min_B V(B)\tau + \gamma \tau B.
\]

### 4. Fluid-Based Formulation

This section studies a two-stage optimization problem that can be regarded as a deterministic approximation of the original BSS battery purchasing and charging problem. We describe the model in §4.1. §4.2 provides analytical characterization of the fluid-based solution. In 4.3 we derive an upper bound for the optimal amount of battery fluid to be purchased to gain additional insights.

#### 4.1. System Equations & Model Formulation

The fluid model has deterministic and continuous dynamics, and is obtained by replacing the discrete stochastic demand process by its demand (arrival-rate) function \( \lambda(t) \) and the stochastic charging-completion process by \( \mu m(t) \) where \( \mu \) and \( m(t) \) denote the (constant) charging rate and the amount of DB fluid in the charger at time \( t \). Let \( x(t) \) denote the amount of FB fluid at time \( t \). Then the fluid model can be rigorously justified as a limit under a law-of-large-number type of scaling as the potential demand and the charging capacity grow large proportionally.
Let \( p(t) \) denote the electricity price at time \( t \). Paralleling the discrete-time formulation described in the previous section, we take \( \lambda(t) \) and \( p(t) \) to be jointly periodic functions in time with the cycle length equal to \( \tau \); i.e., \( \lambda(t) = \lambda(t + \tau) \) and \( p(t) = p(t + \tau) \) for all \( t \geq 0 \).

Here the demand function \( \lambda(t) \) and the price function \( p(t) \) are considered given, and we regard \( m(t) \) and \( x(t) \) as our control and system state, respectively. There are two types of resource constraints, \( \kappa \) and \( b \), representing the maximum charger fluid (i.e., maximum possible amount of DB fluid that can be charged simultaneously) and the total amount of battery fluid in circulation, respectively. These two quantities correspond to the number of charging bays \( K \) and the number of batteries \( B \) in the MDP described in the previous section. With this notation, we can state the second-stage problem which we also refer to as the charging problem:

\[
\min_{x_0 \leq b, m \in \mathbb{L}} \int_0^\tau p(t)m(t)dt + c \int_0^\tau x^-(t)dt
\]

\[
\text{s.t. } \dot{x}(t) = \mu m(t) - \lambda(t), \quad 0 \leq t \leq \tau, \tag{7}
\]

\[
0 \leq m(t) \leq \kappa, \quad 0 \leq t \leq \tau, \tag{8}
\]

\[
m(t) + x^+(t) \leq b, \quad 0 \leq t \leq \tau, \tag{9}
\]

\[
x(0) = x(\tau) = x_0, \tag{10}
\]

where we have used \( \mathbb{L} \) to denote the space of integrable functions over \([0, \tau]\). The goal is to determine a charging policy \( m^* \) and an initial FB fluid content \( x_0^* \) that minimize the the sum of the charging cost and the cost of waiting within one cycle. The state \( x(t) \) can take both positive and negative values. Positive values indicate that we have an amount of \( x(t) \) FB fluid in stock whereas negative values occur when demand for FB fluid exceeds its supply in which case there is an amount of \(-x(t)\) EV fluid in queue. Constraint (7) is the basic flow equation derived of conservation laws. Constraint (8) stems from the fact the amount of charger fluid at any time is nonnegative and cannot exceed the maximum charger fluid \( \kappa \). Constraint (9) states that the amount of DB fluid being charged and the amount of FB fluid combined cannot exceed the total amount of battery fluid in system. Finally, we impose the terminal condition (10) which is primarily motivated by
the existence of a periodic-stationary optimal policy \( \pi^* \) to the MDP in (4) whose induced FB inventory has a periodic-stationary distribution; i.e., \( x_t \) equals in distribution to \( x_{t+\tau} \).

Since our fluid model can be seen as a deterministic approximation for the MDP, it would be reasonable to have \( x(t) = x(t + \tau) \) in the fluid model with a periodic-stationary control. Loosely speaking, by adding constraint (10), we restrict ourselves to the space of periodic-stationary solutions.

Note that constraint (10) requires the amount of FB fluid produced over \([0, \tau]\) to be equal to the demand occurred over the same cycle. Use \( \Lambda(t) \equiv \int_0^t \lambda(u)du \) to denote the cumulative demand up to time \( t \). Then combining (7) and (10) yields

\[
\Lambda(\tau) = \mu \int_0^\tau m(t)dt. \tag{11}
\]

Further let \( \theta \) be such that \( \mu \kappa \theta = \Lambda(\tau) \), or simply \( \theta = \Lambda(\tau)/(\mu \kappa) \). From (8) and (11), it is readily seen that \( \theta \) is the minimum amount of time that the BSS has to spend on charging batteries within a cycle. In order to ensure that the optimization problem given by (6) - (10) has at least one (feasible) solution, we make the following assumption that will be maintained in the rest of the paper.

**Assumption 1.** The quantity \( \theta \) satisfies \( \theta \leq \tau \).

The result below guarantees the existence of an optimal solution over the decision space \( \mathbb{R} \times L \) for the problem specified by (6) - (10). The proof makes use of an equivalent formulation and is deferred to the appendix.

