APPENDIX TO ESTIMATING WAITING TIMES WITH THE TIME-VARYING LITTLE'S LAW

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Abstract

When waiting times cannot be observed directly, Little's law can be applied to estimate the average waiting time by the average number in system divided by the average arrival rate, but that simple indirect estimator tends to be biased significantly when the arrival rates are time-varying and the service times are relatively long. This study shows that the bias in that indirect estimator can be estimated and reduced by applying the time-varying Little's law (TVLL). The new methods are shown to be effective in estimating the bias in the indirect estimator and reducing it, using simulations of multi-server queues and data from a call center. This appendix provides additional details about those experiments.

Keywords: Little's law; time-varying Little's law; $L = \lambda W$; estimation; estimation bias; estimating the average wait

Contents

1 Overview

We present supporting material in this appendix to the main paper. In §2 we present additional results for the simulation experiments discussed in Section 8. In §3 we present additional results for the call center data examined in Section 9. Throughout this appendix, we refer to equations as numbered in the main paper.

2 Additional Results for the Simulation Experiments in Section 8

In this section, we provide additional information about the simulation experiments summarized in Section 8 of the main paper. Recall that we consider the $M_t/GI/s_t + M$ multi-server queueing model, having a nonhomogeneous Poisson arrival process (the M_t), i.i.d. service times distributed according to a random variable S with a general distribution (the GI), a time-varying staffing level (number of servers, the s_t) and customer abandonment with i.i.d. exponentially random patience times (the $+M$). The arrival process, service times and patience times are assumed to be mutually independent. Consistent with many call centers, we let the mean patience time be 2. We simulated each model specification using matlab, performing 100 replications.

In §2.1, we first describe the three types of nonhomogeneous Poisson arrival process used in our experiments and their arrival rate approximations by constant, linear and quadratic functions. We then explain the different service time distributions and staffing method we use and show the performance of different models in §2.2. Comparison of the performance of different estimators follows in §2.3. §2.4 gives additional simulation results when we do 1000, instead of 100 replications for the most variable H_2 service time distribution. As in the main paper, we look further into three special cases to gain more insights into these different estimators: longer service times in §2.5, decreasing arrival rate in §2.6 and sinusoidal arrival rate in §2.7.

2.1 The Three Arrival Rate Processes

As the actual arrival processes, we initially consider three nonhomogeneous Poisson arrival process, having constant ($\lambda(t) = 45$), linear ($\lambda(t) = 36 + 3t$) and quadratic ($\lambda(t) = 53.333 + 2.222t - 0.185t^2$) arrival rate functions. We assume that the system starts empty at time $t = -12$, and generate arrivals according to these processes over the interval $[-12, 12]$. Figure 1 shows the three different arrival rate functions. We generated these arrival processes by thinning a homogeneous arrival process with rate λ^* for $\lambda^* \geq \lambda(t)$, $-12 \leq t \leq 12$. The homogeneous Poisson process generates potential arrivals. We then let a potential arrival at time t be an actual arrival in the nonhomogeneous arrival process with probability $\lambda(t)/\lambda^*$.

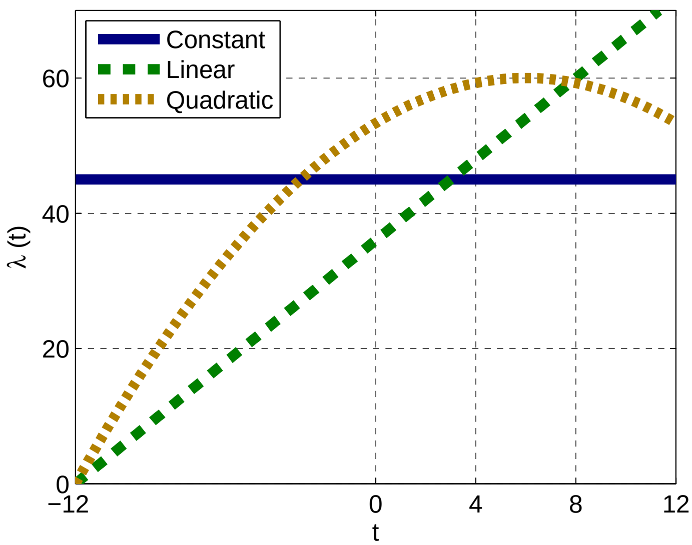

Figure 1: The constant, linear and quadratic arrival rate functions used in simulation experiments.

In any application, we fit candidate nonhomogeneous Poisson processes to the arrival data. Accordingly, with these three arrival processes, we also fit all three of the arrival processes, having constant, linear and quadratic arrival rate functions. Since the constant arrival rate function is a special case of the linear and quadratic arrival rate functions, for the constant arrival rate function, we see the impact of having more parameters than actually would be needed. On the other hand, for nonlinear quadratic arrival rate functions, we see the impact of fitting fewer parameters than needed. Presumably, fitting parameters to the specified process is best, but we see how important that is.

We generated 100 replications of each of the three time-varying arrival processes. Our goal is to compare the performance of different waiting time estimators over the interval $[0, t]$ for $t = 4$ and $t = 8$. Hence, in each replication, we need to approximate the arrival rate by constant, linear and quadratic functions over these subintervals. In order to better estimate the arrival rate function and to account for the time lag in the approximations, we estimate the arrival rate over $[-4, 4]$ and $[-8, 8]$ for the performance intervals $[0, 4]$ and $[0, 8]$, respectively.

Table 1 shows the average of the estimated parameters for the arrival rate functions over 100 replications in each case. When we approximate the arrival rate by constant function, we consider intervals [0, 4] and [0, 8] instead of [−4, 4] and [−8, 8]. The halfwidths of 95% confidence intervals for all estimates are also reported.

		Constant	Linear		Quadratic		
Int.	*Arrival*	$\bar{\lambda}(t)$	a	b	a	b	c
[−4, 4]	*Constant*	45.2 ± 0.7	44.9 ± 0.5	0.099 ± 0.197	44.9 ± 0.7	0.099 ± 0.197	-0.013 ± 0.084
	Linear	41.7 ± 0.6	35.8 ± 0.5	2.907 ± 0.162	35.4 ± 0.6	2.907 ± 0.162	0.069 ± 0.083
	Quadratic	56.6 ± 0.7	52.1 ± 0.5	2.167 ± 0.212	52.8 ± 0.8	2.167 ± 0.212	-0.120 ± 0.112
[−8, 8]	*Constant*	45.1 ± 0.5	45.0 ± 0.4	0.017 ± 0.058	44.7 ± 0.6	0.017 ± 0.058	0.011 ± 0.018
	Linear	48.0 ± 0.5	35.9 ± 0.3	3.025 ± 0.064	35.6 ± 0.5	3.025 ± 0.064	0.016 ± 0.016
	Quadratic	58.3 ± 0.6	49.5 ± 0.4	2.185 ± 0.071	53.1 ± 0.5	2.185 ± 0.071	-0.167 ± 0.015

Table 1: Fitting constant, linear and quadratic arrival rate functions over the intervals [−4, 4] and [−8, 8] to the arrival data for each arrival process; estimates and halfwidths of 95% confidence intervals over 100 replications.

2.2 Average Performance of Different Models

We let the mean service time be $E[S] = 1$, so that we are measuring time in units of mean service times. We consider three service time distributions: exponential (M, having parameters $\gamma_W^2 = \theta_W^3 = 1$), Erlang E_4 (less variable, a sum of four i.i.d. exponentials, having parameters $\gamma_W^2 = 0.6125$ and $\theta_W^3 = 0.3125$) and hyperexponential H_2 (more variable, a mixture of two exponentials, having parameters $\gamma_W^2 = 3.0$ and $\theta_W^3 = 15.0$). The third H_2 parameter is chosen to produce balanced means as in (3.7) of [6]; the cdf is $P(S \leq x) \equiv 1 - p_1 e^{-\lambda_1 x} - p_2 e^{-\lambda_2 x}$, where $p_1 = 0.0918$ and $p_2 = 0.9082$, while $\lambda_i = 2p_i$, yielding $p_i/\lambda_i = 1/2$ for $i = 1, 2$, $c^2 = 5$ and $E[S^3] = 90$.

The time-varying staffing is chosen to stabilize the performance at typical performance levels, following the method of [3] and [2]. In particular, the staffing is set using the square root staffing formula in (33), i.e., $s(t) \equiv \lceil m(t) + \beta\sqrt{m(t)} \rceil$, where $m(t)$ is the offered load and $\lceil x \rceil$ is the least integer greater than or equal to x. The offered load is $m(t) \equiv E[L(t)]$ in the associated IS model, which has formula (8) with the service time S playing the role of the waiting time W there. We consider three cases for the quality-of-service (QoS) parameter β: 0, 1 and 2. With abandonment in the model, the first two cases produce typical performance, while $\beta = 2$ corresponding to high QoS, producing performance close to the IS model. For the three arrival rate functions, we have explicit expressions for the offered load $m(t)$ and thus the staffing via (33). For the linear arrival rate function $36 + 3t$, the offered load is $m(t) = 36 + 3t - 3\gamma_W^2$. For the quadratic arrival rate, the offered load is $m(t) = 53.333 - 2.222\gamma_W^2 - 0.370\theta_W^3 + (2.222 + 0.370\gamma_W^2)t - 0.185t^2$. For the constant arrival rate, $m(t) = \lambda(t) = 45$.

As mentioned above, we generate 100 replications of each of the three time-varying arrival processes. For the service times, we again generate 100 replications of the three different distributions. We then combine each of the arrival process with service time process based on three different staffing with varying quality-of-service (QoS) parameter β in (33): 0, 1 and 2. This design gives us 27 different models in total, which is all of the possible combinations of the 3 arrival rate functions, 3 service time distributions and 3 QoS parameters. To minimize randomness in our experiments, we generate one set of patience times and use that for all models.

Table 2 shows the average performance of each of the models in terms of the average waiting time ($E[W]$), percent of arrivals delayed, and percent of arrivals abandoning. In each replication, we average the performance measures over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. The reported numbers are their averages over the 100 replications and the halfwidths of 95% confidence intervals.

Since we have staffed in order to stabilize performance, these performance measures should be close to corresponding steady-state values. Specifically, in Section 8.2, we analyzed the performance of constant arrival rate and exponential service times analytically using the algorithm described in [7]. Using the constant arrival rate $\lambda = 45$ and $E[S] = 1$, the stationary offered load is $m = \lambda E[S] = 45$, so that the staffing level with Qos parameter $\beta = 0$, 1 and 2 is $s = 45$, 52 and 59. In these three cases, the mean waiting time (in system) is 1.043, 1.0077 and 1.0008; the variance of the waiting time is 0.923, 0.938 and 0.986; the probability of delay is 0.602, 0.185 and 0.028; the probability of abandonment is 0.049, 0.0084 and 0.0008. Comparing these theoretical values to the experiment results in Table 2, we see that the performance is very similar; the numbers match almost exactly when we consider constant arrival rate and exponential service times (the first three rows of Table 2), but the performance in other models are also similar for the same value of the QoS parameter β.

We now further investigate whether the performance is indeed stabilized over the target intervals $[0, 4]$ and $[0, 8]$. Paralleling Figures 2-5 in the main paper, Figures 2 - 55 provide more information on the performance of different models over time. Performance of models with constant arrival rate is shown in Figures 2 - 19, with linear arrival rates in Figures 20 - 37 and with quadratic arrival rates in Figures 38 - 55. From these plots, we see that the performance is indeed typically stabilized approximately by time $t = 0$. However, with H_2 service time distribution (where the service time is highly variable) the plots (for instance, see Figure 11) suggest that we might need more time to

Performance Interval			[0, 4]			[0, 8]		
Arrival	*GI*	β	$E[W]$	*%Delayed*	*%Aban.*	$E[W]$	*%Delayed*	*%Aban.*
Constant	M	0	1.06 ± 0.02	61.3 ± 5.7	4.73 ± 0.94	1.08 ± 0.02	62.8 ± 4.8	4.18 ± 0.63
		1	1.01 ± 0.01	17.3 ± 4.2	0.89 ± 0.33	1.02 ± 0.01	16.8 ± 3.0	0.60 ± 0.19
		2	1.00 ± 0.01	3.6 ± 1.7	0.09 ± 0.08	1.01 ± 0.01	2.7 ± 1.0	0.05 ± 0.04
	H_2	0	1.08 ± 0.04	48.6 ± 7.0	4.35 ± 1.11	1.08 ± 0.03	52.6 ± 5.9	4.20 ± 0.86
		1	1.03 ± 0.04	13.0 ± 4.2	0.68 ± 0.38	1.03 ± 0.03	14.5 ± 3.3	0.58 ± 0.24
		2	1.03 ± 0.03	2.2 ± 1.3	0.05 ± 0.04	1.02 ± 0.03	2.0 ± 0.9	0.03 ± 0.03
	E_4	0	1.05 ± 0.01	64.1 ± 4.9	4.09 ± 0.74	1.06 ± 0.01	63.0 ± 4.2	3.41 ± 0.48
		1	1.01 ± 0.01	17.1 ± 3.8	0.54 ± 0.19	1.01 ± 0.01	16.2 ± 2.6	0.38 ± 0.11
		2	1.00 ± 0.01	3.4 ± 1.4	0.03 ± 0.03	1.00 ± 0.01	2.5 ± 0.8	0.02 ± 0.02
Linear	M	0	1.04 ± 0.02	52.3 ± 5.6	4.48 ± 0.88	1.05 ± 0.01	54.1 ± 4.4	3.80 ± 0.61
		1	1.00 ± 0.02	15.6 ± 3.7	0.70 ± 0.28	1.01 ± 0.01	16.4 ± 3.0	0.54 ± 0.16
		2	1.00 ± 0.02	2.4 ± 1.1	0.08 ± 0.06	1.00 ± 0.01	2.3 ± 0.9	0.05 ± 0.03
	H_2	0	1.04 ± 0.04	52.3 ± 5.6	5.28 ± 1.11	1.06 ± 0.03	52.8 ± 4.7	4.56 ± 0.78
		1	1.01 ± 0.04	17.0 ± 4.2	1.00 ± 0.46	1.02 ± 0.03	18.4 ± 3.5	0.79 ± 0.29
		2	1.00 ± 0.04	3.5 ± 1.8	0.16 ± 0.11	1.01 ± 0.02	3.1 ± 1.3	0.09 ± 0.06
	E_4	0	1.04 ± 0.01	53.4 ± 5.1	3.91 ± 0.68	1.05 ± 0.01	56.1 ± 4.1	3.40 ± 0.47
		1	1.01 ± 0.01	14.7 ± 3.1	0.49 ± 0.18	1.01 ± 0.01	16.1 ± 2.4	0.42 ± 0.12
		2	1.01 ± 0.01	1.8 ± 0.8	0.05 ± 0.05	1.01 ± 0.00	2.3 ± 0.8	0.03 ± 0.03
Quadratic	M	0	1.06 ± 0.02	56.4 ± 5.7	2.81 ± 0.54	1.06 ± 0.01	59.6 ± 4.6	3.31 ± 0.51
		1	1.02 ± 0.01	15.8 ± 3.6	0.33 ± 0.13	1.02 ± 0.01	17.7 ± 3.1	0.44 ± 0.12
		2	1.01 ± 0.01	1.8 ± 0.9	0.01 ± 0.01	1.01 ± 0.01	2.6 ± 0.9	0.03 ± 0.02
	H_2	0	1.09 ± 0.04	61.7 ± 5.8	4.65 ± 0.94	1.08 ± 0.03	61.6 ± 5.3	4.98 ± 0.90
		1	1.03 ± 0.03	20.6 ± 4.1	0.71 ± 0.31	1.03 ± 0.02	22.3 ± 4.0	0.95 ± 0.34
		2	1.02 ± 0.03	3.6 ± 1.8	0.06 ± 0.05	1.02 ± 0.02	4.7 ± 1.8	0.13 ± 0.08
	E_4	0	1.05 ± 0.01	57.3 ± 5.4	2.43 ± 0.45	1.05 ± 0.01	59.8 ± 4.3	2.71 ± 0.41
		1	1.01 ± 0.01	14.1 ± 3.1	0.27 ± 0.12	1.01 ± 0.01	15.4 ± 2.7	0.32 ± 0.11
		2	1.00 ± 0.01	2.5 ± 1.4	0.02 ± 0.02	1.00 ± 0.00	2.6 ± 1.0	0.03 ± 0.02

Table 2: Average performance of the 27 different models, averaged over periods of length 0.5 in the intervals [0, 4] and [0, 8].

reach the steady state.

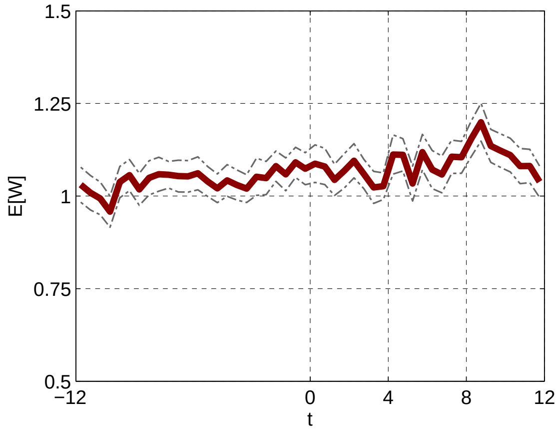

Figure 2: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

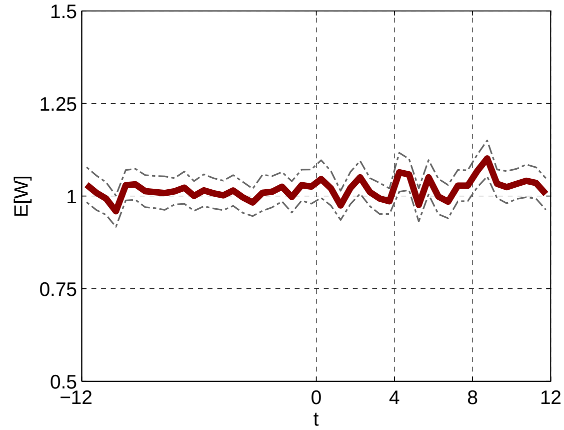

Figure 3: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

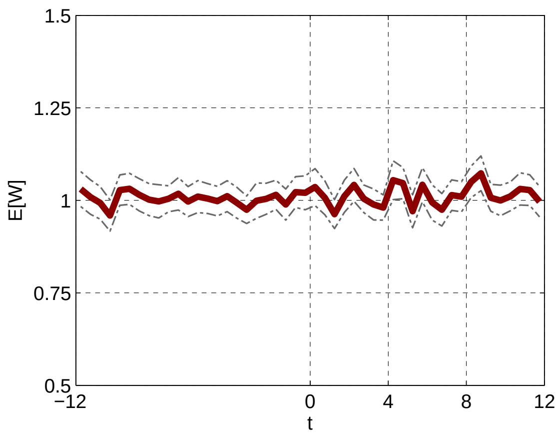

Figure 4: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

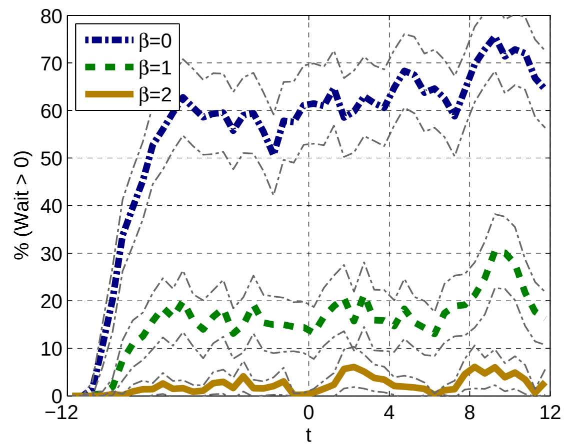

Figure 5: Average percent of arrivals delayed over periods of length 0.5.

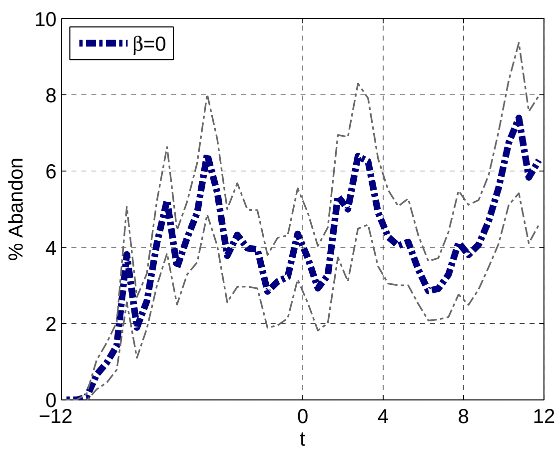

Figure 6: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

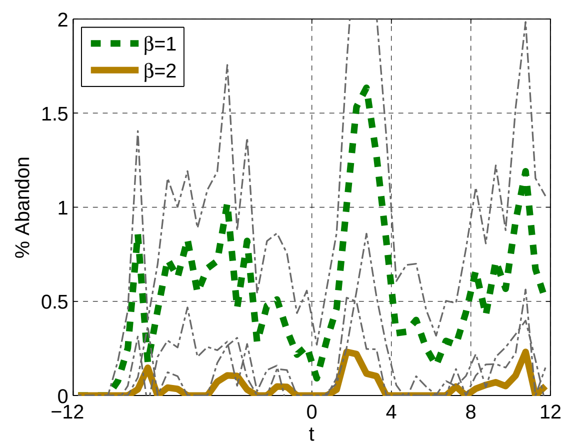

Figure 7: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 2-7: Constant arrival rate and M service time distribution

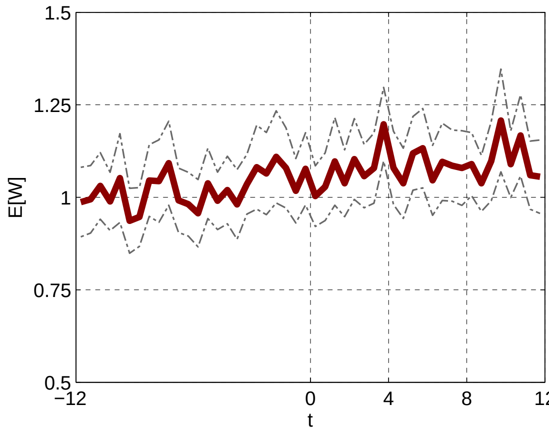

Figure 8: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

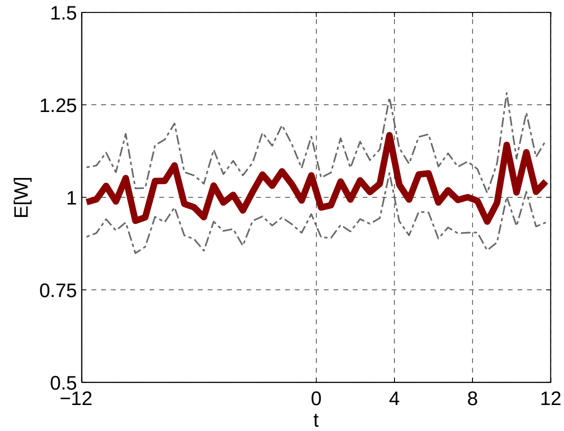

Figure 9: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

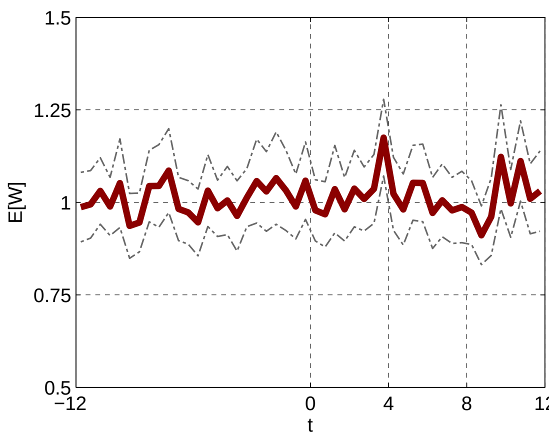

Figure 10: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

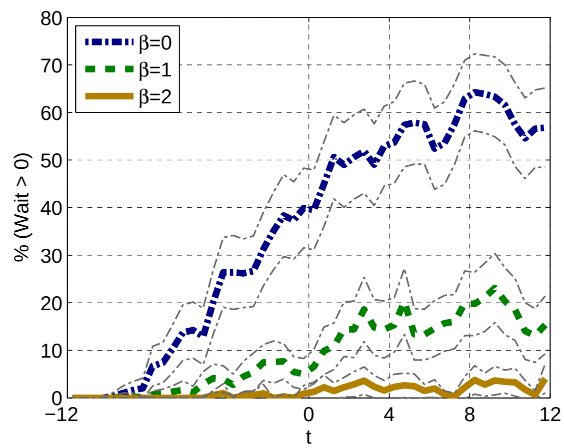

Figure 11: Average percent of arrivals delayed over periods of length 0.5.

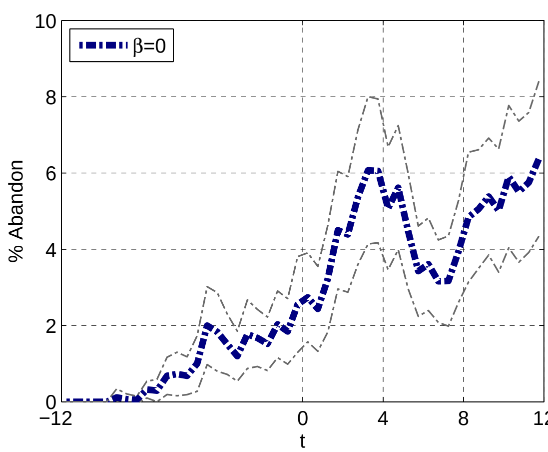

Figure 12: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

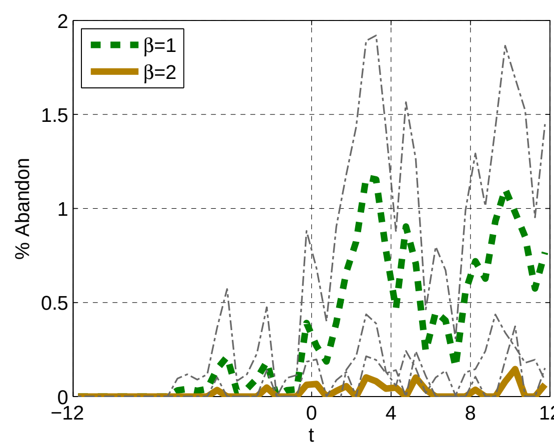

Figure 13: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 8-13: Constant arrival rate and H_2 service time distribution

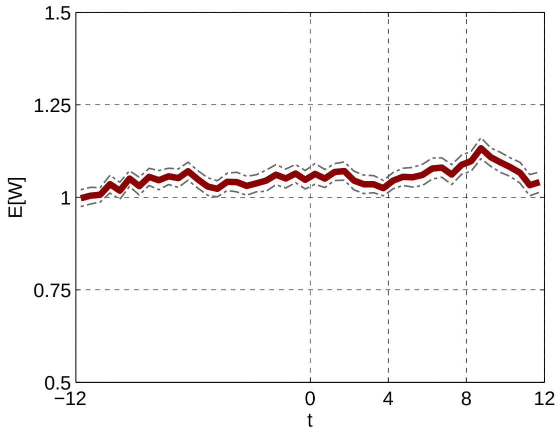

Figure 14: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

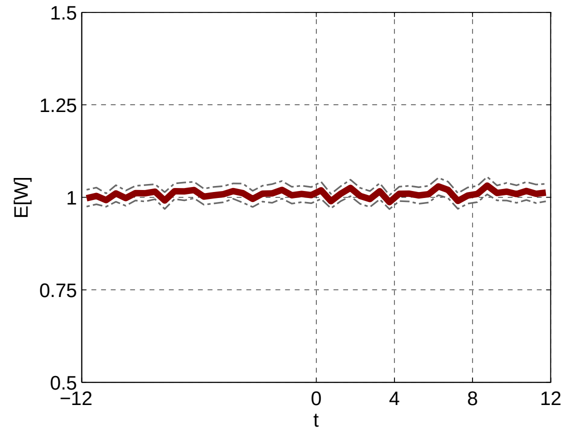

Figure 15: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

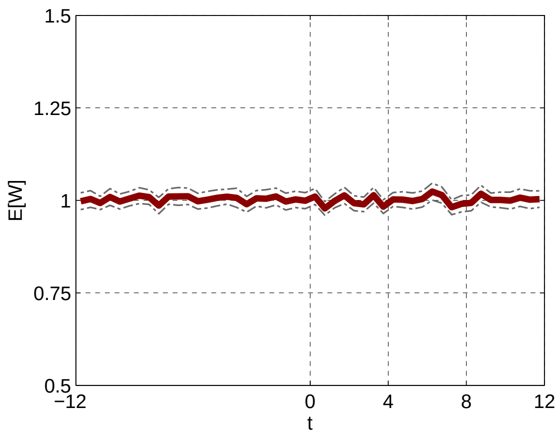

Figure 16: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

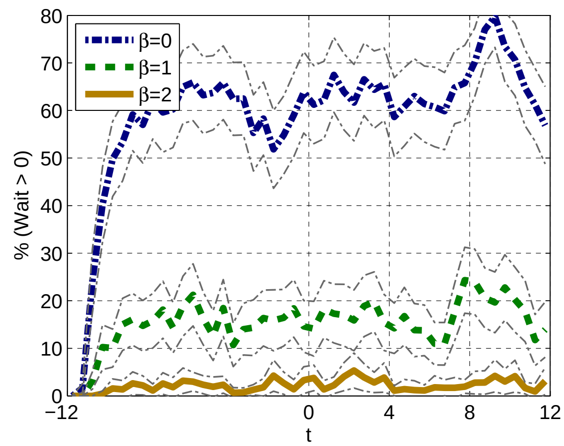

Figure 17: Average percent of arrivals delayed over periods of length 0.5.

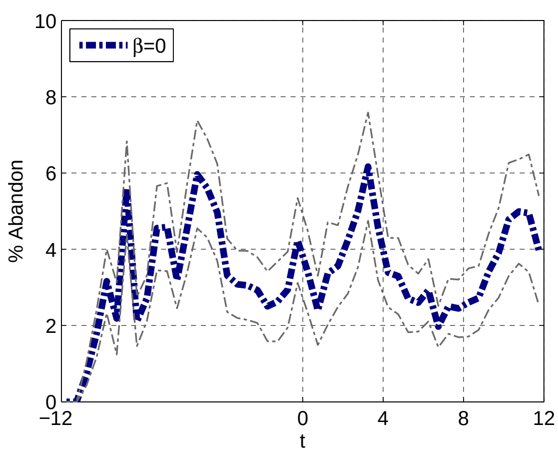

Figure 18: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

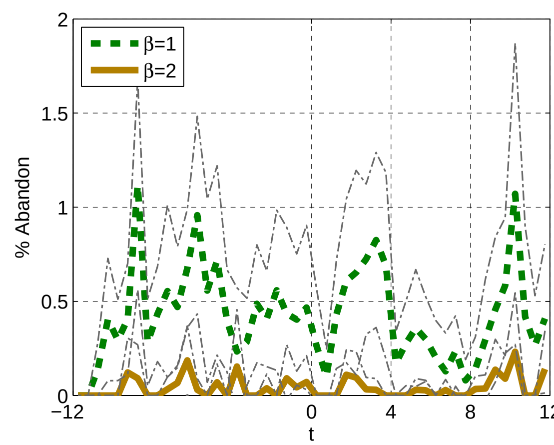

Figure 19: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 14-19: Constant arrival rate and E_4 service time distribution

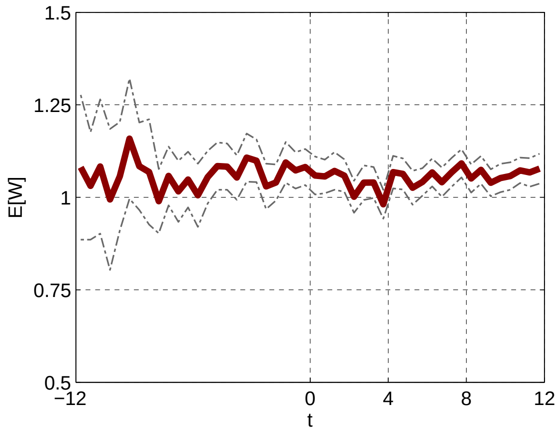

Figure 20: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

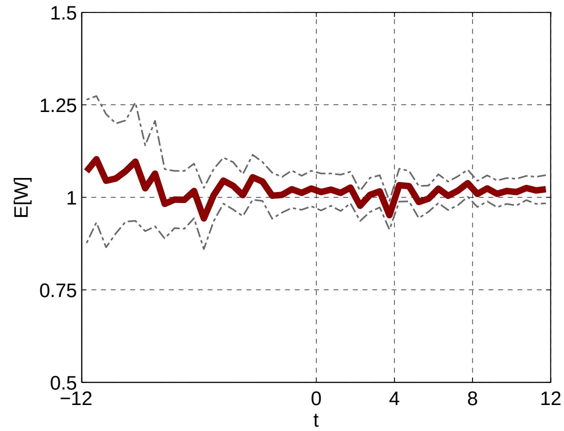

Figure 21: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

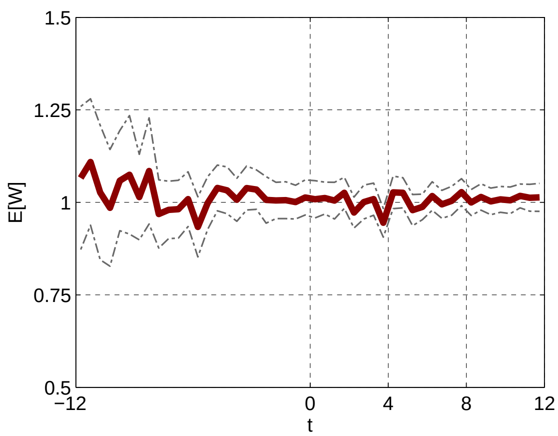

Figure 22: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

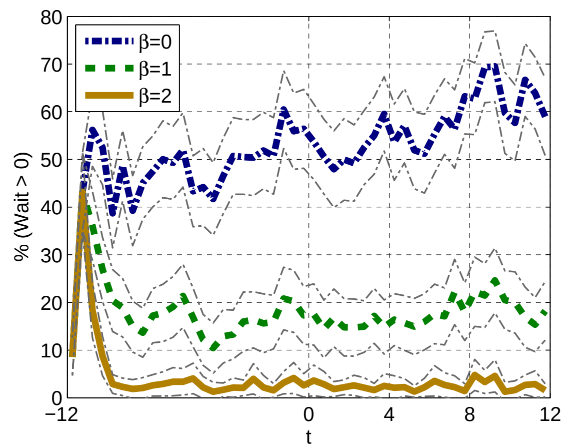

Figure 23: Average percent of arrivals delayed over periods of length 0.5.