**Theorem 2.** Suppose Assumption 1 holds. Then there exists at least one optimal solution \( (x_0^*, m^*) \) to the second-stage problem given by (6) - (10).

In what follows, we denote the optimal objective value of the charging problem by \( V(b) \) so as to indicate the dependence of the solution on the value of \( b \). The next result provides convexity of the charging problem in \( b \), i.e., the amount of battery fluid \( b \) used by the system.

**Theorem 3.** The function \( V(b) \) is convex.
Paralleling (5), we can write down the first-stage problem:

$$
\min_b V(b) + \gamma \tau b. \quad (12)
$$

Theorem 3 suggests that although adding an additional battery can help reduce the charging and waiting costs altogether, the marginal gain of doing so diminishes. Moreover, Theorem 3 shows that the first-stage problem given by (12) is convex and its solution is guaranteed to exist.

### 4.2. Solution by the Maximum Principle

In this section, we solve the charging problem (i.e., the second-stage problem) using Lagrangian form of Pontryagin’s maximum principle; see Chapter 3 of Sethi and Thompson (2000); see also Chapter 3 of Bertsekas et al. (1995). To do that, we associate an adjoint function $\alpha$ with equation (7) and write down the Hamiltonian function

$$
H \equiv H(x(t), m(t), \alpha(t)) \equiv -p(t)m(t) - cx^-(t) + \alpha(t)(\mu m(t) - \lambda(t)). \quad (13)
$$

In (13), we have used the negative of the integrand in (6), since the minimization of $V(b)$ in (6) is equivalent to the maximization of $-V(b)$. To apply the Pontryagin’s maximum principle, we differentiate (13) to get

$$
\frac{\partial H}{\partial m} = -p + \mu \alpha,
$$

so that the optimal control is of the bang-bang form:

$$
m^* = \begin{cases} 
0 & \text{if } p > \mu \alpha, \\
\min(\lambda/\mu, \kappa, b - x^+) & \text{if } p = \mu \alpha, \\
\min(\kappa, b - x^+) & \text{if } p < \mu \alpha.
\end{cases} \quad (14)
$$

The adjoint variable $\alpha(t)$ can be interpreted as the future value (at time $\tau$) of one unit of FB at time $t$. Therefore, the decision rule (14) has a clear economic interpretation: The BSS is willing to charge DB fluid at the maximum (possible) capacity if the marginal benefit of producing an additional FB fluid exceeds the associated cost; similarly, if the
marginal benefit falls short of cost, it is beneficial for the BSS not to charge battery at all. To proceed, we form the Lagrangian

\[
L \equiv L(x(t), m(t), \alpha(t), \nu(t)) \equiv H + \nu_1(t)m(t) + \nu_2(t)(\kappa - m(t)) + \nu_3(t)(b - m(t) - x^+(t)),
\]

(15)

where the set of Lagrange multipliers \( \nu \equiv (\nu_1, \nu_2, \nu_3) \) satisfy the complementary slackness (CS) conditions:

\[
\begin{align*}
\nu_1(t) & \geq 0, \quad \nu_1(t)m(t) = 0, \\
\nu_2(t) & \geq 0, \quad \nu_2(t)(\kappa - m(t)) = 0, \\
\nu_3(t) & \geq 0, \quad \nu_3(t)(b - m(t) - x^+(t)) = 0.
\end{align*}
\]

(16)

In addition, the optimal state trajectory, optimal control, and the corresponding Lagrange multipliers must satisfy

\[
\begin{align*}
\frac{\partial L}{\partial m} &= -p(t) + \mu \alpha(t) + \nu_1(t) - \nu_2(t) - \nu_3(t) = 0 \quad \text{and} \\
\dot{\alpha}(t) &= -\frac{\partial L}{\partial x} = -c_1 \mathbb{1}_{\{x(t)<0\}} + \mu^{-1} \dot{p}(t) \mathbb{1}_{\{x(t)=0, -\mu c \leq \dot{p}(t) \leq 0\}} + \nu_3(t) \mathbb{1}_{\{x(t)>0\}}.
\end{align*}
\]

(17)

(18)

One can easily check by combining the CS conditions, (17) and (18) that \( \nu_3(t) = \mu \alpha(t) - p(t) \) if \( x(t) + m(t) = b \) and \( \nu_3(t) = 0 \) otherwise.

In canonical control problems, it is usually assumed that the initial state \( x(0) \) is fixed. Here the initial state is free but the trajectory must return to it. It turns out that this set-up can be easily handled by using a version of the transversality condition, which involves the values of the adjoint function both at the initial time and at the terminal time; see equation (4.46) and the discussion above at page 107 in Liberzon (2011). Namely, this condition requires that \( (\alpha(0), -\alpha(\tau)) \) must be orthogonal to \( (x(0), x(\tau)) \); i.e.,

\[
\alpha(0) = \alpha(\tau).
\]

(19)