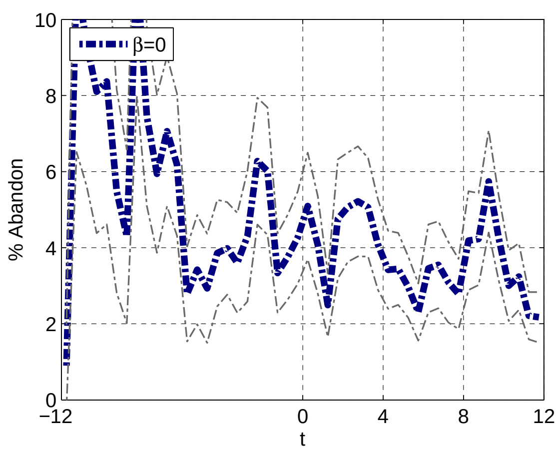

Figure 24: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

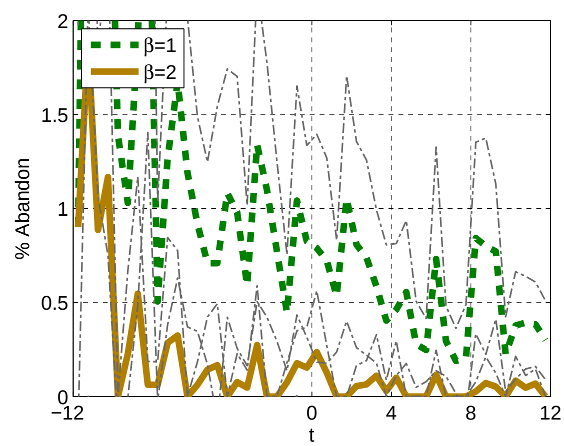

Figure 25: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 20-25: Linear arrival rate and M service time distribution

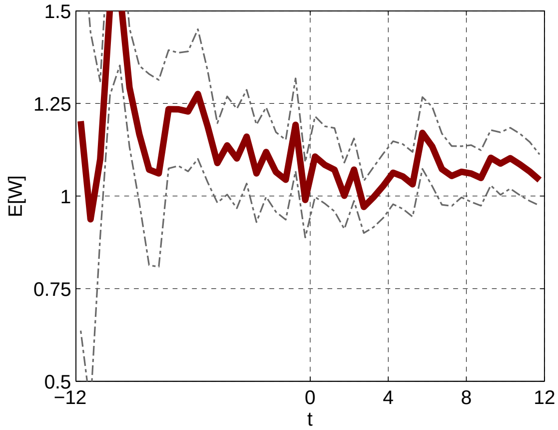

Figure 26: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

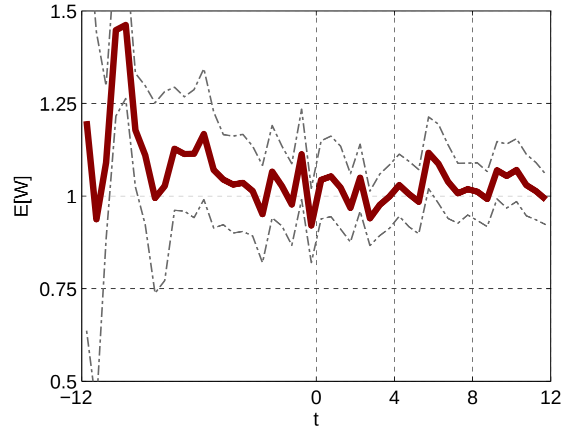

Figure 27: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

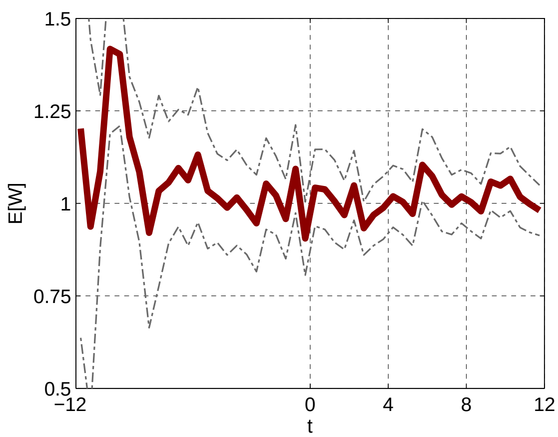

Figure 28: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

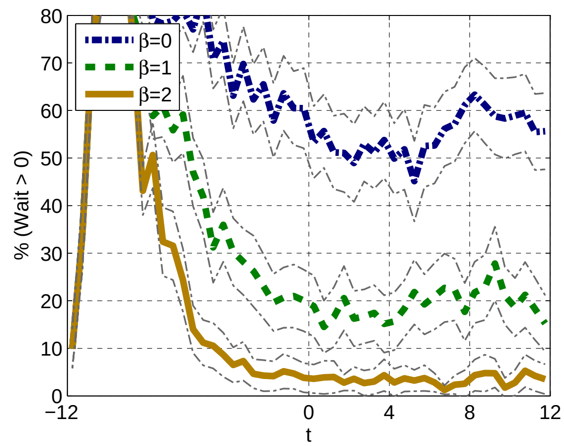

Figure 29: Average percent of arrivals delayed over periods of length 0.5.

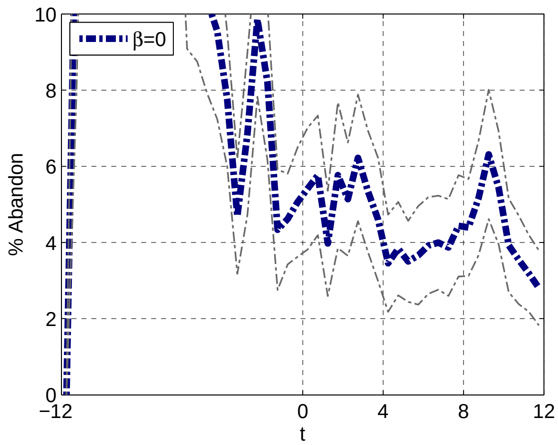

Figure 30: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

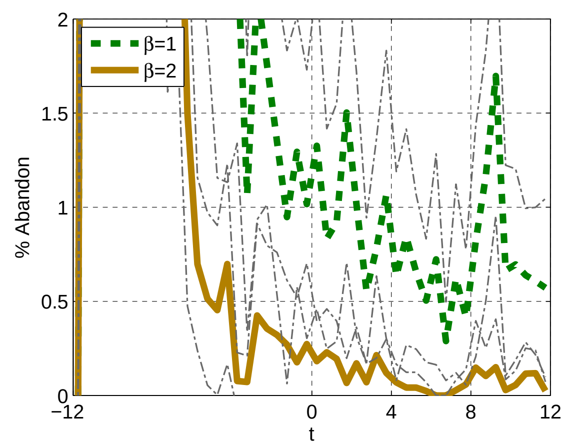

Figure 31: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 26-31: Linear arrival rate and H_2 service time distribution

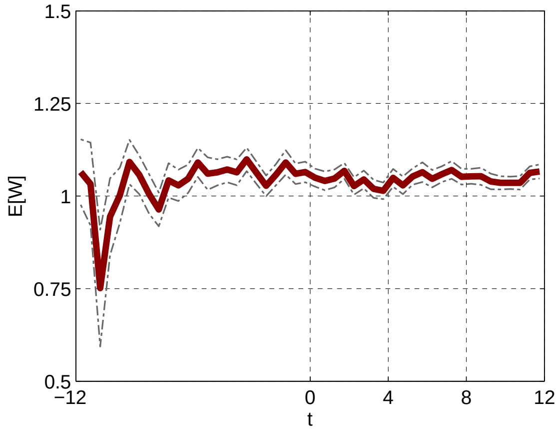

Figure 32: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

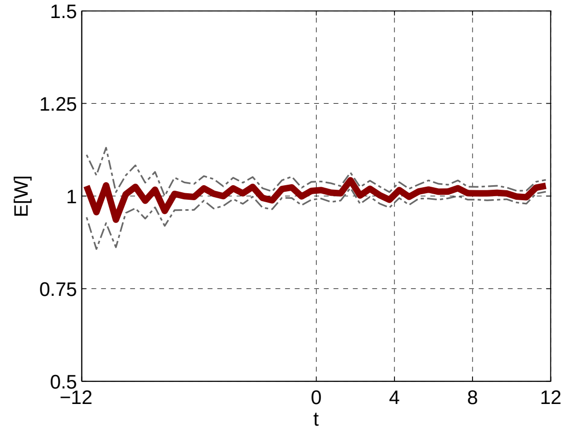

Figure 33: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

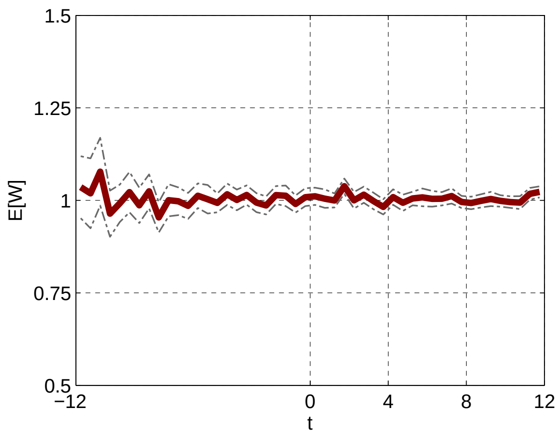

Figure 34: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

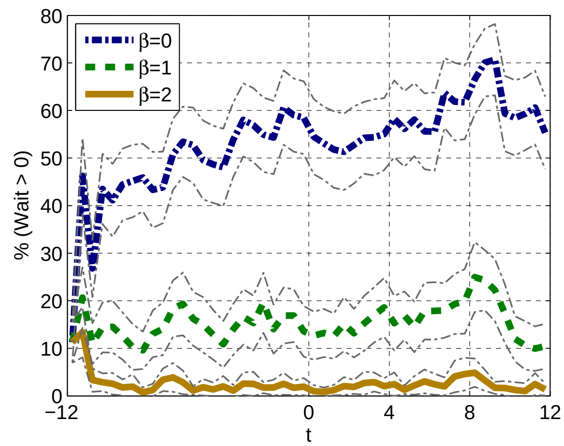

Figure 35: Average percent of arrivals delayed over periods of length 0.5.

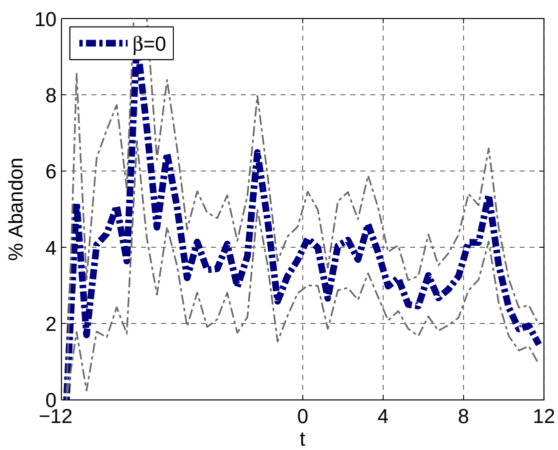

Figure 36: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

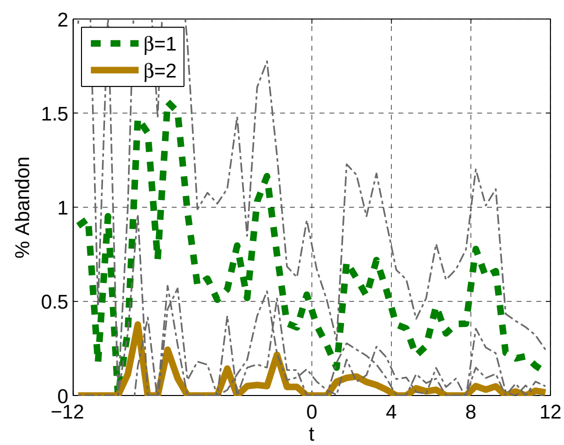

Figure 37: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 32-37: Linear arrival rate and E_4 service time distribution

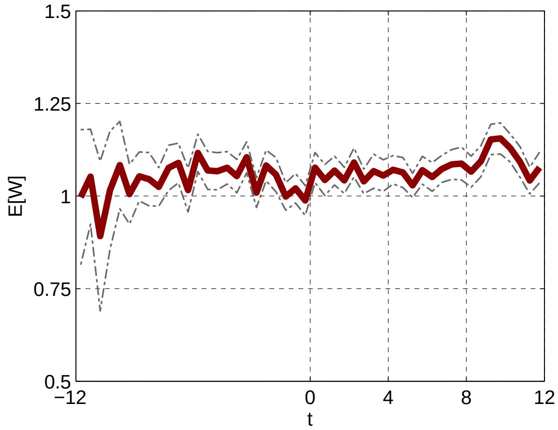

Figure 38: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

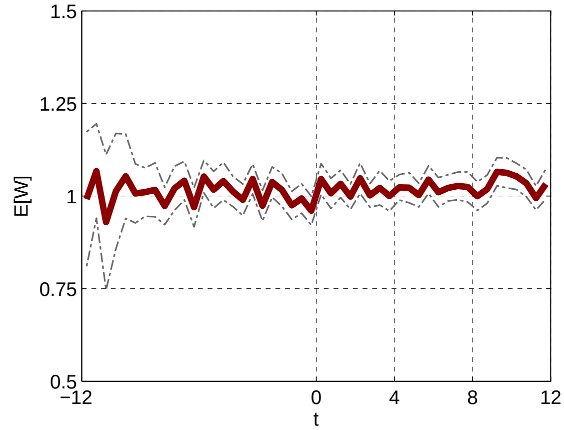

Figure 39: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

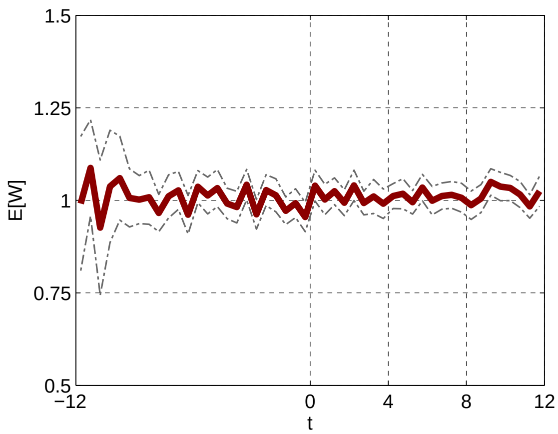

Figure 40: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

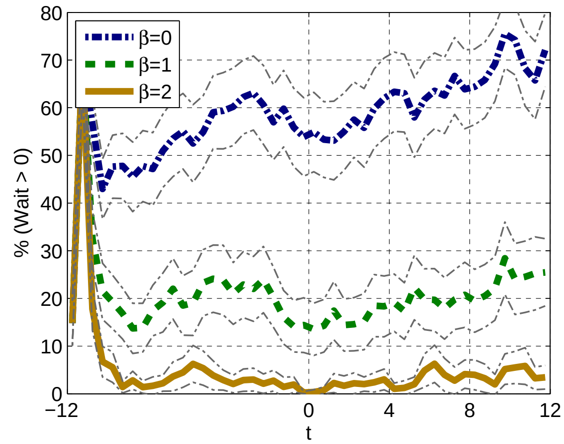

Figure 41: Average percent of arrivals delayed over periods of length 0.5.

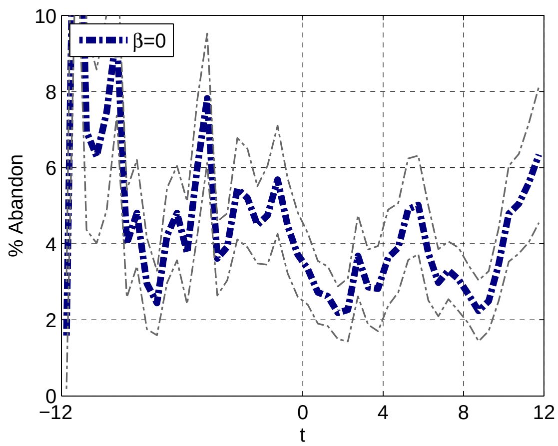

Figure 42: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

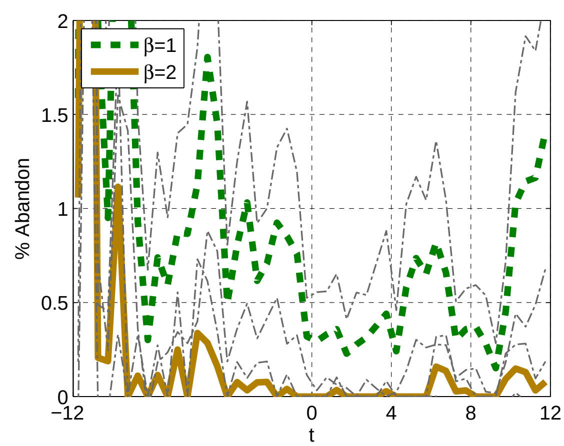

Figure 43: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 38-43: Quadratic arrival rate and M service time distribution

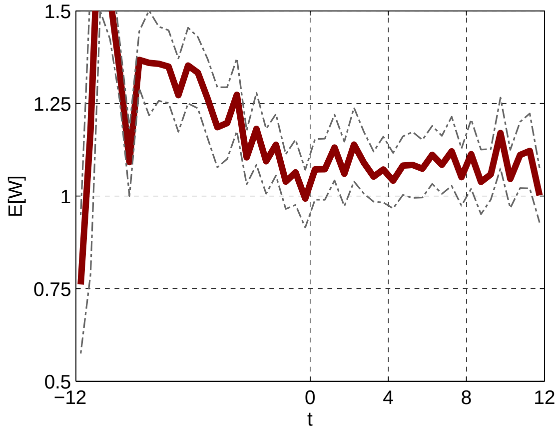

Figure 44: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

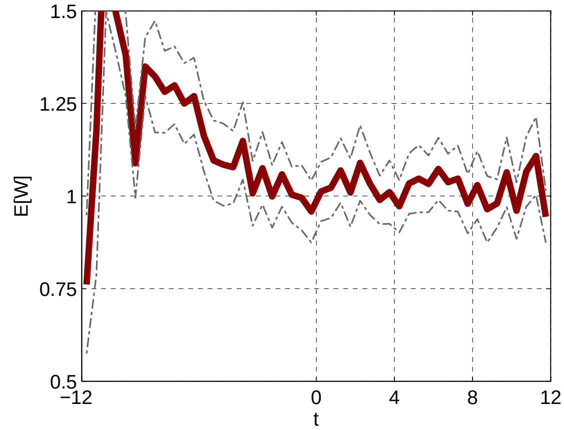

Figure 45: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

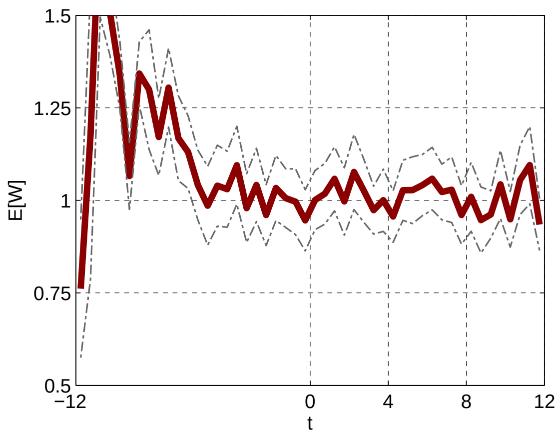

Figure 46: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

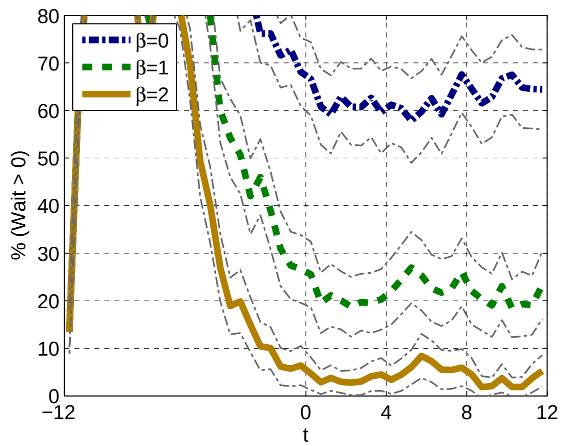

Figure 47: Average percent of arrivals delayed over periods of length 0.5.