When the final state is not fixed, there must also be a terminal condition for the adjoint equation. Here that condition is not necessary because we have the extra boundary condition (10) maintaining the balance between boundary conditions and unknowns. To summarize, the optimal solution \( (x^*_0, m^*) \) to the second-stage problem is completely characterized by (10), (14), and (16) - (19).
It is clear from (7), (14) and (18) that (for a fixed initial state $x(0)$) the terminal value $x(\tau)$ is increasing in the value of $\alpha(0)$. This suggests an efficient trial-and-error approach for computing the correct $\alpha(0)$ with which the boundary condition (10) is satisfied: If $x(\tau) < x(0)$ one should increase the current value of $\alpha(0)$ and decrease it if the inequality is reversed. To optimize over $x_0$, i.e., to find among all possible values of $x_0$ the one that produces the lowest operating cost, we can rely on the sign of $d \equiv \alpha(0) - \alpha(\tau)$ as a searching guide. Specifically, if $d$ is positive (under the current choice of $x_0$) meaning that one unit of FB has more worth at the beginning of the cycle, then we increase the current value of $x_0$; similarly, if the sign is negative (i.e., one unit of FB is less valuable at the initial time than compared with its value at the terminal time), we decrease its present value.

![Comparison of the FB fluid and charging control based on the DP and fluid approximation approaches.](image)

We conclude the section by numerically testing the accuracy of the fluid model approximation. To that end, we solve the MDP in §3 and the second-stage fluid optimization under the same settings, and then compare the results from the two approaches. To make the problem manageable for the MDP, we run our experiments on a medium-size system with $K = 20$ chargers, $B = 40$ batteries, price function $p(t) = 2.45 - 1.05 \sin(\pi t/12)$ and arrival-rate function $\lambda(t) = 16 - 8 \sin(\pi t/12)$. The cycle length is thus $\tau = 24$ hours.
The effectiveness of the fluid-model approximation is visually confirmed from Figure 2, which reports the mean FB inventory level and the number of working chargers (averaged over 1000 sample paths under the optimal policy from the MDP) together with the time-varying FB fluid $x$ and the working-charger fluid $m$ computed from the fluid-model approximation.

### 4.3. Upper-Bound Analysis

To gain greater managerial insights into the joint impact of energy price and demand functions on battery investment cost, here we construct an explicit upper bound for the optimal amount of battery fluid $b^*$ the arises in solving the fluid-based cost minimization problem. We start by focusing on the charging cost. To save on energy costs, the BSS would like to operate at full capacity at those times when the electricity price is among the lowest. To explicitly characterize such a policy, define the set-valued function

$$\phi(\zeta) \equiv \{0 \leq t \leq \tau : p(t) \leq \zeta\}, \tag{20}$$

that maps each real number to a level set. Figure 3 presents a graphical illustration of the function $\phi$. If the price function $p$ is Borel measurable, $\phi(\zeta)$ are Borel sets. Denote by $\ell$ the Lebesgue measure and define

$$g(\zeta) \equiv \ell(\phi(\zeta)). \tag{21}$$

From (20) it is easily verifiable that $g$ is a nondecreasing function yet not necessarily continuous. Indeed, if $p$ happens to be a step function, then $g$ has points of discontinuity.

**Assumption 2.** There exists a unique $\zeta^* \geq 0$ such that $g(\zeta^*) = \theta$ for $g$ given by (21).

**Remark 2.** Assumption 2 is likely to be violated if the price function is stair-wise. Indeed, in the staircase price function case, there typically exist two price levels $\zeta_1$ and $\zeta_2$ such that $g(\zeta_1) \leq \theta \leq g(\zeta_2)$, $\zeta_1 \leq \zeta_2$. To minimize the charging cost, the BSS must charge batteries at full capacity when the price is lower than or equal to $\zeta_1$, and charge the remaining batteries at the price $\zeta_2$. Moreover, there exist multiple charging policies that can achieve the minimum charging cost without loss of demand.
Suppose for the moment that there is unlimited amount of battery fluid in the system; i.e., $b = \infty$. Then (9) is no longer a real constraint. Consider the following charging policy:

$$m^*(t) = \begin{cases} \kappa & \text{if } t \in \phi(\zeta^*), \\ 0 & \text{if } t \notin \phi(\zeta^*), \end{cases} \quad (22)$$

where $\zeta^*$ is defined by Assumption 2. The result below indicates that $m^*$ is optimal as far as the charging cost is concerned.

**Proposition 1.** Under Assumptions 1 and 2, if the amount of battery fluid in circulation is unlimited (i.e., $b = \infty$), then the charging policy $m^*$ given in (22) achieves the minimum (possible) charging cost; i.e., there exists no such a charging policy that gives a lower charging cost.

We next consider the cost associated with waiting. Note that the cost of waiting will be completely eliminated if we choose $(x(0), m)$ in such a way that

$$x(t) \geq 0 \quad \text{for all } 0 \leq t \leq \tau. \quad (23)$$

Indeed, for a charging policy $m$, we can choose

$$x(0) \equiv \sup_{0 \leq t \leq \tau} \left[ \mu \int_0^t m(u)du - \Lambda(t) \right]^- = \sup_{0 \leq t \leq \tau} \left[ \Lambda(t) - \mu \int_0^t m(u)du \right]^+, \quad (24)$$
to ensure that condition (23) holds. To see that this is indeed the case, notice
\[
x(t) = x(0) + \mu \int_0^t m(u)du - \Lambda(t) \geq \left[ \mu \int_0^t m(u)du - \Lambda(t) \right]^- + \mu \int_0^t m(u)du - \Lambda(t) \geq 0
\] (25)
where the first inequality is due to (24) and the second inequality follows from the simple relation \( x^+ = x^- + x \). In particular, we set, for the optimal charging policy \( m^* \) given in (22)
\[
x^*(0) \equiv \sup_{0 \leq t \leq \tau} \left[ \Lambda(t) - \mu \int_0^t m^*(u)du \right]^+.
\] (26)
By Proposition 1 and (25), the solution \((x^*(0), m^*)\) yields the lowest charging cost without causing any congestion, provided the total amount of battery fluid in system is sufficient. i.e., solution \((x^*(0), m^*)\) is optimal for the fluid-model optimization problem given by (6) - (10) with \( b = \infty \). With the control function \( m^* \) and the initial state \( x^*(0) \) given by (22) and (26) respectively, the state dynamics, denoted by \( x^*(t) \), is uniquely determined and is given by
\[
x^*(t) = x^*(0) + \mu \int_0^t m^*(u)du - \Lambda(t), \quad \text{for all} \quad 0 \leq t \leq \tau.
\] (27)
Using the state \( x^* \) and control \( m^* \) specified by (22) and (27) respectively, we define
\[
b^* \equiv \sup_{t \leq \tau} \{m^*(t) + x^*(t)\}.
\] (28)
By choosing \( b = b^* \), we make sure that constraint (9) is not violated. Indeed, \( b^* \) is the minimum amount of battery fluid content with which both components of the objective function (6) reach the lowest possible value. The theorem below summarizes the main results in this section.

**Theorem 4.** Suppose Assumptions 1 and 2 hold. Then there exists a threshold value \( b^* \) such that the value \( V(b) \) with respect to \( b \) is (i) strictly decreasing for \( b < b^* \) and (ii) constant for \( b \geq b^* \).

Theorem 4 immediately implies that the objective value of problem (12) is monotonically increasing for \( b > b^* \). Consequently, \( b^* \) is an upper bound of the optimal amount of battery fluid in the two-stage optimization problem (12). Moreover, this bound depends on the energy price \( p \) and demand \( \lambda \) only. To analytically evaluate the joint impact
of energy price and demand functions on battery investment cost, we illustrate by an example how the degree of synchronization between \( p \) and \( \lambda \) can affect the value of \( b^* \). For simplicity, we stipulate that both price and demand follow sinusoidal functions with cycle length \( \tau \). Specifically, we assume the price and demand functions to follow

\[
p(t) = \bar{p} + A_p \sin(2\pi t/\tau) \quad \text{and} \quad \lambda(t) = \bar{\lambda} + A_\lambda \sin(2\pi (t - \psi)/\tau)
\]

respectively, where \( \bar{p} \) and \( \bar{\lambda} \) are the vertical shifts, \( A_p \) and \( A_\lambda \) represents the amplitudes, and \( \psi \) denotes the phase shift. Here we regard \( \psi \) as a design parameter quantifying the degree of synchronization between \( p \) and \( \lambda \). Note that \( \psi = 0 \) and \( \psi = \pi \) represent the cases that demand function is unsynchronized and synchronized with price function, respectively. It is readily checked that the total demand \( \Lambda(\tau) = \bar{\lambda} \tau \). In addition, let \( \mu = 1 \) and \( \kappa = 2\bar{\lambda} \). With these parameters, we can calculate the minimum amount of time that the BSS has to spend on charging batteries within a cycle, yielding \( \theta = \Lambda(\tau)/\kappa = \tau/2 \).

Using (22), (26), (27) and (29), we deduce

\[
b^* = \kappa + \bar{\lambda} \tau - \int_{\tau/2}^\tau \lambda(t)dt = \kappa + \frac{\bar{\lambda} \tau}{2} - A_\lambda \int_{\tau/2}^\tau \sin(2\pi (t - \psi)/\tau)dt = 2\bar{\lambda} + \frac{\bar{\lambda} \tau}{2} + \frac{A_\lambda \tau}{\pi} \cos(2\pi \psi/\tau).
\]

It is immediate by the analytical expression that \( b^* \), as a function of \( \psi \), attains its maximum at \( \psi = 0 \), and keeps decreasing until reaching its minimum value at \( \psi = \pi \). This carries practical implications that are important for the BSS operator to be aware of in order to determine the optimal number of batteries in circulation. In particular, when the demand function and the electricity price function are unsynchronized, the BSS can keep its charging and waiting cost down using a smaller number of batteries. The reason is that when \( \lambda \) is synchronized with \( p \), the BSS tends not to recharge batteries over the high-demand period so as to keep its charging cost down, but it has to built high FB inventory over the low-demand (low-price) period to avoid shortages of FBs over the high-demand (high-price) period. In contrast, in the unsynchronized case, high price coincides with low demand allowing the BSS to maintain lower FB inventory levels.
5. Numerical Studies & Discussion

In this section, we numerically solve the two-stage fluid-based cost minimization problem based on real-world data to gain managerial insights into (i) how the optimal charging control trades off the charging and waiting costs in the second-stage problem (6)-(10), and (ii) how the number of batteries trades off the battery capital cost and the BSS operating cost (including charging and waiting costs) in the first-stage problem (12). Although our discussion here is based on the solution to the fluid model optimization, we will interpret $b$, $x$ and $m$ as the number of batteries in system, the number of FBs and the number of working chargers, respectively. Throughout the section we fix the system parameters as follows: the number of charging bays is $K = 50$, the average charging time is $\mu^{-1} = 1$ hour, the power rate of each charging bay is 26.8 kW.