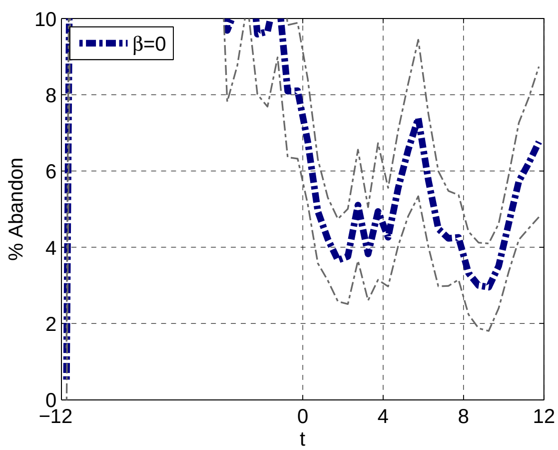

Figure 48: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

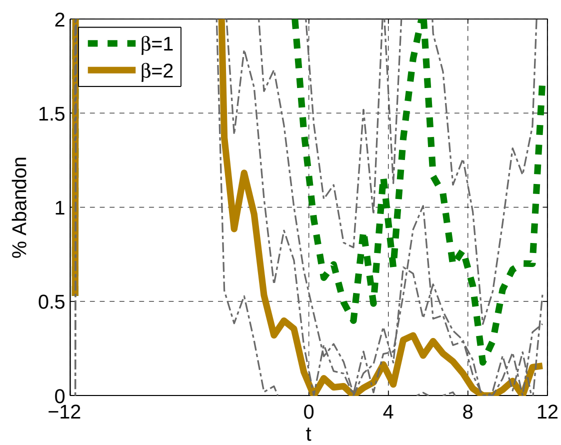

Figure 49: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 44-49: Quadratic arrival rate and H_2 service time distribution

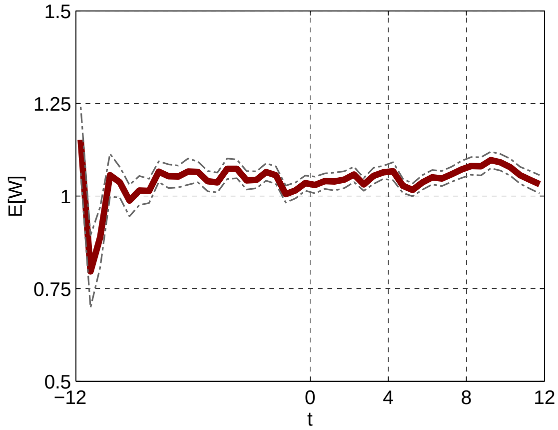

Figure 50: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

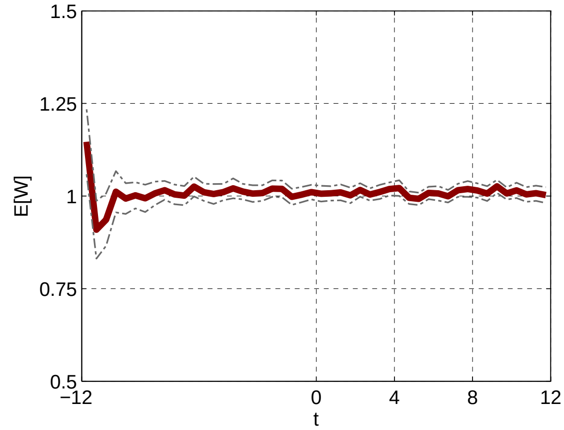

Figure 51: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

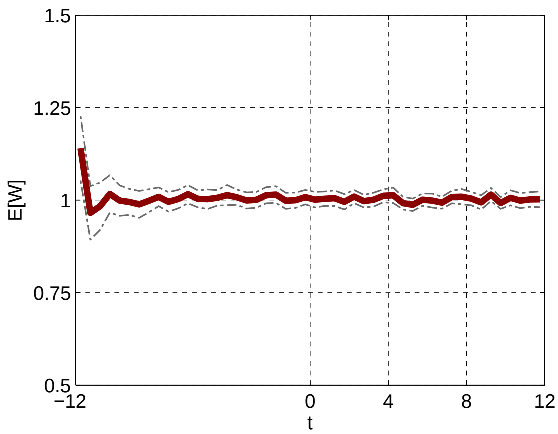

Figure 52: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

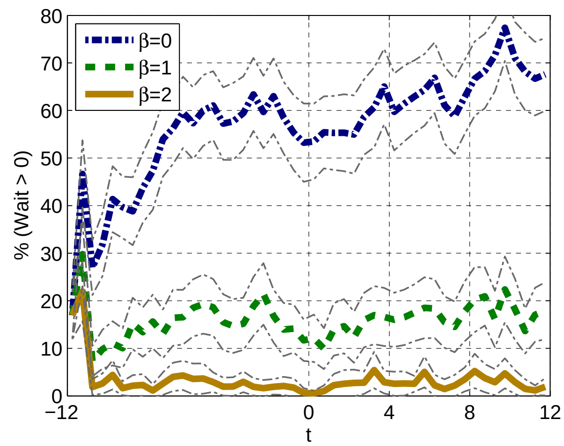

Figure 53: Average percent of arrivals delayed over periods of length 0.5.

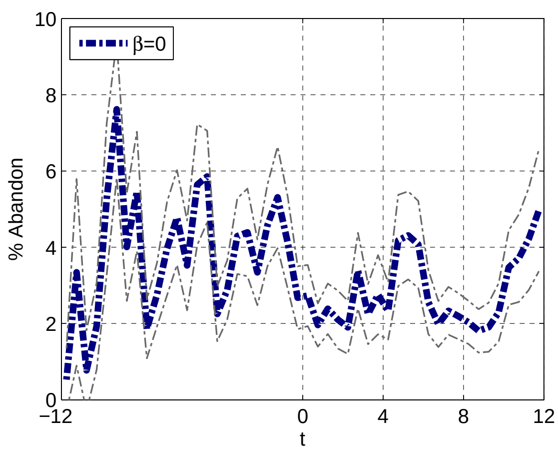

Figure 54: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

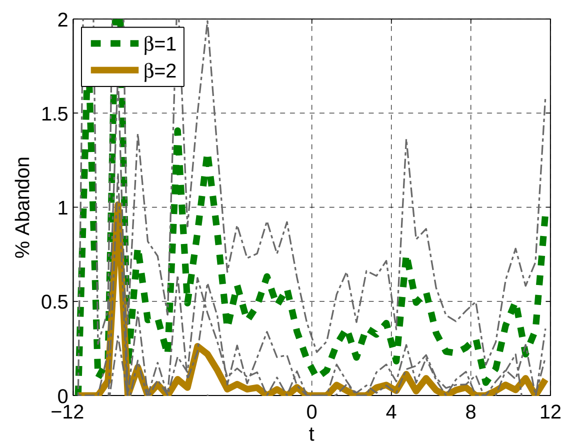

Figure 55: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 50-55: Quadratic arrival rate and E_4 service time distribution

2.3 Estimation Results

We now describe the estimation results. We consider ten different methods for the constant, linear and quadratic arrival rate functions. The first estimator is the direct average $\bar{W}(t)$ in (1), which we could not use if the waiting times were not actually observed. The second is the indirect estimator $\bar{W}_{L,\lambda}(t)$ in (2) based on LL, whose bias we want to reduce. Then we give the estimators $\bar{W}_{L,\lambda,r}(t)$ in (4) from [4] and its extension $\bar{W}_{L,\lambda,r,\gamma}(t)$ in (27), which are based on the sample path relation in (3). Next we consider estimators based on the TVLL. We do not consider the direct estimator $\bar{W}_{tvll}(t)$ provided by Theorem 2 here, because we consider it covered by the linear approximation, as we demonstrated in §8.4. For the TVLL-based estimators, we consider the estimator $\bar{W}_{L,\lambda,l}(t)$ from §4 based on the fitted linear arrival rate function, its perturbation refinement $\bar{W}_{L,\lambda,l,p}(t)$ from §5 and the estimated best of these two, $\bar{W}_{L,\lambda,l,b}(t)$, chosen as the one with the smaller confidence interval. Finally there are the corresponding three estimators from §7 based on the fitted quadratic arrival rate function.

Since we consider multi-server queues with reasonable staffing (specified below), the waiting times (time spent in system) do not differ greatly from the service times. For customers that are served, the waiting times are somewhat longer because of the time spent in queue, but that usually is relatively short compared to the service times. Longer waiting times in queue are reduced by customer abandonment. Thus, in our TVLL linear and quadratic estimation procedures, we approximate the unknown (γ_W^2, θ_W^3) by the specified (γ_S^2, θ_S^3).

Tables 3 - 5 show the estimated waiting times by these ten methods.

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.062 ± 0.019	1.066 ± 0.019	1.064 ± 0.019	1.064 ± 0.019	1.069 ± 0.019	1.065 ± 0.018	1.065 ± 0.018	1.068 ± 0.018	1.062 ± 0.017	1.062 ± 0.017
		1	1.014 ± 0.015	1.015 ± 0.015	1.017 ± 0.015	1.017 ± 0.015	1.018 ± 0.015	1.015 ± 0.015	1.015 ± 0.015	1.017 ± 0.014	1.013 ± 0.014	1.013 ± 0.014
		2	1.005 ± 0.013	1.005 ± 0.014	1.006 ± 0.013	1.006 ± 0.013	1.008 ± 0.013	1.005 ± 0.013	1.008 ± 0.013	1.007 ± 0.013	1.003 ± 0.013	1.007 ± 0.013
	[0, 8]	0	1.077 ± 0.017	1.068 ± 0.016	1.079 ± 0.018	1.079 ± 0.018	1.070 ± 0.016	1.068 ± 0.016	1.068 ± 0.016	1.072 ± 0.016	1.069 ± 0.015	1.069 ± 0.015
		1	1.019 ± 0.012	1.014 ± 0.012	1.020 ± 0.012	1.020 ± 0.012	1.016 ± 0.011	1.014 ± 0.011	1.016 ± 0.011	1.017 ± 0.011	1.015 ± 0.011	1.017 ± 0.011
		2	1.009 ± 0.011	1.005 ± 0.011	1.010 ± 0.011	1.010 ± 0.011	1.006 ± 0.010	1.005 ± 0.011	1.006 ± 0.010	1.008 ± 0.010	1.006 ± 0.010	1.008 ± 0.010
	Avg		1.031	1.029	1.033	1.033	1.031	1.029	1.030	1.032	1.028	1.030
H_2	[0, 4]	0	1.069 ± 0.036	1.020 ± 0.031	1.036 ± 0.033	1.068 ± 0.046	1.033 ± 0.034	1.021 ± 0.033	1.021 ± 0.033	0.201 ± 0.273	0.965 ± 0.063	0.965 ± 0.063
		1	1.023 ± 0.032	0.974 ± 0.024	0.986 ± 0.024	1.010 ± 0.034	0.984 ± 0.025	0.974 ± 0.024	0.974 ± 0.024	0.281 ± 0.255	0.927 ± 0.049	0.927 ± 0.049
		2	1.019 ± 0.032	0.966 ± 0.021	0.977 ± 0.022	0.999 ± 0.032	0.977 ± 0.022	0.967 ± 0.022	0.967 ± 0.022	0.277 ± 0.254	0.919 ± 0.046	0.919 ± 0.046
	[0, 8]	0	1.078 ± 0.031	1.040 ± 0.026	1.058 ± 0.027	1.093 ± 0.035	1.043 ± 0.027	1.040 ± 0.026	1.040 ± 0.026	1.047 ± 0.029	1.031 ± 0.026	1.031 ± 0.026
		1	1.022 ± 0.026	0.988 ± 0.019	0.998 ± 0.019	1.018 ± 0.025	0.990 ± 0.019	0.987 ± 0.019	0.987 ± 0.019	0.992 ± 0.021	0.980 ± 0.019	0.980 ± 0.019
		2	1.013 ± 0.025	0.979 ± 0.017	0.987 ± 0.017	1.004 ± 0.021	0.981 ± 0.017	0.979 ± 0.017	0.979 ± 0.017	0.982 ± 0.019	0.972 ± 0.017	0.972 ± 0.017
	Avg		1.037	0.994	1.007	1.032	1.001	0.995	0.995	0.630	0.966	0.966
E_4	[0, 4]	0	1.052 ± 0.011	1.058 ± 0.014	1.051 ± 0.013	1.054 ± 0.012	1.060 ± 0.014	1.058 ± 0.014	1.058 ± 0.014	1.059 ± 0.012	1.056 ± 0.011	1.056 ± 0.011
		1	1.008 ± 0.008	1.011 ± 0.010	1.007 ± 0.010	1.008 ± 0.009	1.014 ± 0.010	1.012 ± 0.010	1.012 ± 0.010	1.013 ± 0.009	1.010 ± 0.008	1.010 ± 0.008
		2	0.999 ± 0.007	1.002 ± 0.009	0.998 ± 0.010	0.999 ± 0.008	1.004 ± 0.009	1.002 ± 0.008	1.002 ± 0.008	1.004 ± 0.007	1.001 ± 0.007	1.001 ± 0.007
	[0, 8]	0	1.060 ± 0.010	1.054 ± 0.010	1.061 ± 0.011	1.058 ± 0.010	1.056 ± 0.011	1.054 ± 0.010	1.054 ± 0.010	1.057 ± 0.011	1.055 ± 0.010	1.055 ± 0.010
		1	1.009 ± 0.006	1.007 ± 0.007	1.010 ± 0.007	1.009 ± 0.006	1.009 ± 0.008	1.007 ± 0.007	1.007 ± 0.007	1.010 ± 0.008	1.008 ± 0.006	1.008 ± 0.006
		2	1.001 ± 0.005	0.999 ± 0.006	1.002 ± 0.006	1.001 ± 0.006	1.001 ± 0.007	0.999 ± 0.006	0.999 ± 0.006	1.002 ± 0.007	1.000 ± 0.005	1.000 ± 0.005
	Avg		1.021	1.022	1.021	1.022	1.024	1.022	1.022	1.024	1.021	1.021

Table 3: CONSTANT arrival rate: waiting time estimates by different methods with 95% confidence intervals.