![Graphs](image.png)

**Figure 4** Illustrating the battery-swapping demand and the energy price.

We choose the cycle length $\tau$ as one week to take into account the periodicity of price and demand in the time of day and the day of week. Due to lacking the real data of battery-swapping demand, we follow the same data and approach as Widrick et al. (2018) and Nurre et al. (2014), using the refueling demand of vehicles at gasoline stations to estimate the battery-swapping demand. Figure 4(a) illustrates the percentage of the average hourly demand over one week. We assume the demand within each hour
follows Poisson process with an average rate determined by the product of the weekly total demand and the hourly demand percentage. We adopt the real energy prices of New York City (NYC) from LCG Consulting (2018). Specifically, we use the hourly prices in Jul. 17-23, 2017, Oct. 17-23, 2017, Jan. 15-21, 2018, and Apr. 16-22, 2018 to represent the energy prices in summer, autumn, winter and spring. As illustrated in Figure 4(b), the energy prices of weekdays are generally higher than those of weekends due to the electricity load reduction in weekends, and the weekly average prices in summer and winner are higher than those in spring and autumn due to extremely hot/cold weather conditions.

5.1. Trade-offs between the Battery Capital Cost and the BSS Operating Cost

This section studies the first-stage battery purchasing problem (12). To expose the trade-offs between the amortized battery cost $\gamma \tau b$ and the operating cost $V(b)$, we show the operating cost against the battery capital cost with the increase of the number of batteries from 50 to 300 (which is called cost curve in this part) in Figure 5. It can be observed that the marginal reduction of the operating cost decreases with the increase of the battery cost for all cost curves in Figure 5. This observation verifies that $V(b)$ is convex in $b$ as claimed in Theorem 3. In the following, we discuss in details about how the key factors affect the trade-offs between battery capital cost and the BSS operating cost in the battery purchasing problem.

We start with the impact of the battery purchasing cost. Recent years have witnessed the rapid decrease in EV battery price due to the increasing production scale and the advance of the battery manufacturing technology. Battery capital cost (including battery cell and pack costs) has fallen from 1000 $/kWh in 2010 to 209 $/kWh in 2017; see Mark Chediak (2017), and is expected to reach 125-150 $/kWh around 2025 by Union of Concerned Scientists (2017). Because most battery manufacturers are providing eight-year warranties, we assume that each battery has an eight-year lifespan in expectation. Then we can estimate the amortized battery capital cost per week. For example, the amortized cost in 2017 is $26.8 \times 209/8/365 \times 7 = 13.43$ $$/week/battery$. Figure
5(a) illustrates the cost curves with the battery prices in different years. To single out the effect of battery purchasing cost, we take the electricity price and the demand pattern to be fixed. Note that this keeps the charging cost and the waiting cost unchanged so long as the number of batteries does not change over the years. But we see that the falling battery cost drives the cost curves to the cost-efficient regime (left-bottom corner). Furthermore, the optimal number of batteries to be purchased (i.e., the red star) increases from 74 in 2015 to 123 in 2017. Thus, with the continuously falling battery...
cost, it becomes more and more cost-efficient to purchase more batteries to reduce the cost.

Next, we fix the battery capital price to be 209 $/kWh, and continue to examine the impact of the key factors that affect the operating cost. Figure 5(b) illustrates the cost curves of different waiting cost factors $c$ given the same price and demand functions. As the waiting cost factor increases, the operating cost increases mainly due to the large increase of charging cost. This is because the BSS has to recharge batteries during the high-price period in order to avoid high waiting cost. Furthermore, to mitigate the negative impact of high penalty due to customer waiting, BSS is also motivated to purchase more batteries. Interestingly, the cost curves for $c = 0.5$ and $c = 1$ are close to each other. A close scrutiny reveals that in these two cases, the number of backlogged demands for battery swap has reached a relatively small value, and hence the waiting cost only takes a small portion of the operating cost. Since the waiting cost indicates the service quality of the BSSs and is the foundation for a successful business, we prioritize the waiting cost over the charging cost for the BSS operations. To do so, we set the waiting cost factor $c = 1$ in the following numerical tests to ensure the service quality.

Figure 5(c) depicts the cost curves with different weekly total demand for battery swap. It can be observed that the marginal gain (in terms of reducing the operating cost) of adding an additional battery is higher when the demand becomes higher. With the increasing penetration of EVs in the future, the battery-swapping demand is expected to increase accordingly. Thus, the BSS operators are encouraged to purchase more batteries to reduce the operating cost in the future. The weekly total demand is set to be 5000 EVs in the following tests to represent a medial-level penetration of EVs.

Figure 5(d) compares the cost curves when the operating cost is evaluated based on the energy prices in different seasons. It can be observed that the energy prices in different seasons greatly affect the battery purchasing decisions. For prices with a large mean and variation in winter, BSS operators need to maintain 123 batteries in circulation to best trade off the battery cost and the operating cost. However, with a smaller and flatter
price in autumn, only 60 batteries are needed to achieve the minimum total cost. To take into account the seasonality of energy prices in the battery purchasing problem, we can reformulate the first-stage problem as follows. We index the four seasons (i.e., winter, spring, summer, and autumn) by \( i = 1, \ldots, 4 \), respectively. Assume that the operating cost per week in each season can be represented by the periodic-stationary cost in the second-stage problem (6)-(10). Then denote the operating cost per week in four seasons by \( V_i(b) \), \( i = 1, \ldots, 4 \). Let \( \omega_i \) be the ratio of the number of weeks of season \( i \) to the total number of weeks of one year. We can determine the optimal number of batteries for purchasing by solving

\[
\min_b \sum_{i=1}^{4} \omega_i V_i(b) + \gamma \tau b. \tag{29}
\]

Set \( \omega_i = 0.25 \), \( \forall i \). Then based on the energy price data in Figure 4(b), we can solve problem (29) and the optimal number of batteries to be purchased is 81.

To summarize, the seasonality of the operating cost indicates that BSS operators prefer to maintain different number of batteries to minimize their total cost. Thus, instead of purchasing batteries, BSS operators may prefer to lease batteries from companies (which can be independent third-party companies or the battery manufacturing companies) based on the amortized battery cost and adjust the number of batteries in system over different seasons to minimize their overall costs.

5.2. Impact of Demand and Price Functions on the Optimal Charging Control

This section focuses on the second-stage charging operation problem (6)-(10) to trade off the charging and waiting costs. In order to reduce the charging cost, the BSS has the incentive to build up FB inventory in low-price periods and then use the on-hand inventory to satisfy the demand in high-price periods. On the other hand, to reduce the waiting cost, the BSS would like to build up FB inventory over underloaded periods (i.e., the time period that \( \lambda(t) < \mu \kappa \)), and then use the inventory together with the real-time production of FBs to satisfy the demand over overload periods. Intuitively, to avoid holding too much FB inventory (which necessarily requires greater number of batteries in circulation), it is beneficial for the BSS to fulfill the demand of overload periods with
best effort (i.e., charging at full capacity) rather using the on-hand FB inventory built in the underloaded period only. For this reason, there exists an important trade-off between achieving low charging cost and reducing waiting cost, especially when the overload period is not overlapping with the low-price period.

Figure 6 Illustration of the optimal charging control and FB inventory level under the time-varying energy price (Jan. 15-21, 2018) and demand (5000 EVs per week).

We obtain the optimal charging control $m$ and the optimal FB inventory level $x$ by solving the second-stage problem (6)-(10). In this numerical test, we set the weekly total demand as 5000 EVs/week, the energy price as the price data of NYC in Jan. 15-21, the waiting cost factor as $c = 1$ $$/\text{min/EV}$, and total number battery as 123. Figure 6 illustrates the optimal charging control with reference to price function, demand function and the FB inventory. Consistent with the optimal charging control (14) derived based on Maximum Principle, the charging control in Figure 6 can be divided into three types of operations: (i) the BSS stops charging when the price is high; (ii) the BSS charges at
its maximum capacity \( \min\{k, b - x^+\} \) when the price is low; (iii) the BSS charges at the rate of the offered load \( \lambda/\mu \) when the price is too high to charge at the maximum capacity but the price decreasing rate \( |\dot{p}| \) is smaller than the waiting cost factor \( c \), preventing the state variable \( x \) transiting to negative values.

We also see that the optimal charging control builds up a large FB inventory before the overload period (i.e., peak of the FB inventory precedes peak of the demand) to reduce the backlogged demand (i.e., the waiting cost) except on Saturday. This is because the energy prices on Saturday and Sunday are much lower than weekdays, and thus to reduce the charging cost the BSS prefers to charge at the full capacity during Saturday to satisfy the demand on the same day. In addition, the low energy prices on weekends motivate the BSS to build up the highest FB inventory before the overload period of Monday to reduce the charging cost.

5.3. Impact of the Synchronization between the Price and Demand Functions on the Operating Cost

![Figure 7 Illustrating the optimal charging operation under the same demand and different energy prices.](image)

Theorem 4 and the example followed show that the degree of synchronization between the demand and price functions can greatly affect the upper bound of the optimal number of batteries. Inspired by the this result, we conjecture that given the number of batteries, the degree of synchronization can also make a difference in the operating cost.
To verify this conjecture, we numerically compute the operating cost when the demand function is shifted over time and compare the resulting cost with that before shifting the demand. We do this numerical test in three scenarios with different energy prices as shown in Figure 7(a). In each scenario, we periodically shift the demand function forward and afterward compared the result to that of the original demand in Figure 7(a). For each shifted demand function, we solve the second-stage problem and obtain the corresponding operating cost. Then we center this operating cost by subtracting the operating cost with the original demand function. The centered operating cost is illustrated in Figure 7(b) for different energy price scenarios. We can readily observe that the shifted demand can lead to either increase (positive centered cost) or decrease (negative centered cost) of the operating cost, and the centered cost changes differently under different price scenarios. Moreover, the operating cost is always reduced when the demand function is shifted to be unsynchronized with the price function. This phenomenon naturally motivates the introduction of demand-management strategies such as pricing to shift the demand to be unsynchronized with the corresponding energy price.