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.038 ± 0.019	0.980 ± 0.020	1.047 ± 0.020	1.047 ± 0.020	1.058 ± 0.023	1.046 ± 0.022	1.046 ± 0.022	1.062 ± 0.021	1.046 ± 0.021	1.046 ± 0.021
		1	1.002 ± 0.016	0.939 ± 0.015	1.005 ± 0.015	1.005 ± 0.015	1.011 ± 0.016	1.000 ± 0.016	1.011 ± 0.016	1.014 ± 0.015	1.000 ± 0.015	1.014 ± 0.015
		2	0.996 ± 0.016	0.933 ± 0.014	1.000 ± 0.014	1.000 ± 0.014	1.003 ± 0.015	0.993 ± 0.015	1.003 ± 0.015	1.007 ± 0.013	0.993 ± 0.014	1.007 ± 0.013
	[0, 8]	0	1.051 ± 0.013	0.983 ± 0.015	1.050 ± 0.015	1.050 ± 0.015	1.052 ± 0.017	1.044 ± 0.016	1.044 ± 0.016	1.054 ± 0.016	1.045 ± 0.016	1.045 ± 0.016
		1	1.010 ± 0.010	0.944 ± 0.011	1.006 ± 0.011	1.006 ± 0.011	1.008 ± 0.012	1.001 ± 0.012	1.001 ± 0.012	1.009 ± 0.011	1.002 ± 0.011	1.009 ± 0.011
		2	1.003 ± 0.009	0.938 ± 0.010	0.998 ± 0.010	0.998 ± 0.010	1.000 ± 0.011	0.993 ± 0.011	0.993 ± 0.011	1.002 ± 0.010	0.994 ± 0.010	1.002 ± 0.010
	Avg		1.017	0.953	1.018	1.018	1.022	1.013	1.016	1.025	1.013	1.021
H_2	[0, 4]	0	1.041 ± 0.035	0.854 ± 0.026	0.909 ± 0.029	1.017 ± 0.043	1.157 ± 0.061	1.007 ± 0.036	1.007 ± 0.036	0.348 ± 1.511	1.230 ± 0.568	1.230 ± 0.568
		1	1.006 ± 0.035	0.811 ± 0.020	0.868 ± 0.021	0.981 ± 0.033	1.058 ± 0.041	0.948 ± 0.027	0.948 ± 0.027	0.567 ± 1.488	-11.3 ± 24.2	0.567 ± 1.488
		2	0.998 ± 0.035	0.802 ± 0.017	0.858 ± 0.018	0.971 ± 0.028	1.043 ± 0.038	0.935 ± 0.021	0.935 ± 0.021	0.520 ± 1.490	1.444 ± 1.305	1.444 ± 1.305
	[0, 8]	0	1.063 ± 0.027	0.873 ± 0.019	0.931 ± 0.022	1.048 ± 0.029	1.116 ± 0.040	1.018 ± 0.026	1.018 ± 0.026	0.852 ± 0.244	1.009 ± 0.026	1.009 ± 0.026
		1	1.021 ± 0.026	0.831 ± 0.014	0.884 ± 0.015	0.991 ± 0.021	1.038 ± 0.027	0.962 ± 0.019	0.962 ± 0.019	0.853 ± 0.202	0.954 ± 0.019	0.954 ± 0.019
		2	1.013 ± 0.025	0.822 ± 0.012	0.874 ± 0.013	0.980 ± 0.018	1.020 ± 0.021	0.949 ± 0.015	0.949 ± 0.015	0.970 ± 0.095	0.942 ± 0.015	0.942 ± 0.015
	Avg		1.024	0.832	0.887	0.998	1.072	0.970	0.970	0.685	-0.959	1.024
E_4	[0, 4]	0	1.039 ± 0.009	0.997 ± 0.012	1.069 ± 0.010	1.042 ± 0.011	1.045 ± 0.013	1.040 ± 0.013	1.040 ± 0.013	1.048 ± 0.012	1.041 ± 0.012	1.048 ± 0.012
		1	1.010 ± 0.008	0.963 ± 0.010	1.039 ± 0.008	1.011 ± 0.008	1.008 ± 0.010	1.004 ± 0.010	1.008 ± 0.010	1.012 ± 0.009	1.005 ± 0.009	1.012 ± 0.009
		2	1.005 ± 0.007	0.959 ± 0.009	1.033 ± 0.008	1.005 ± 0.008	1.003 ± 0.010	0.998 ± 0.010	1.003 ± 0.010	1.006 ± 0.008	1.000 ± 0.009	1.006 ± 0.008
	[0, 8]	0	1.048 ± 0.008	1.001 ± 0.008	1.070 ± 0.008	1.044 ± 0.008	1.043 ± 0.010	1.040 ± 0.009	1.040 ± 0.009	1.044 ± 0.009	1.041 ± 0.008	1.041 ± 0.008
		1	1.011 ± 0.005	0.967 ± 0.005	1.033 ± 0.006	1.008 ± 0.005	1.006 ± 0.006	1.003 ± 0.005	1.003 ± 0.005	1.007 ± 0.006	1.005 ± 0.005	1.005 ± 0.005
		2	1.004 ± 0.005	0.960 ± 0.005	1.026 ± 0.005	1.001 ± 0.005	0.999 ± 0.005	0.997 ± 0.005	0.997 ± 0.005	1.000 ± 0.005	0.998 ± 0.005	0.998 ± 0.005
	Avg		1.020	0.974	1.045	1.018	1.017	1.014	1.015	1.020	1.015	1.018

Table 4: LINEAR arrival rate: waiting time estimates by different methods with 95% confidence intervals.

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.059 ± 0.015	1.014 ± 0.015	1.056 ± 0.016	1.056 ± 0.016	1.058 ± 0.017	1.052 ± 0.016	1.052 ± 0.016	1.053 ± 0.015	1.046 ± 0.014	1.046 ± 0.014
		1	1.017 ± 0.012	0.977 ± 0.011	1.014 ± 0.012	1.014 ± 0.012	1.018 ± 0.013	1.013 ± 0.012	1.013 ± 0.012	1.013 ± 0.011	1.007 ± 0.011	1.007 ± 0.011
		2	1.009 ± 0.012	0.971 ± 0.011	1.006 ± 0.012	1.006 ± 0.012	1.011 ± 0.012	1.006 ± 0.011	1.006 ± 0.011	1.006 ± 0.010	1.000 ± 0.010	1.000 ± 0.010
	[0, 8]	0	1.064 ± 0.013	1.033 ± 0.012	1.059 ± 0.013	1.059 ± 0.013	1.077 ± 0.014	1.073 ± 0.013	1.073 ± 0.013	1.056 ± 0.013	1.054 ± 0.013	1.054 ± 0.013
		1	1.020 ± 0.010	0.993 ± 0.009	1.015 ± 0.010	1.015 ± 0.010	1.034 ± 0.010	1.030 ± 0.010	1.030 ± 0.010	1.015 ± 0.010	1.013 ± 0.009	1.013 ± 0.009
		2	1.010 ± 0.009	0.985 ± 0.008	1.006 ± 0.009	1.006 ± 0.009	1.025 ± 0.009	1.021 ± 0.008	1.021 ± 0.008	1.006 ± 0.009	1.005 ± 0.008	1.005 ± 0.008
	Avg		1.030	0.995	1.026	1.026	1.037	1.033	1.033	1.025	1.021	1.021
H_2	[0, 4]	0	1.087 ± 0.036	0.914 ± 0.025	0.948 ± 0.027	1.016 ± 0.042	1.048 ± 0.035	1.008 ± 0.029	1.008 ± 0.029	−0.265 ± 0.316	0.985 ± 0.072	0.985 ± 0.072
		1	1.031 ± 0.034	0.860 ± 0.017	0.895 ± 0.019	0.963 ± 0.032	0.973 ± 0.022	0.942 ± 0.019	0.942 ± 0.019	−0.144 ± 0.298	0.916 ± 0.053	0.916 ± 0.053
		2	1.020 ± 0.033	0.854 ± 0.015	0.889 ± 0.017	0.959 ± 0.026	0.964 ± 0.020	0.935 ± 0.017	0.935 ± 0.017	−0.147 ± 0.297	0.907 ± 0.051	0.907 ± 0.051
	[0, 8]	0	1.088 ± 0.027	0.954 ± 0.023	0.985 ± 0.024	1.048 ± 0.031	1.091 ± 0.031	1.057 ± 0.027	1.057 ± 0.027	0.537 ± 0.294	1.078 ± 0.031	1.078 ± 0.031
		1	1.032 ± 0.024	0.903 ± 0.017	0.931 ± 0.018	0.988 ± 0.023	1.021 ± 0.021	0.994 ± 0.020	0.994 ± 0.020	0.931 ± 0.140	1.007 ± 0.022	1.007 ± 0.022
		2	1.021 ± 0.023	0.894 ± 0.014	0.921 ± 0.015	0.976 ± 0.019	1.009 ± 0.018	0.983 ± 0.017	0.983 ± 0.017	0.960 ± 0.114	0.995 ± 0.019	0.995 ± 0.019
	Avg		1.046	0.897	0.928	0.992	1.018	0.987	0.987	0.312	0.981	0.981
E_4	[0, 4]	0	1.045 ± 0.010	1.020 ± 0.011	1.057 ± 0.012	1.043 ± 0.011	1.047 ± 0.012	1.044 ± 0.011	1.044 ± 0.011	1.042 ± 0.010	1.039 ± 0.010	1.039 ± 0.010
		1	1.009 ± 0.007	0.989 ± 0.008	1.019 ± 0.009	1.008 ± 0.008	1.014 ± 0.009	1.012 ± 0.009	1.012 ± 0.009	1.010 ± 0.008	1.007 ± 0.008	1.007 ± 0.008
		2	1.002 ± 0.007	0.983 ± 0.008	1.012 ± 0.008	1.001 ± 0.007	1.009 ± 0.008	1.006 ± 0.008	1.006 ± 0.008	1.004 ± 0.007	1.001 ± 0.007	1.004 ± 0.007
	[0, 8]	0	1.048 ± 0.008	1.031 ± 0.008	1.055 ± 0.009	1.046 ± 0.008	1.058 ± 0.010	1.056 ± 0.009	1.056 ± 0.009	1.043 ± 0.009	1.042 ± 0.008	1.042 ± 0.008
		1	1.009 ± 0.005	0.996 ± 0.006	1.015 ± 0.006	1.008 ± 0.006	1.021 ± 0.007	1.019 ± 0.006	1.019 ± 0.006	1.007 ± 0.006	1.006 ± 0.006	1.006 ± 0.006
		2	1.002 ± 0.004	0.989 ± 0.005	1.008 ± 0.005	1.001 ± 0.005	1.014 ± 0.006	1.012 ± 0.005	1.012 ± 0.005	1.001 ± 0.006	1.000 ± 0.005	1.000 ± 0.005
	Avg		1.019	1.001	1.028	1.018	1.027	1.025	1.025	1.018	1.016	1.016

Table 5: QUADRATIC arrival rate: waiting time estimates by different methods with 95% confidence intervals.

Tables 6, 7 and 8 provide additional information: the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (19) and (32)). We can again compare our simulation results with the theoretical reference points discussed in Section 8.2. We note that the linear arrival rate case is like the IS reference linear case. We computed that for the M, E_4 and H_2 service time distributions, respectively, we have $E[\bar{L}(t)] = 39.0$, 40.125 and 33.0 over $[0,4]$ and 45, 46.125 and 39.0 over $[0,8]$. The indirect estimator $\bar{W}_{L,\lambda}(t)$ in (2) takes the values $39/42 = 0.929$, $40.125/42 = 0.955$ and $33/42 = 0.786$ over $[0,4]$, and $45/48 = 0.938$, $46.125/48 = 0.961$ and $39.0/48 = 0.813$ over $[0,8]$. These numbers translate to the estimation bias in $\bar{W}_{L,\lambda}(t)$ of 7.1%, 4.5% and 21.4% over $[0,4]$, and 6.2%, 3.9% and 18.7% over $[0,8]$. The estimated bias in Table 7 closely matches the results for the M, H_2 and E_4 service with $\beta = 2$, which is very similar to the infinite server (IS) case. Similarly, for the quadratic arrival rate function, we have $E[\bar{L}(t)] = 56.173$, 58.140 and 55.432 over $[0,4]$ and 62.840, 65.363 and 60.617 over $[0,8]$ for the M, E_4 and H_2 service time distributions, respectively. The indirect estimator $\bar{W}_{L,\lambda}(t)$ takes the values $56.173/58.765 = 0.956$, $58.140/58.765 = 0.989$ and $55.432/58.765 = 0.943$ over $[0,4]$, $62.840/66.173 = 0.950$, $65.363/66.173 = 0.988$ and $60.617/66.173 = 0.916$ over $[0,8]$. That means that the estimation bias in $\bar{W}_{L,\lambda}(t)$ is 4.4%, 1.1% and 5.7% over $[0,4]$, and 5.0%, 1.2% and 9.2% over $[0,8]$. These numbers are in agreement with the estimated bias in Table 8.

GI	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
M	$[0,4]$	0	48.1 ± 1.2	$6.08 \times 10^{-4} \pm 4.59 \times 10^{-3}$	$-2.27 \times 10^{-4} \pm 4.19 \times 10^{-3}$
		1	45.8 ± 0.9	$6.77 \times 10^{-4} \pm 4.35 \times 10^{-3}$	$3.09 \times 10^{-5} \pm 3.77 \times 10^{-3}$
		2	45.3 ± 0.8	$6.73 \times 10^{-4} \pm 4.29 \times 10^{-3}$	$1.73 \times 10^{-4} \pm 3.70 \times 10^{-3}$
	$[0,8]$	0	48.3 ± 1.0	$1.64 \times 10^{-4} \pm 1.43 \times 10^{-3}$	$5.78 \times 10^{-4} \pm 9.58 \times 10^{-4}$
		1	45.8 ± 0.7	$1.31 \times 10^{-4} \pm 1.36 \times 10^{-3}$	$5.85 \times 10^{-4} \pm 8.45 \times 10^{-4}$
		2	45.3 ± 0.6	$1.16 \times 10^{-4} \pm 1.36 \times 10^{-3}$	$5.80 \times 10^{-4} \pm 8.28 \times 10^{-4}$
	Avg		46.4	3.95×10^{-4}	2.87×10^{-4}
H_2	$[0,4]$	0	46.1 ± 1.6	$2.20 \times 10^{-3} \pm 1.29 \times 10^{-2}$	$5.22 \times 10^{-2} \pm 6.85 \times 10^{-2}$
		1	43.9 ± 1.2	$1.92 \times 10^{-3} \pm 1.23 \times 10^{-2}$	$4.60 \times 10^{-2} \pm 6.07 \times 10^{-2}$
		2	43.6 ± 1.0	$1.80 \times 10^{-3} \pm 1.23 \times 10^{-2}$	$4.64 \times 10^{-2} \pm 5.96 \times 10^{-2}$
	$[0,8]$	0	47.0 ± 1.3	$2.22 \times 10^{-4} \pm 4.14 \times 10^{-3}$	$1.42 \times 10^{-2} \pm 1.47 \times 10^{-2}$
		1	44.5 ± 0.9	$1.52 \times 10^{-4} \pm 3.94 \times 10^{-3}$	$1.26 \times 10^{-2} \pm 1.29 \times 10^{-2}$
		2	44.1 ± 0.8	$1.78 \times 10^{-4} \pm 3.90 \times 10^{-3}$	$1.24 \times 10^{-2} \pm 1.26 \times 10^{-2}$
	Avg		44.9	1.08×10^{-3}	3.06×10^{-2}
E_4	$[0,4]$	0	47.8 ± 1.0	$4.56 \times 10^{-4} \pm 2.81 \times 10^{-3}$	$-4.30 \times 10^{-4} \pm 1.31 \times 10^{-3}$
		1	45.6 ± 0.8	$5.60 \times 10^{-4} \pm 2.67 \times 10^{-3}$	$-2.91 \times 10^{-4} \pm 1.20 \times 10^{-3}$
		2	45.2 ± 0.7	$5.60 \times 10^{-4} \pm 2.65 \times 10^{-3}$	$-2.19 \times 10^{-4} \pm 1.16 \times 10^{-3}$
	$[0,8]$	0	47.6 ± 0.8	$1.09 \times 10^{-4} \pm 8.67 \times 10^{-4}$	$1.53 \times 10^{-4} \pm 2.83 \times 10^{-4}$
		1	45.4 ± 0.6	$9.29 \times 10^{-5} \pm 8.32 \times 10^{-4}$	$1.57 \times 10^{-4} \pm 2.56 \times 10^{-4}$
		2	45.1 ± 0.5	$9.39 \times 10^{-5} \pm 8.27 \times 10^{-4}$	$1.61 \times 10^{-4} \pm 2.50 \times 10^{-4}$
	Avg		46.1	3.12×10^{-4}	-7.84×10^{-5}

Table 6: CONSTANT arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

GI	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
M	[0, 4]	0	40.8 ± 1.1	$6.72 \times 10^{-2} \pm 3.06 \times 10^{-3}$	$3.91 \times 10^{-3} \pm 4.79 \times 10^{-3}$
		1	39.1 ± 0.8	$6.46 \times 10^{-2} \pm 2.94 \times 10^{-3}$	$3.69 \times 10^{-3} \pm 4.32 \times 10^{-3}$
		2	38.8 ± 0.8	$6.42 \times 10^{-2} \pm 2.91 \times 10^{-3}$	$3.72 \times 10^{-3} \pm 4.26 \times 10^{-3}$
	[0, 8]	0	47.2 ± 0.9	$6.17 \times 10^{-2} \pm 1.11 \times 10^{-3}$	$5.35 \times 10^{-4} \pm 7.53 \times 10^{-4}$
		1	45.3 ± 0.7	$5.93 \times 10^{-2} \pm 9.63 \times 10^{-4}$	$5.69 \times 10^{-4} \pm 6.81 \times 10^{-4}$
		2	45.0 ± 0.6	$5.89 \times 10^{-2} \pm 9.16 \times 10^{-4}$	$5.80 \times 10^{-4} \pm 6.69 \times 10^{-4}$
	Avg		42.7	6.27×10^{-2}	2.17×10^{-3}
H_2	[0, 4]	0	35.6 ± 1.2	$1.76 \times 10^{-1} \pm 9.31 \times 10^{-3}$	$-2.21 \times 10^{-1} \pm 6.04 \times 10^{-1}$
		1	33.8 ± 0.9	$1.67 \times 10^{-1} \pm 8.44 \times 10^{-3}$	$1.34 \times 10^{1} \pm 2.63 \times 10^{1}$
		2	33.4 ± 0.8	$1.65 \times 10^{-1} \pm 8.00 \times 10^{-3}$	$-6.00 \times 10^{-1} \pm 1.57 \times 10^{0}$
	[0, 8]	0	41.9 ± 1.1	$1.65 \times 10^{-1} \pm 4.38 \times 10^{-3}$	$1.65 \times 10^{-2} \pm 1.34 \times 10^{-2}$
		1	39.9 ± 0.8	$1.57 \times 10^{-1} \pm 3.54 \times 10^{-3}$	$1.52 \times 10^{-2} \pm 1.16 \times 10^{-2}$
		2	39.4 ± 0.7	$1.55 \times 10^{-1} \pm 3.11 \times 10^{-3}$	$1.45 \times 10^{-2} \pm 1.10 \times 10^{-2}$
	Avg		37.3	1.64×10^{-1}	2.10×10^{0}
E_4	[0, 4]	0	41.5 ± 0.8	$4.30 \times 10^{-2} \pm 2.02 \times 10^{-3}$	$1.03 \times 10^{-3} \pm 1.38 \times 10^{-3}$
		1	40.1 ± 0.7	$4.16 \times 10^{-2} \pm 1.95 \times 10^{-3}$	$1.02 \times 10^{-3} \pm 1.29 \times 10^{-3}$
		2	39.9 ± 0.6	$4.14 \times 10^{-2} \pm 1.94 \times 10^{-3}$	$1.03 \times 10^{-3} \pm 1.27 \times 10^{-3}$
	[0, 8]	0	48.1 ± 0.7	$3.93 \times 10^{-2} \pm 6.11 \times 10^{-4}$	$1.95 \times 10^{-4} \pm 2.33 \times 10^{-4}$
		1	46.4 ± 0.6	$3.80 \times 10^{-2} \pm 5.70 \times 10^{-4}$	$1.98 \times 10^{-4} \pm 2.14 \times 10^{-4}$
		2	46.1 ± 0.5	$3.77 \times 10^{-2} \pm 5.54 \times 10^{-4}$	$2.00 \times 10^{-4} \pm 2.10 \times 10^{-4}$
	Avg		43.7	4.02×10^{-2}	6.12×10^{-4}