6. Conclusions

In this paper we studied the problem of battery purchasing and charging at an EV BSS. In §3, we formulated a two-stage decision model to concurrently determine the optimal number of batteries to be purchased and the optimal charging control. The solution of the two-stage decision model requires solving an infinite-horizon MDP which is computationally challenging for real-world instances. To reduce the computational complexity and identify the key relations of various parameters, we proposed in §4 a fluid approximation of the original two-stage decision model and numerically validated the effectiveness of such an approximation in §5. Based on the fluid model analysis, we gained important managerial insights for determining medium-term decisions (i.e., minimum number of batteries) and how system parameters affect the optimal charging policy. We found that it is tremendously helpful for the BSS to run in an environment where the demand and the electricity price function are unsynchronous. Particularly,
when the demand function is out of sync with the price function, charging at full capacity during the low-price period can reduce both charging cost and cost of waiting at the same time. In contrast, when the demand and the price function are highly synchronous, it becomes difficult to achieve low charging cost and low cost of waiting at the same time. This leads us to the consideration of demand management that complements the supply management view adopted by the present paper.

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Online Supplement

to

_optimal battery purchasing and charging at an electric vehicle battery swap station_

In this online supplement, we provide the technical proofs omitted in the main paper.

Proof of Theorem 2. For the ease of mathematical analysis, we turn the original fluid-
model optimization into an equivalent problem. Loosely speaking, two optimization prob-
lems are equivalent if an optimal solution to one can easily be “translated” into an opti-
mal solution for the other. Here we substitute (7) into the objective (6) and constraints
(8) - (9) to get

\[
\min_{x \in C_1} \mu^{-1} \int_0^\tau p(t)(\lambda(t) + \dot{x}(t))dt + c \int_0^\tau x^-(t)dt, \\
s.t. \quad 0 \leq \mu^{-1}(\lambda(t) + \dot{x}(t)) \leq \kappa, \quad 0 \leq t \leq \tau, \\
\mu^{-1}(\lambda(t) + \dot{x}(t)) + x^+(t) \leq b, \quad 0 \leq t \leq \tau, \\
x(0) = x(\tau),
\]

where we have used \( C_1 \) to denote the space of differentiable functions over \([0, \tau]\). The rest of the proof proceeds in two steps. We first show that the equivalent problem specified by (30) - (33) is a convex optimization problem. We then argue that we are minimizing over a compact set of continuous functions. Noting that there exists at least one optimal solution to a convex optimization problem over a compact set (see, e.g., Beck (2014), p. 149), we complete the proof.

Let us define \( F^+ : C \rightarrow C, x \mapsto x^+ \) and \( F^- : C \rightarrow C, x \mapsto x^- \) to be two mappings from the space of continuous functions \( C \) to itself. It is evident that both \( F^+ \) and \( F^- \) are convex. Also define \( A : C_1 \rightarrow \mathbb{L}, x \mapsto \mu^{-1}(\lambda + \dot{x}) \). It is immediate that \( A \) is affine and hence convex. Next define

\[
G : \mathbb{L} \rightarrow \mathbb{R}, y \mapsto \int_0^\tau p(u)y(u)du \quad \text{and} \quad H : \mathbb{C} \rightarrow \mathbb{R}, x \mapsto \int_0^\tau x(u)du.
\]
That $G$ and $H$ are linear mappings implies they are convex. Finally, let us use $P_t : \mathbb{C} \to \mathbb{R}$, $x \mapsto x(t)$ to denote the projection mapping at time $t$. The objective (30) is convex in $x$ due to the fact that $G \circ A + H \circ F$ is convex. Similarly, we can write (31), (32) and (33) as $0 \leq A(x) \leq \kappa$, $(A + F^+)(x) \leq b$ and $(P_0 - P_\tau)(x) = 0$ respectively. Hence all constraints are convex.

To argue that the feasible regime $X$ is compact, we apply Arzela-Ascoli theorem. To that end, we show that (i) functions in $X$ are uniformly bounded, and (ii) they are equicontinuous, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x(t) - x(s)| < \epsilon$ uniformly over $X$ whenever $|t - s| < \delta$. From (32), it follows $-\lambda^+ \leq \dot{x} \leq \mu(\kappa - \lambda^+)$ where $\lambda^+ \equiv \sup_{0 \leq t \leq \tau} \lambda(t)$ and $\lambda^+ \equiv \inf_{0 \leq t \leq \tau} \lambda(t)$. Hence, condition (ii) is automatically satisfied. By the same token, condition (i) reduces to the statement that $x(0)$ is bounded uniformly over $X$. From (31) and (32) it follows easily that $x(0)$ is upper bounded uniformly over $X$. We can also impose a (finite) lower bound for $x(0)$ without affecting the optimal solution because the value of the objective function goes to infinity as $x(0)$ approaches $-\infty$. This shows that the feasible region is essentially compact. The proof is thus complete.