Table 7: LINEAR arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

GI	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
M	[0, 4]	0	57.3 ± 1.1	$3.79 \times 10^{-2} \pm 3.45 \times 10^{-3}$	$-4.26 \times 10^{-3} \pm 4.49 \times 10^{-3}$
		1	55.2 ± 0.9	$3.66 \times 10^{-2} \pm 3.31 \times 10^{-3}$	$-3.78 \times 10^{-3} \pm 4.12 \times 10^{-3}$
		2	54.9 ± 0.8	$3.63 \times 10^{-2} \pm 3.27 \times 10^{-3}$	$-3.61 \times 10^{-3} \pm 4.06 \times 10^{-3}$
	[0, 8]	0	60.2 ± 1.0	$3.86 \times 10^{-2} \pm 1.09 \times 10^{-3}$	$-6.34 \times 10^{-3} \pm 5.75 \times 10^{-4}$
		1	57.9 ± 0.8	$3.71 \times 10^{-2} \pm 1.00 \times 10^{-3}$	$-5.84 \times 10^{-3} \pm 5.16 \times 10^{-4}$
		2	57.4 ± 0.7	$3.68 \times 10^{-2} \pm 9.77 \times 10^{-4}$	$-5.73 \times 10^{-3} \pm 5.03 \times 10^{-4}$
	Avg		57.2	3.72×10^{-2}	-4.93×10^{-3}
H_2	[0, 4]	0	51.7 ± 1.5	$1.02 \times 10^{-1} \pm 9.16 \times 10^{-3}$	$8.77 \times 10^{-3} \pm 8.38 \times 10^{-2}$
		1	48.6 ± 1.0	$9.56 \times 10^{-2} \pm 8.47 \times 10^{-3}$	$1.21 \times 10^{-2} \pm 7.23 \times 10^{-2}$
		2	48.2 ± 0.9	$9.49 \times 10^{-2} \pm 8.35 \times 10^{-3}$	$1.39 \times 10^{-2} \pm 7.12 \times 10^{-2}$
	[0, 8]	0	55.6 ± 1.5	$1.07 \times 10^{-1} \pm 3.59 \times 10^{-3}$	$-8.65 \times 10^{-2} \pm 9.21 \times 10^{-3}$
		1	52.6 ± 1.1	$1.01 \times 10^{-1} \pm 2.97 \times 10^{-3}$	$-7.64 \times 10^{-2} \pm 7.65 \times 10^{-3}$
		2	52.1 ± 0.9	$9.99 \times 10^{-2} \pm 2.80 \times 10^{-3}$	$-7.44 \times 10^{-2} \pm 7.13 \times 10^{-3}$
	Avg		51.5	1.00×10^{-1}	-3.38×10^{-2}
E_4	[0, 4]	0	57.7 ± 1.0	$2.39 \times 10^{-2} \pm 2.16 \times 10^{-3}$	$-1.47 \times 10^{-3} \pm 1.33 \times 10^{-3}$
		1	55.9 ± 0.8	$2.32 \times 10^{-2} \pm 2.10 \times 10^{-3}$	$-1.29 \times 10^{-3} \pm 1.25 \times 10^{-3}$
		2	55.6 ± 0.8	$2.30 \times 10^{-2} \pm 2.08 \times 10^{-3}$	$-1.25 \times 10^{-3} \pm 1.23 \times 10^{-3}$
	[0, 8]	0	60.1 ± 0.9	$2.41 \times 10^{-2} \pm 6.94 \times 10^{-4}$	$-1.97 \times 10^{-3} \pm 1.89 \times 10^{-4}$
		1	58.0 ± 0.7	$2.33 \times 10^{-2} \pm 6.40 \times 10^{-4}$	$-1.82 \times 10^{-3} \pm 1.70 \times 10^{-4}$
		2	57.6 ± 0.6	$2.31 \times 10^{-2} \pm 6.26 \times 10^{-4}$	$-1.80 \times 10^{-3} \pm 1.67 \times 10^{-4}$
	Avg		57.5	2.34×10^{-2}	-1.60×10^{-3}

Table 8: QUADRATIC arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

We now estimate the bias reduction achieved by our estimators by two performance measures. The first performance measure computes the absolute difference between (i) the average of the estimate of interest over the 100 replications and (ii) the average of the direct estimate $\bar{W}(t)$ over the same 100 replications. Tables 9, 10 and 11 illustrate these results when constant, linear and quadratic arrival rates are used, respectively.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	0.3	0.2	0.2	0.6	0.3	0.3	0.5	0.0	0.0
		1	0.1	0.2	0.2	0.3	0.1	0.1	0.3	0.2	0.2
		2	0.0	0.1	0.1	0.3	0.0	0.3	0.2	0.2	0.2
	$[0,8]$	0	0.9	0.2	0.2	0.7	0.9	0.9	0.5	0.8	0.8
		1	0.5	0.1	0.1	0.4	0.5	0.4	0.2	0.4	0.2
		2	0.4	0.1	0.1	0.3	0.4	0.3	0.1	0.3	0.1
	Avg		0.4	0.2	0.2	0.4	0.4	0.4	0.3	0.3	0.2
H_2	$[0,4]$	0	4.9	3.3	0.1	3.6	4.8	4.8	86.8	10.4	10.4
		1	4.9	3.7	1.3	3.8	4.8	4.8	74.2	9.5	9.5
		2	5.2	4.2	2.0	4.2	5.2	5.2	74.2	10.0	10.0
	$[0,8]$	0	3.7	2.0	1.6	3.4	3.7	3.7	3.1	4.7	4.7
		1	3.4	2.4	0.4	3.2	3.4	3.4	3.0	4.1	4.1
		2	3.4	2.6	0.9	3.2	3.4	3.4	3.1	4.1	4.1
	Avg		4.3	3.0	1.0	3.6	4.2	4.2	40.7	7.1	7.1
E_4	$[0,4]$	0	0.6	0.1	0.2	0.8	0.6	0.6	0.7	0.4	0.4
		1	0.4	0.1	0.1	0.6	0.4	0.4	0.5	0.2	0.2
		2	0.3	0.1	0.0	0.5	0.3	0.3	0.5	0.2	0.2
	$[0,8]$	0	0.6	0.1	0.2	0.4	0.6	0.6	0.3	0.5	0.5
		1	0.2	0.1	0.0	0.0	0.2	0.2	0.1	0.1	0.1
		2	0.2	0.1	0.0	0.0	0.2	0.2	0.1	0.1	0.1
	Avg		0.4	0.1	0.1	0.4	0.4	0.4	0.4	0.2	0.2

Table 9: CONSTANT arrival rate: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	5.9	0.8	0.8	2.0	0.7	0.7	2.3	0.7	0.7
		1	6.3	0.3	0.3	0.8	0.2	0.8	1.2	0.2	1.2
		2	6.3	0.4	0.4	0.7	0.3	0.7	1.1	0.3	1.1
	$[0,8]$	0	6.8	0.1	0.1	0.1	0.7	0.7	0.3	0.6	0.6
		1	6.6	0.5	0.5	0.3	1.0	1.0	0.1	0.9	0.1
		2	6.5	0.4	0.4	0.3	1.0	1.0	0.1	0.8	0.1
	Avg		6.4	0.4	0.4	0.7	0.7	0.8	0.8	0.6	0.6
H_2	$[0,4]$	0	18.7	13.2	2.4	11.6	3.4	3.4	69.3	18.9	18.9
		1	19.4	13.8	2.5	5.2	5.7	5.7	43.9	1233.8	43.9
		2	19.6	14.0	2.7	4.5	6.3	6.3	47.8	44.6	44.6
	$[0,8]$	0	19.0	13.2	1.6	5.3	4.5	4.5	21.1	5.4	5.4
		1	19.0	13.7	3.0	1.7	5.9	5.9	16.8	6.7	6.7
		2	19.1	13.8	3.2	0.7	6.3	6.3	4.2	7.0	7.0
	Avg		19.1	13.6	2.6	4.8	5.4	5.4	33.9	219.4	21.1
E_4	$[0,4]$	0	4.2	3.0	0.3	0.6	0.1	0.1	0.9	0.2	0.9
		1	4.7	2.9	0.0	0.2	0.7	0.2	0.1	0.5	0.1
		2	4.7	2.8	0.0	0.2	0.7	0.2	0.1	0.5	0.1
	$[0,8]$	0	4.7	2.2	0.4	0.5	0.8	0.8	0.4	0.7	0.7
		1	4.5	2.1	0.3	0.6	0.8	0.8	0.4	0.7	0.7
		2	4.4	2.2	0.3	0.5	0.7	0.7	0.4	0.6	0.6
	Avg		4.5	2.5	0.2	0.4	0.6	0.5	0.4	0.5	0.5

Table 10: LINEAR arrival rate: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

Another performance measure of interest is the absolute relative error of the estimate in each

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	4.5	0.4	0.4	0.1	0.7	0.7	0.7	1.4	1.4
		1	4.0	0.3	0.3	0.1	0.4	0.4	0.4	1.0	1.0
		2	3.8	0.4	0.4	0.2	0.3	0.3	0.3	0.9	0.9
	$[0,8]$	0	3.1	0.4	0.4	1.3	0.9	0.9	0.8	0.9	0.9
		1	2.7	0.5	0.5	1.4	1.0	1.0	0.6	0.7	0.7
		2	2.5	0.4	0.4	1.5	1.1	1.1	0.4	0.6	0.6
	Avg		3.5	0.4	0.4	0.8	0.7	0.7	0.5	0.9	0.9
H_2	$[0,4]$	0	17.3	13.9	7.1	3.9	7.9	7.9	135.2	10.2	10.2
		1	17.0	13.6	6.7	5.8	8.8	8.8	117.4	11.5	11.5
		2	16.5	13.0	6.0	5.5	8.5	8.5	116.6	11.3	11.3
	$[0,8]$	0	13.3	10.2	4.0	0.4	3.0	3.0	55.1	1.0	1.0
		1	13.0	10.1	4.4	1.1	3.8	3.8	10.2	2.5	2.5
		2	12.7	10.0	4.5	1.1	3.7	3.7	6.1	2.6	2.6
	Avg		15.0	11.8	5.5	3.0	6.0	6.0	73.4	6.5	6.5
E_4	$[0,4]$	0	2.6	1.2	0.2	0.2	0.1	0.1	0.3	0.7	0.7
		1	2.0	1.0	0.1	0.5	0.3	0.3	0.1	0.2	0.2
		2	1.9	1.0	0.1	0.7	0.4	0.4	0.2	0.1	0.2
	$[0,8]$	0	1.7	0.7	0.2	1.0	0.8	0.8	0.5	0.6	0.6
		1	1.3	0.6	0.1	1.2	1.0	1.0	0.2	0.3	0.3
		2	1.2	0.6	0.1	1.2	1.1	1.1	0.1	0.2	0.2
	Avg		1.8	0.9	0.1	0.8	0.6	0.6	0.2	0.3	0.4

Table 11: QUADRATIC arrival rate: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

sample path. We average this relative error over 100 replications. Tables 12, 13 and 14 report the results. These results measure the ability of the estimator to match the direct estimator $\bar{W}(t)$ over each sample path, and thus strongly favor the sample-path based estimator $\bar{W}_{L,\lambda,r,\gamma}(t)$. Indeed, that is the only estimator that provides significant improvement over the indirect estimator $\bar{W}_{L,\lambda}(t)$ from this perspective.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	5.6 ± 0.7	4.3 ± 0.7	4.3 ± 0.7	5.1 ± 0.7	4.9 ± 0.7	5.2 ± 0.7	5.2 ± 0.7
		1	5.3 ± 0.7	4.4 ± 0.6	4.4 ± 0.6	4.9 ± 0.7	4.8 ± 0.7	5.1 ± 0.7	5.1 ± 0.7
		2	5.2 ± 0.7	4.4 ± 0.6	4.4 ± 0.6	5.0 ± 0.7	4.8 ± 0.7	5.2 ± 0.7	5.1 ± 0.7
	$[0,8]$	0	3.1 ± 0.5	2.2 ± 0.3	2.2 ± 0.3	3.3 ± 0.5	3.0 ± 0.5	3.1 ± 0.5	2.8 ± 0.4
		1	2.9 ± 0.4	2.2 ± 0.3	2.2 ± 0.3	3.0 ± 0.5	2.8 ± 0.4	2.9 ± 0.5	2.7 ± 0.4
		2	2.7 ± 0.4	2.2 ± 0.3	2.2 ± 0.3	3.0 ± 0.5	2.7 ± 0.4	2.9 ± 0.5	2.6 ± 0.4
	Avg		4.1	3.3	3.3	4.1	3.8	4.1	3.9
H_2	$[0,4]$	0	12.6 ± 1.7	11.2 ± 1.6	13.1 ± 2.1	12.5 ± 1.7	12.6 ± 1.8	83.7 ± 24.1	21.1 ± 3.4
		1	12.9 ± 1.9	11.9 ± 1.7	12.9 ± 2.0	13.0 ± 1.9	13.1 ± 1.9	77.3 ± 23.6	19.1 ± 3.1
		2	12.9 ± 1.8	11.7 ± 1.7	12.6 ± 1.9	12.7 ± 1.9	13.0 ± 1.9	77.5 ± 23.6	18.7 ± 3.0
	$[0,8]$	0	9.4 ± 1.3	7.9 ± 1.1	8.2 ± 1.2	9.4 ± 1.2	9.5 ± 1.3	10.0 ± 1.4	9.9 ± 1.4
		1	9.8 ± 1.3	8.7 ± 1.1	7.8 ± 1.2	9.8 ± 1.2	9.8 ± 1.3	10.2 ± 1.3	10.1 ± 1.3
		2	9.7 ± 1.3	8.7 ± 1.1	8.0 ± 1.1	9.7 ± 1.2	9.7 ± 1.2	10.0 ± 1.3	10.0 ± 1.3
	Avg		11.2	10.0	10.4	11.2	11.3	44.8	14.8
E_4	$[0,4]$	0	4.5 ± 0.6	1.9 ± 0.3	2.0 ± 0.3	3.6 ± 0.5	3.7 ± 0.6	2.9 ± 0.5	3.1 ± 0.5
		1	3.7 ± 0.5	2.3 ± 0.3	1.7 ± 0.3	3.3 ± 0.5	3.3 ± 0.5	2.7 ± 0.4	2.6 ± 0.4
		2	3.6 ± 0.5	2.5 ± 0.4	1.8 ± 0.3	3.3 ± 0.5	3.2 ± 0.5	2.8 ± 0.4	2.6 ± 0.4
	$[0,8]$	0	2.2 ± 0.3	1.1 ± 0.2	1.1 ± 0.2	2.3 ± 0.3	2.0 ± 0.3	2.0 ± 0.3	1.7 ± 0.3
		1	1.7 ± 0.3	1.3 ± 0.2	1.1 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.9 ± 0.3	1.4 ± 0.2
		2	1.7 ± 0.2	1.4 ± 0.2	1.0 ± 0.2	2.0 ± 0.3	1.6 ± 0.2	1.9 ± 0.3	1.4 ± 0.2
	Avg		2.9	1.7	1.5	2.8	2.6	2.4	2.2

Table 12: CONSTANT arrival rate: average of the absolute relative error of the estimate from the direct estimate in each sample path.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	7.0 ± 1.1	4.3 ± 0.6	4.3 ± 0.6	6.6 ± 0.9	6.3 ± 0.8	6.2 ± 0.9	5.8 ± 0.8
		1	7.2 ± 1.1	4.4 ± 0.6	4.4 ± 0.6	6.1 ± 0.9	5.8 ± 0.8	5.7 ± 0.8	5.3 ± 0.7
		2	7.1 ± 1.1	4.4 ± 0.6	4.4 ± 0.6	6.0 ± 0.8	5.7 ± 0.7	5.6 ± 0.8	5.2 ± 0.7
	$[0,8]$	0	6.6 ± 0.7	2.0 ± 0.3	2.0 ± 0.3	3.4 ± 0.6	3.3 ± 0.6	3.0 ± 0.5	3.1 ± 0.5
		1	6.6 ± 0.7	1.9 ± 0.3	1.9 ± 0.3	3.1 ± 0.5	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
		2	6.5 ± 0.6	1.9 ± 0.3	1.9 ± 0.3	3.0 ± 0.4	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
	Avg		6.8	3.2	3.2	4.7	4.5	4.3	4.1
H_2	$[0,4]$	0	19.6 ± 2.1	15.4 ± 2.0	14.6 ± 2.2	20.7 ± 4.6	14.2 ± 2.4	198.9 ± 120.2	55.5 ± 34.2
		1	20.2 ± 2.2	16.0 ± 2.0	14.4 ± 2.0	16.9 ± 3.2	14.2 ± 2.4	184.8 ± 122.2	759.9 ± 1408.2
		2	20.3 ± 2.2	16.0 ± 2.1	14.3 ± 1.9	16.6 ± 3.3	13.7 ± 2.3	188.7 ± 122.4	90.1 ± 119.7
	$[0,8]$	0	17.3 ± 1.8	12.5 ± 1.5	8.2 ± 1.0	12.0 ± 2.1	9.8 ± 1.3	31.6 ± 20.0	10.2 ± 1.4
		1	18.0 ± 1.7	13.2 ± 1.6	8.0 ± 1.0	9.8 ± 1.7	9.8 ± 1.3	24.3 ± 16.5	10.1 ± 1.4
		2	18.2 ± 1.7	13.3 ± 1.6	8.0 ± 1.0	9.4 ± 1.4	9.9 ± 1.3	14.1 ± 8.8	10.2 ± 1.4
	Avg		18.9	14.4	11.2	14.2	11.9	107.1	156.0
E_4	$[0,4]$	0	5.2 ± 0.6	3.2 ± 0.5	2.3 ± 0.4	3.9 ± 0.5	3.6 ± 0.6	3.2 ± 0.5	3.0 ± 0.5
		1	5.3 ± 0.7	3.2 ± 0.4	2.2 ± 0.3	3.7 ± 0.5	3.7 ± 0.5	2.9 ± 0.5	3.0 ± 0.4
		2	5.2 ± 0.6	3.1 ± 0.5	2.2 ± 0.3	3.6 ± 0.5	3.6 ± 0.5	2.9 ± 0.5	2.9 ± 0.4
	$[0,8]$	0	4.5 ± 0.5	2.1 ± 0.2	1.2 ± 0.2	2.4 ± 0.4	2.2 ± 0.3	2.0 ± 0.3	1.9 ± 0.3
		1	4.4 ± 0.4	2.2 ± 0.2	1.1 ± 0.2	2.2 ± 0.4	1.9 ± 0.3	1.8 ± 0.3	1.6 ± 0.3
		2	4.4 ± 0.4	2.3 ± 0.2	1.1 ± 0.2	2.1 ± 0.3	1.7 ± 0.3	1.8 ± 0.3	1.6 ± 0.2
	Avg		4.8	2.7	1.7	3.0	2.8	2.4	2.3