\textit{Proof of Theorem 3.} It suffices to show for arbitrary $b_1$ and $b_2$,

$$V(\varrho b_1 + \bar{\varrho} b_2) \leq \varrho V(b_1) + \bar{\varrho} V(b_2) \quad \text{for} \quad 0 < \varrho < 1 \quad \text{and} \quad \bar{\varrho} \equiv 1 - \varrho.$$  

Let $(x_i(0), m_i)$ denote an optimal solution associated with $b_i$, and let $x_i$ denote the optimal trajectory of the state, $i = 1, 2$. Consider $m \equiv \varrho m_1 + \bar{\varrho} m_2$ and $x \equiv \varrho x_1 + \bar{\varrho} x_2$. We argue that $(x(0), m)$ is feasible for the problem with $b \equiv \varrho b_1 + \bar{\varrho} b_2$. Note that constraints (7), (8), and (10) are trivially satisfied. For the third constraint, we have

$$m(t) + x^+(t) = \varrho m_1(t) + \bar{\varrho} m_2(t) + [\varrho x_1(t) + \bar{\varrho} x_2(t)]^+$$

$$\leq \varrho (m_1(t) + x_1^+(t)) + \bar{\varrho} (m_2(t) + x_2^+(t))$$

$$\leq \varrho b_1 + \bar{\varrho} b_2 = b.$$
This shows that constraint (9) is indeed satisfied and therefore the solution \((x(0), m)\) is feasible. The proof is complete by observing that

\[
V(b) \leq \int_0^T p(t)m(t)dt + c \int_0^T x^{-}(t)dt \\
\leq \varrho \left( \int_0^T p(t)m_1(t)dt + c \int_0^T x_1^{-}(t)dt \right) + \bar{\varrho} \left( \int_0^T p(t)m_2(t)dt + c \int_0^T x_2^{-}(t)dt \right) \\
= \varrho V(b_1) + \bar{\varrho} V(b_2).
\]

\[\square\]

**Proof of Proposition 1.** Suppose, by way of contradiction, that there exists another charging policy \(\tilde{m}\) such that

\[
\int_0^T p(t)\tilde{m}(t)dt < \int_0^T p(t)m^*(t)dt. \tag{34}
\]

By (22), it holds that

\[
\tilde{m}(t) - m^*(t) = \begin{cases} 
\tilde{m}(t) - \kappa \leq 0 & \text{for } t \in \phi(\zeta^*), \\
\tilde{m}(t) - 0 \geq 0 & \text{for } t \notin \phi(\zeta^*). 
\end{cases} \tag{35}
\]

Note that \(\phi \equiv \phi(\zeta^*)\) is the collection of time points at which the electricity price \(p\) is less than or equal to \(\zeta^*\). Thus,

\[
\int_0^T p(t)\tilde{m}(t)dt - \int_0^T p(t)m^*(t)dt = \int_\phi p(t)(\tilde{m}(t) - m^*(t))dt + \int_{\phi^c} p(t)(\tilde{m}(t) - m^*(t))dt. \tag{36}
\]

Using (35), we deduce that the right hand side of (36) is at least

\[
\int_\phi \zeta^*(\tilde{m}(t) - m^*(t))dt + \int_{\phi^c} \zeta^*(\tilde{m}(t) - m^*(t))dt = \zeta^* \int_0^T \tilde{m}(t)dt - \zeta^* \int_0^T m^*(t)dt = 0, \tag{37}
\]

where the last equality in (37) is owing to (11). This implies

\[
\int_0^T p(t)\tilde{m}(t)dt - \int_0^T p(t)m^*(t)dt \geq 0,
\]

which contradicts the assumption (34). Hence, there is no feasible charging policy that can beat \(m^*\) in terms of minimizing the charging cost. \[\square\]
Proof of Theorem 4. Note that the first case follows directly from Proposition 1 and the construction (26) - (29). It remains to show the second case. Suppose for the sake of contradiction that there exists $b < b^*$ for which $(\tilde{x}(0), m^*)$ is an optimal solution. Then it must be the case that $\tilde{x}(t) \geq 0$ for all $0 \leq t \leq \tau$. Note that $x^*(0)$ in (26) is the smallest value that makes $x(t) \geq 0$ for $0 \leq t \leq \tau$. Therefore, $\tilde{x}(t) \geq 0$ implies $\tilde{x}(0) \geq x^*(0)$. Also note that $\tilde{x}$ satisfies

$$\tilde{x}(t) = \tilde{x}(0) + \mu \int_0^t m^*(u) du - \Lambda(t).$$

Combining with (27) yields

$$\tilde{x}(t) - x^*(t) = \tilde{x}(0) - x^*(0) \geq 0 \quad \text{for all} \quad 0 \leq t \leq \tau.$$  \hspace{1cm} (39)

To proceed, suppose the right hand side of (29) reaches its maximum at time $u$, i.e., $m^*(u) + x^*(u) = b^*$. Then

$$m^*(u) \leq b - \tilde{x}(u) < b^* - x^*(u) = m^*(u),$$

where the first inequality follows from the constraint (9) and the assumption $\tilde{x}(t) \geq 0$ for all $0 \leq t \leq \tau$, the second inequality is due to (39) and the assumption $b < b^*$. But this leads to a contradiction and thus completes the proof. \qed