Table 13: LINEAR arrival rate: average of the absolute relative error of the estimate from the direct estimate in each sample path.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	5.8 ± 0.7	3.2 ± 0.5	3.2 ± 0.5	4.8 ± 0.7	4.7 ± 0.6	4.6 ± 0.6	4.5 ± 0.6
		1	5.6 ± 0.7	3.2 ± 0.5	3.2 ± 0.5	5.1 ± 0.8	4.6 ± 0.7	4.7 ± 0.7	4.3 ± 0.6
		2	5.5 ± 0.7	3.1 ± 0.5	3.1 ± 0.5	5.1 ± 0.8	4.6 ± 0.7	4.7 ± 0.7	4.3 ± 0.6
	$[0,8]$	0	3.5 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.9 ± 0.4	2.5 ± 0.4	2.6 ± 0.3	2.5 ± 0.3
		1	3.2 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.9 ± 0.4	2.4 ± 0.4	2.4 ± 0.3	2.3 ± 0.3
		2	3.1 ± 0.4	1.7 ± 0.2	1.7 ± 0.2	2.8 ± 0.4	2.4 ± 0.4	2.3 ± 0.3	2.2 ± 0.3
	Avg		4.5	2.4	2.4	3.9	3.5	3.6	3.4
H_2	$[0,4]$	0	16.9 ± 2.2	14.0 ± 2.0	14.2 ± 1.9	13.7 ± 2.0	13.8 ± 2.0	128.6 ± 28.2	23.0 ± 4.9
		1	17.2 ± 2.2	14.4 ± 2.0	12.3 ± 1.8	13.3 ± 2.0	14.1 ± 1.9	118.4 ± 28.6	20.6 ± 4.3
		2	17.2 ± 2.2	14.4 ± 2.0	11.7 ± 1.8	13.3 ± 2.0	14.1 ± 2.0	119.2 ± 28.9	20.4 ± 4.3
	$[0,8]$	0	12.4 ± 1.5	9.9 ± 1.3	7.6 ± 1.0	8.5 ± 1.2	8.2 ± 1.1	56.8 ± 24.0	8.6 ± 1.1
		1	12.6 ± 1.4	10.2 ± 1.2	7.1 ± 1.0	8.0 ± 1.1	8.0 ± 1.1	19.7 ± 11.5	8.2 ± 1.1
		2	12.6 ± 1.4	10.3 ± 1.3	7.0 ± 1.0	8.2 ± 1.1	8.1 ± 1.2	16.6 ± 9.7	8.3 ± 1.1
	Avg		14.8	12.2	10.0	10.8	11.0	76.6	14.8
E_4	$[0,4]$	0	3.5 ± 0.5	2.2 ± 0.3	1.8 ± 0.3	3.2 ± 0.4	2.8 ± 0.4	2.6 ± 0.4	2.4 ± 0.3
		1	2.9 ± 0.5	2.2 ± 0.3	1.6 ± 0.2	3.0 ± 0.4	2.5 ± 0.4	2.4 ± 0.4	2.0 ± 0.3
		2	2.8 ± 0.4	2.2 ± 0.3	1.6 ± 0.2	3.1 ± 0.4	2.5 ± 0.3	2.5 ± 0.4	2.0 ± 0.3
	$[0,8]$	0	1.9 ± 0.3	1.1 ± 0.2	0.8 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
		1	1.6 ± 0.3	1.1 ± 0.2	0.9 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
		2	1.6 ± 0.2	1.1 ± 0.2	0.9 ± 0.1	2.1 ± 0.3	1.6 ± 0.2	1.6 ± 0.3	1.2 ± 0.2
	Avg		2.4	1.7	1.3	2.6	2.1	2.0	1.7

Table 14: QUADRATIC arrival rate: average of the absolute relative error of the estimate from the direct estimate in each sample path.

2.4 Estimation Results - 1000 replications for H_2 distribution

The results in Section 2.3 show that the estimators for the H_2 service time distribution perform the worst. However, we anticipate that is due to high variability in that service time distribution. We explore further in this section by doing $1,000$ replications of the models with the H_2 service time distribution. The results below for this case show that much of the poor performance for the H_2 service time distribution is due to the small sample size.

A	$Interval$	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
C	[0, 4]	0	1.034 ± 0.011	1.010 ± 0.009	1.011 ± 0.009	1.014 ± 0.013	1.012 ± 0.009	1.003 ± 0.009	1.003 ± 0.009	0.170 ± 0.082	1.016 ± 0.207	0.170 ± 0.082
		1	0.999 ± 0.010	0.967 ± 0.007	0.972 ± 0.007	0.980 ± 0.010	0.970 ± 0.007	0.962 ± 0.007	0.962 ± 0.007	0.240 ± 0.076	0.854 ± 0.051	0.854 ± 0.051
		2	0.994 ± 0.010	0.961 ± 0.006	0.966 ± 0.006	0.975 ± 0.009	0.963 ± 0.006	0.956 ± 0.006	0.956 ± 0.006	0.254 ± 0.075	0.863 ± 0.029	0.863 ± 0.029
	[0, 8]	0	1.057 ± 0.010	1.031 ± 0.009	1.042 ± 0.009	1.062 ± 0.012	1.032 ± 0.009	1.031 ± 0.009	1.032 ± 0.009	1.038 ± 0.009	1.028 ± 0.009	1.028 ± 0.009
		1	1.006 ± 0.008	0.980 ± 0.006	0.985 ± 0.006	0.996 ± 0.008	0.980 ± 0.006	0.979 ± 0.006	0.980 ± 0.006	0.983 ± 0.006	0.976 ± 0.006	0.976 ± 0.006
		2	0.996 ± 0.007	0.971 ± 0.005	0.975 ± 0.005	0.983 ± 0.007	0.972 ± 0.005	0.971 ± 0.005	0.972 ± 0.005	0.974 ± 0.006	0.967 ± 0.005	0.967 ± 0.005
	Avg		1.014	0.987	0.992	1.002	0.988	0.983	0.984	0.610	0.951	0.810
L	[0, 4]	0	1.052 ± 0.012	0.863 ± 0.008	0.915 ± 0.009	1.018 ± 0.013	1.177 ± 0.018	1.019 ± 0.011	1.019 ± 0.011	-0.65 ± 0.23	0.969 ± 0.219	0.969 ± 0.219
		1	1.008 ± 0.011	0.815 ± 0.006	0.868 ± 0.006	0.974 ± 0.010	1.070 ± 0.013	0.952 ± 0.007	0.952 ± 0.007	-0.47 ± 0.18	0.366 ± 0.888	-0.47 ± 0.18
		2	0.999 ± 0.011	0.806 ± 0.005	0.859 ± 0.006	0.965 ± 0.009	1.051 ± 0.011	0.941 ± 0.006	0.941 ± 0.006	-0.40 ± 0.17	0.852 ± 0.048	0.852 ± 0.048
	[0, 8]	0	1.058 ± 0.009	0.885 ± 0.007	0.940 ± 0.008	1.051 ± 0.011	1.135 ± 0.014	1.033 ± 0.010	1.033 ± 0.010	0.695 ± 0.106	1.029 ± 0.010	1.029 ± 0.010
		1	1.009 ± 0.008	0.835 ± 0.005	0.885 ± 0.005	0.986 ± 0.007	1.039 ± 0.008	0.966 ± 0.006	0.966 ± 0.006	0.942 ± 0.049	0.962 ± 0.007	0.962 ± 0.007
		2	0.999 ± 0.007	0.825 ± 0.004	0.875 ± 0.005	0.974 ± 0.006	1.022 ± 0.007	0.953 ± 0.005	0.953 ± 0.005	1.005 ± 0.022	0.949 ± 0.006	0.949 ± 0.006
	Avg		1.021	0.838	0.890	0.995	1.082	0.977	0.977	0.188	0.855	0.715
Q	[0, 4]	0	1.080 ± 0.011	0.931 ± 0.009	0.968 ± 0.010	1.042 ± 0.014	1.074 ± 0.013	1.029 ± 0.011	1.029 ± 0.011	-0.53 ± 0.11	1.035 ± 0.021	1.035 ± 0.021
		1	1.016 ± 0.010	0.868 ± 0.006	0.903 ± 0.007	0.972 ± 0.010	0.985 ± 0.009	0.953 ± 0.007	0.953 ± 0.007	-0.34 ± 0.10	0.953 ± 0.014	0.953 ± 0.014
		2	1.001 ± 0.009	0.856 ± 0.005	0.890 ± 0.005	0.957 ± 0.008	0.967 ± 0.007	0.938 ± 0.006	0.938 ± 0.006	-0.30 ± 0.10	0.935 ± 0.012	0.935 ± 0.012
	[0, 8]	0	1.068 ± 0.008	0.953 ± 0.007	0.980 ± 0.008	1.034 ± 0.010	1.092 ± 0.010	1.058 ± 0.009	1.058 ± 0.009	0.425 ± 0.098	1.080 ± 0.010	1.080 ± 0.010
		1	1.012 ± 0.007	0.900 ± 0.005	0.924 ± 0.005	0.974 ± 0.007	1.019 ± 0.007	0.992 ± 0.006	0.992 ± 0.006	0.825 ± 0.059	1.005 ± 0.007	1.005 ± 0.007
		2	0.999 ± 0.007	0.889 ± 0.004	0.913 ± 0.004	0.961 ± 0.006	1.004 ± 0.006	0.979 ± 0.005	0.979 ± 0.005	0.922 ± 0.042	0.989 ± 0.006	0.989 ± 0.006
	Avg		1.029	0.899	0.930	0.990	1.024	0.991	0.991	0.167	0.999	0.999

Table 15: H_2 service time distribution with 1000 replications: waiting time estimates by different methods with 95% confidence intervals.

A	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
C	[0, 4]	0	45.5 ± 0.4	$-4.09 \times 10^{-3} \pm 4.09 \times 10^{-3}$	$1.92 \times 10^{-2} \pm 1.51 \times 10^{-1}$
		1	43.5 ± 0.3	$-3.79 \times 10^{-3} \pm 3.91 \times 10^{-3}$	$1.09 \times 10^{-1} \pm 4.67 \times 10^{-2}$
		2	43.2 ± 0.3	$-3.70 \times 10^{-3} \pm 3.88 \times 10^{-3}$	$9.86 \times 10^{-2} \pm 3.22 \times 10^{-2}$
	[0, 8]	0	46.5 ± 0.4	$-4.89 \times 10^{-5} \pm 1.57 \times 10^{-3}$	$-1.42 \times 10^{-4} \pm 3.95 \times 10^{-3}$
		1	44.1 ± 0.3	$-1.59 \times 10^{-4} \pm 1.49 \times 10^{-3}$	$5.25 \times 10^{-4} \pm 3.54 \times 10^{-3}$
		2	43.7 ± 0.3	$-1.97 \times 10^{-4} \pm 1.47 \times 10^{-3}$	$5.98 \times 10^{-4} \pm 3.45 \times 10^{-3}$
	Avg		44.4	-2.00×10^{-3}	3.80×10^{-2}
L	[0, 4]	0	36.1 ± 0.4	$1.78 \times 10^{-1} \pm 3.12 \times 10^{-3}$	$5.18 \times 10^{-2} \pm 2.47 \times 10^{-1}$
		1	34.0 ± 0.3	$1.69 \times 10^{-1} \pm 2.80 \times 10^{-3}$	$6.99 \times 10^{-1} \pm 1.06 \times 10^{0}$
		2	33.6 ± 0.2	$1.67 \times 10^{-1} \pm 2.72 \times 10^{-3}$	$1.09 \times 10^{-1} \pm 5.86 \times 10^{-2}$
	[0, 8]	0	42.4 ± 0.4	$1.65 \times 10^{-1} \pm 1.48 \times 10^{-3}$	$4.26 \times 10^{-3} \pm 4.05 \times 10^{-3}$
		1	39.9 ± 0.3	$1.56 \times 10^{-1} \pm 1.09 \times 10^{-3}$	$3.81 \times 10^{-3} \pm 3.33 \times 10^{-3}$
		2	39.5 ± 0.2	$1.54 \times 10^{-1} \pm 9.66 \times 10^{-4}$	$3.95 \times 10^{-3} \pm 3.19 \times 10^{-3}$
	Avg		37.6	1.65×10^{-1}	1.45×10^{-1}
Q	[0, 4]	0	52.7 ± 0.6	$1.05 \times 10^{-1} \pm 3.24 \times 10^{-3}$	$-3.82 \times 10^{-2} \pm 2.18 \times 10^{-2}$
		1	49.1 ± 0.4	$9.76 \times 10^{-2} \pm 2.95 \times 10^{-3}$	$-3.17 \times 10^{-2} \pm 1.80 \times 10^{-2}$
		2	48.4 ± 0.3	$9.62 \times 10^{-2} \pm 2.87 \times 10^{-3}$	$-2.84 \times 10^{-2} \pm 1.73 \times 10^{-2}$
	[0, 8]	0	55.5 ± 0.5	$1.08 \times 10^{-1} \pm 1.21 \times 10^{-3}$	$-9.43 \times 10^{-2} \pm 3.14 \times 10^{-3}$
		1	52.3 ± 0.3	$1.02 \times 10^{-1} \pm 1.01 \times 10^{-3}$	$-8.24 \times 10^{-2} \pm 2.52 \times 10^{-3}$
		2	51.6 ± 0.3	$1.01 \times 10^{-1} \pm 9.21 \times 10^{-4}$	$-7.98 \times 10^{-2} \pm 2.34 \times 10^{-3}$
	Avg		51.6	1.02×10^{-1}	-5.91×10^{-2}

Table 16: H_2 service time distribution with 1000 replications: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

A	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
C	[0, 4]	0	2.4	2.3	2.0	2.1	3.1	3.1	86.4	1.7	86.4
		1	3.2	2.8	1.9	3.0	3.8	3.8	76.0	14.5	14.5
		2	3.3	2.8	1.9	3.1	3.8	3.8	74.0	13.1	13.1
	[0, 8]	0	2.6	1.6	0.5	2.5	2.6	2.5	1.9	3.0	3.0
		1	2.6	2.0	1.0	2.5	2.6	2.5	2.2	3.0	3.0
		2	2.5	2.1	1.4	2.5	2.6	2.5	2.2	2.9	2.9
	Avg		2.8	2.3	1.4	2.6	3.1	3.0	40.4	6.4	20.5
L	[0, 4]	0	18.9	13.7	3.4	12.5	3.3	3.3	169.9	8.3	8.3
		1	19.3	14.0	3.4	6.2	5.5	5.5	147.9	64.2	147.9
		2	19.3	14.0	3.3	5.2	5.8	5.8	139.3	14.6	14.6
	[0, 8]	0	17.3	11.8	0.7	7.7	2.5	2.5	36.3	2.9	2.9
		1	17.4	12.4	2.3	3.0	4.3	4.3	6.7	4.7	4.7
		2	17.4	12.4	2.5	2.3	4.6	4.6	0.6	4.9	4.9
	Avg		18.3	13.0	2.6	6.2	4.3	4.3	83.5	16.6	30.5
Q	[0, 4]	0	14.9	11.2	3.8	0.6	5.1	5.1	161.1	4.5	4.5
		1	14.8	11.4	4.4	3.1	6.3	6.3	135.7	6.4	6.4
		2	14.5	11.1	4.4	3.4	6.3	6.3	129.8	6.6	6.6
	[0, 8]	0	11.4	8.7	3.4	2.5	0.9	0.9	64.3	1.2	1.2
		1	11.3	8.8	3.8	0.7	2.0	2.0	18.8	0.8	0.8
		2	11.1	8.6	3.8	0.5	2.0	2.0	7.8	1.0	1.0
	Avg		13.0	10.0	3.9	1.8	3.8	3.8	86.2	3.4	3.4

Table 17: H_2 service time distribution with 1000 replications: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

A	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
C	$[0,4]$	0	13.1 ± 0.6	11.7 ± 0.5	14.7 ± 0.7	13.2 ± 0.6	13.1 ± 0.6	89.1 ± 7.7	35.5 ± 17.1
		1	13.3 ± 0.6	12.1 ± 0.6	13.4 ± 0.7	13.4 ± 0.6	13.4 ± 0.6	84.1 ± 7.6	27.2 ± 4.6
		2	13.3 ± 0.6	12.1 ± 0.6	13.1 ± 0.6	13.5 ± 0.6	13.4 ± 0.6	82.9 ± 7.6	25.7 ± 2.6
	$[0,8]$	0	8.6 ± 0.4	7.6 ± 0.4	8.8 ± 0.4	8.5 ± 0.4	8.4 ± 0.4	8.8 ± 0.4	8.7 ± 0.4
		1	8.8 ± 0.4	8.0 ± 0.4	8.2 ± 0.4	8.6 ± 0.4	8.6 ± 0.4	8.7 ± 0.4	8.8 ± 0.4
		2	8.8 ± 0.4	8.0 ± 0.4	8.0 ± 0.4	8.6 ± 0.4	8.6 ± 0.4	8.7 ± 0.4	8.7 ± 0.4
	Avg		11.0	9.9	11.0	11.0	10.9	47.1	19.1
L	$[0,4]$	0	18.4 ± 0.7	14.5 ± 0.6	13.4 ± 0.6	21.0 ± 1.2	13.4 ± 0.6	170.6 ± 19.4	50.1 ± 22.2
		1	19.1 ± 0.7	15.1 ± 0.6	12.7 ± 0.6	17.6 ± 1.0	13.4 ± 0.6	157.6 ± 16.0	72.6 ± 84.7
		2	19.2 ± 0.7	15.2 ± 0.6	12.5 ± 0.6	17.1 ± 0.9	13.3 ± 0.6	151.3 ± 15.5	28.3 ± 4.4
	$[0,8]$	0	16.0 ± 0.5	11.4 ± 0.4	7.8 ± 0.4	12.0 ± 0.6	8.5 ± 0.4	45.2 ± 8.5	8.9 ± 0.4
		1	16.8 ± 0.5	12.3 ± 0.4	7.5 ± 0.3	9.6 ± 0.5	8.6 ± 0.4	18.5 ± 4.2	8.9 ± 0.4
		2	16.8 ± 0.5	12.4 ± 0.4	7.5 ± 0.3	9.2 ± 0.4	8.6 ± 0.4	11.7 ± 2.0	8.9 ± 0.4
	Avg		17.7	13.5	10.2	14.4	11.0	92.5	29.6
Q	$[0,4]$	0	15.0 ± 0.6	12.4 ± 0.5	12.6 ± 0.6	12.9 ± 0.6	11.9 ± 0.5	150.5 ± 10.2	20.5 ± 1.3
		1	15.5 ± 0.6	12.8 ± 0.5	11.7 ± 0.5	12.4 ± 0.6	12.1 ± 0.5	138.6 ± 9.5	17.8 ± 1.0
		2	15.4 ± 0.6	12.9 ± 0.5	11.5 ± 0.5	12.2 ± 0.6	12.1 ± 0.5	135.7 ± 9.5	17.3 ± 0.9
	$[0,8]$	0	11.3 ± 0.4	9.1 ± 0.4	7.6 ± 0.3	8.7 ± 0.4	7.8 ± 0.4	67.1 ± 8.2	8.6 ± 0.4
		1	11.6 ± 0.4	9.5 ± 0.4	7.2 ± 0.3	8.0 ± 0.4	7.7 ± 0.4	29.5 ± 5.1	7.9 ± 0.4
		2	11.5 ± 0.4	9.5 ± 0.4	7.2 ± 0.3	7.8 ± 0.4	7.6 ± 0.4	20.2 ± 3.8	7.7 ± 0.4
	Avg		13.4	11.0	9.6	10.4	9.9	90.2	13.3

Table 18: H_2 service time distribution with 1000 replications: average of the absolute relative error of the estimate from the direct estimate in each sample path.

2.5 Longer Service Times

Formulas (19) and (22) show that the bias in $\bar{W}_{L,\lambda}(t)$ should be proportional to $E[W]$. Thus there should be more bias in $\bar{W}_{L,\lambda}(t)$ and we should achieve more bias reduction with longer service times. We illustrate that now by assuming that $E[S] = 4$ instead of 1. However, for these longer service times, the linear and quadratic approximations become less appropriate. Hence, we now use Theorem 2 as well as the other methods to do the estimation. We consider the previous case of the linear arrival rate function with exponential service. Since the system starts empty at time -12, the linear approximation is valid three mean service times in the past, and so should still be reasonable.

Table 19 provides different estimator values and Table 20 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (19) and (32)). Then, Tables 21 and 22 quantify the performance of our estimators by two performance measures. In all tables, we include the results for $E[S] = 1$ as well, to compare with the results for $E[S] = 4$.

$E[S]$	$Interval$	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
1	[0, 4]	0	1.038 ± 0.019	0.980 ± 0.020	1.047 ± 0.020	1.058 ± 0.023	1.058 ± 0.023	1.046 ± 0.022	1.046 ± 0.022	1.062 ± 0.021	1.046 ± 0.021	1.046 ± 0.021
		1	1.002 ± 0.016	0.939 ± 0.015	1.005 ± 0.015	1.011 ± 0.016	1.011 ± 0.016	1.000 ± 0.016	1.011 ± 0.016	1.014 ± 0.015	1.000 ± 0.015	1.014 ± 0.015
		2	0.996 ± 0.016	0.933 ± 0.014	1.000 ± 0.014	1.003 ± 0.015	1.003 ± 0.015	0.993 ± 0.015	1.003 ± 0.015	1.007 ± 0.013	0.993 ± 0.014	1.007 ± 0.013
	[0, 8]	0	1.051 ± 0.013	0.983 ± 0.015	1.050 ± 0.015	1.052 ± 0.017	1.052 ± 0.017	1.044 ± 0.016	1.044 ± 0.016	1.054 ± 0.016	1.045 ± 0.016	1.045 ± 0.016
		1	1.010 ± 0.010	0.944 ± 0.011	1.006 ± 0.011	1.008 ± 0.012	1.008 ± 0.012	1.001 ± 0.012	1.001 ± 0.012	1.009 ± 0.011	1.002 ± 0.011	1.009 ± 0.011
		2	1.003 ± 0.009	0.938 ± 0.010	0.998 ± 0.010	1.000 ± 0.011	1.000 ± 0.011	0.993 ± 0.011	0.993 ± 0.011	1.002 ± 0.010	0.994 ± 0.010	1.002 ± 0.010
	Avg		1.017	0.953	1.018	1.022	1.022	1.013	1.016	1.025	1.013	1.021
4	[0, 4]	0	3.954 ± 0.058	2.841 ± 0.041	3.617 ± 0.043	3.865 ± 0.065	4.000 ± 0.096	3.391 ± 0.047	3.391 ± 0.047	1.173 ± 7.752	3.048 ± 0.371	3.048 ± 0.371
		1	4.015 ± 0.063	2.877 ± 0.044	3.674 ± 0.047	3.938 ± 0.070	4.085 ± 0.102	3.441 ± 0.050	3.441 ± 0.050	1.207 ± 7.753	3.067 ± 0.374	3.067 ± 0.374
		2	4.034 ± 0.064	2.891 ± 0.046	3.696 ± 0.050	3.965 ± 0.072	4.118 ± 0.105	3.460 ± 0.052	3.460 ± 0.052	0.840 ± 7.782	3.085 ± 0.375	3.085 ± 0.375
	[0, 8]	0	3.948 ± 0.040	2.962 ± 0.025	3.702 ± 0.031	3.901 ± 0.036	3.937 ± 0.038	3.513 ± 0.030	3.513 ± 0.030	3.945 ± 0.038	3.490 ± 0.033	3.490 ± 0.033
		1	3.992 ± 0.041	2.998 ± 0.029	3.750 ± 0.035	3.968 ± 0.044	4.009 ± 0.047	3.562 ± 0.035	3.562 ± 0.035	3.838 ± 0.348	3.538 ± 0.038	3.538 ± 0.038
		2	4.003 ± 0.042	3.010 ± 0.030	3.764 ± 0.037	3.991 ± 0.047	4.034 ± 0.050	3.579 ± 0.038	3.579 ± 0.038	3.862 ± 0.350	3.555 ± 0.040	3.555 ± 0.040
	Avg		3.991	2.930	3.700	3.938	4.030	3.491	3.491	2.477	3.297	3.297

Table 19: $E[S] = 1$ and $E[S] = 4$: waiting time estimates by different methods with 95% confidence intervals.

$E[S]$	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
1	[0, 4]	0	40.8 ± 1.1	$6.72 \times 10^{-2} \pm 3.06 \times 10^{-3}$	$3.91 \times 10^{-3} \pm 4.79 \times 10^{-3}$
		1	39.1 ± 0.8	$6.46 \times 10^{-2} \pm 2.94 \times 10^{-3}$	$3.69 \times 10^{-3} \pm 4.32 \times 10^{-3}$
		2	38.8 ± 0.8	$6.42 \times 10^{-2} \pm 2.91 \times 10^{-3}$	$3.72 \times 10^{-3} \pm 4.26 \times 10^{-3}$
	[0, 8]	0	47.2 ± 0.9	$6.17 \times 10^{-2} \pm 1.11 \times 10^{-3}$	$5.35 \times 10^{-4} \pm 7.53 \times 10^{-4}$
		1	45.3 ± 0.7	$5.93 \times 10^{-2} \pm 9.63 \times 10^{-4}$	$5.69 \times 10^{-4} \pm 6.81 \times 10^{-4}$
		2	45.0 ± 0.6	$5.89 \times 10^{-2} \pm 9.16 \times 10^{-4}$	$5.80 \times 10^{-4} \pm 6.69 \times 10^{-4}$
	Avg		42.7	6.27×10^{-2}	2.17×10^{-3}
4	[0, 4]	0	118.0 ± 1.6	$1.95 \times 10^{-1} \pm 8.20 \times 10^{-3}$	$1.38 \times 10^{-1} \pm 1.42 \times 10^{-1}$
		1	119.5 ± 1.8	$1.97 \times 10^{-1} \pm 8.22 \times 10^{-3}$	$1.48 \times 10^{-1} \pm 1.43 \times 10^{-1}$
		2	120.1 ± 1.9	$1.98 \times 10^{-1} \pm 8.21 \times 10^{-3}$	$1.48 \times 10^{-1} \pm 1.43 \times 10^{-1}$
	[0, 8]	0	142.0 ± 1.4	$1.86 \times 10^{-1} \pm 2.28 \times 10^{-3}$	$1.46 \times 10^{-2} \pm 1.06 \times 10^{-2}$
		1	143.8 ± 1.7	$1.88 \times 10^{-1} \pm 2.32 \times 10^{-3}$	$1.46 \times 10^{-2} \pm 1.09 \times 10^{-2}$
		2	144.4 ± 1.8	$1.89 \times 10^{-1} \pm 2.34 \times 10^{-3}$	$1.46 \times 10^{-2} \pm 1.10 \times 10^{-2}$
	Avg		131.3	1.92×10^{-1}	7.97×10^{-2}

Table 20: $E[S] = 1$ and $E[S] = 4$: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

$E[S]$	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
1	[0, 4]	0	5.9	0.8	2.0	2.0	0.7	0.7	2.3	0.7	0.7
		1	6.3	0.3	0.9	0.8	0.2	0.8	1.2	0.2	1.2
		2	6.3	0.4	0.7	0.7	0.3	0.7	1.1	0.3	1.1
	[0, 8]	0	6.8	0.1	0.1	0.1	0.7	0.7	0.3	0.6	0.6
		1	6.6	0.5	0.3	0.3	1.0	1.0	0.1	0.9	0.1
		2	6.5	0.4	0.3	0.3	1.0	1.0	0.1	0.8	0.1
	Avg		6.4	0.4	0.7	0.7	0.7	0.8	0.8	0.6	0.6
4	[0, 4]	0	111.3	33.7	8.8	4.6	56.3	56.3	278.1	90.6	90.6
		1	113.8	34.1	7.7	7.0	57.5	57.5	280.9	94.8	94.8
		2	114.3	33.9	6.9	8.3	57.5	57.5	319.5	95.0	95.0
	[0, 8]	0	98.6	24.6	4.7	1.1	43.5	43.5	0.3	45.8	45.8
		1	99.4	24.2	2.4	1.7	43.0	43.0	15.3	45.3	45.3
		2	99.3	23.9	1.1	3.1	42.4	42.4	14.1	44.8	44.8
	Avg		106.1	29.1	5.3	4.3	50.0	50.0	151.4	69.4	69.4

Table 21: $E[S] = 1$ and $E[S] = 4$: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

$E[S]$	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,Thm2}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
1	[0, 4]	0	7.0 ± 1.1	4.3 ± 0.6	6.6 ± 0.9	6.6 ± 0.9	6.3 ± 0.8	6.2 ± 0.9	5.8 ± 0.8
		1	7.2 ± 1.1	4.4 ± 0.6	6.1 ± 0.9	6.1 ± 0.9	5.8 ± 0.8	5.7 ± 0.8	5.3 ± 0.7
		2	7.1 ± 1.1	4.4 ± 0.6	6.0 ± 0.8	6.0 ± 0.8	5.7 ± 0.7	5.6 ± 0.8	5.2 ± 0.7
	[0, 8]	0	6.6 ± 0.7	2.0 ± 0.3	3.4 ± 0.5	3.4 ± 0.6	3.3 ± 0.6	3.0 ± 0.5	3.1 ± 0.5
		1	6.6 ± 0.7	1.9 ± 0.3	3.1 ± 0.5	3.1 ± 0.5	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
		2	6.5 ± 0.6	1.9 ± 0.3	3.0 ± 0.4	3.0 ± 0.4	2.9 ± 0.5	2.7 ± 0.4	2.7 ± 0.4
	Avg		6.8	3.2	4.7	4.7	4.5	4.3	4.1
4	[0, 4]	0	27.8 ± 1.5	9.9 ± 1.1	9.8 ± 1.4	11.9 ± 2.0	14.7 ± 1.5	267.1 ± 196.4	30.7 ± 8.6
		1	27.9 ± 1.6	10.0 ± 1.1	10.2 ± 1.5	12.3 ± 2.1	15.0 ± 1.5	265.2 ± 196.3	31.4 ± 8.5
		2	27.9 ± 1.6	9.9 ± 1.1	10.1 ± 1.5	12.3 ± 2.2	14.9 ± 1.5	274.2 ± 196.7	31.3 ± 8.5
	[0, 8]	0	24.8 ± 0.9	6.6 ± 0.7	4.9 ± 0.7	4.9 ± 0.8	11.0 ± 0.9	5.0 ± 0.7	11.6 ± 1.0
		1	24.7 ± 0.9	6.5 ± 0.7	4.9 ± 0.8	5.1 ± 0.8	10.9 ± 0.9	9.5 ± 8.6	11.4 ± 1.0
		2	24.7 ± 0.9	6.4 ± 0.7	4.9 ± 0.8	5.1 ± 0.9	10.7 ± 0.9	9.4 ± 8.4	11.3 ± 1.0
	Avg		26.3	8.2	7.5	8.6	12.9	138.4	21.3

Table 22: $E[S] = 1$ and $E[S] = 4$: average of the absolute relative error of the estimate from the direct estimate in each sample path.

2.6 A Decreasing Arrival Rate Function

In this section, we consider a special case of a decreasing arrival rate function and thus decreasing staffing. We use a minor modification of the previous linear arrival rate function, with time reversed. Specifically, we use $\lambda(t) = 48 - 3t$ over [0,4] and [0,8]. Table 23 gives the average of the estimated parameters for the linear decreasing arrival rate function over 100 replications.

	Constant	Linear		Quadratic		
Int.	$\bar{\lambda}(t)$	a	b	a	b	c
$[-4, 4]$	42.6 ± 0.6	48.4 ± 0.5	-2.877 ± 0.187	48.6 ± 0.8	-2.877 ± 0.187	-0.026 ± 0.100
$[-8, 8]$	36.1 ± 0.4	48.2 ± 0.4	-3.018 ± 0.068	48.4 ± 0.5	-3.018 ± 0.068	-0.009 ± 0.018

Table 23: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-4, 4]$ and $[-8, 8]$ to the arrival data for linear and decreasing arrival rate function; Average and halfwidths of 95% confidence intervals over 100 replications.

Int.		[0, 4]						[0, 8]				
GI	β	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$	$\%v$	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$
M	0	12	0.68 ± 0.06	8.18 ± 0.69	180.7 ± 2.7	7.97 ± 0.68	4.36 ± 0.36	24	0.67 ± 0.04	16.09 ± 1.06	314.3 ± 3.7	15.61 ± 1.02
	1	13	0.25 ± 0.05	3.19 ± 0.63	182.0 ± 2.8	3.00 ± 0.64	1.59 ± 0.34	26	0.23 ± 0.04	5.89 ± 0.93	313.4 ± 3.6	5.25 ± 0.90
	2	14	0.05 ± 0.02	0.69 ± 0.24	182.5 ± 2.9	0.64 ± 0.26	0.33 ± 0.14	28	0.04 ± 0.01	1.11 ± 0.34	313.3 ± 3.6	0.96 ± 0.32
H_2	0	12	0.37 ± 0.06	4.43 ± 0.74	177.6 ± 2.5	4.14 ± 0.73	2.30 ± 0.40	24	0.41 ± 0.05	9.83 ± 1.30	307.8 ± 3.5	9.42 ± 1.30
	1	13	0.07 ± 0.02	0.88 ± 0.31	179.2 ± 2.6	0.83 ± 0.31	0.46 ± 0.17	26	0.09 ± 0.03	2.29 ± 0.71	308.6 ± 3.4	2.23 ± 0.77
	2	14	0.01 ± 0.00	0.08 ± 0.06	179.5 ± 2.6	0.07 ± 0.06	0.04 ± 0.03	28	0.01 ± 0.01	0.27 ± 0.19	308.8 ± 3.4	0.29 ± 0.25
E_4	0	12	0.67 ± 0.06	8.04 ± 0.67	181.0 ± 2.2	7.95 ± 0.67	4.33 ± 0.34	24	0.66 ± 0.04	15.86 ± 1.06	314.2 ± 3.4	15.24 ± 1.08
	1	12	0.20 ± 0.04	2.45 ± 0.50	182.2 ± 2.5	2.35 ± 0.52	1.23 ± 0.26	26	0.19 ± 0.03	4.89 ± 0.81	313.2 ± 3.5	4.33 ± 0.74
	2	13	0.04 ± 0.02	0.56 ± 0.21	182.5 ± 2.6	0.53 ± 0.22	0.28 ± 0.11	27	0.03 ± 0.01	0.93 ± 0.30	313.1 ± 3.5	0.71 ± 0.26

Table 24: Early service termination in the 9 different models with linear decreasing arrival rate; *#dec* indicates the number of staffing decreases, *#dep* indicates the number of departures and v indicates violations.

Int.		[0, 4]					[0, 8]				
GI	β	$E[W]$	$\%Delayed$	$\%Aban.$	$\%EarlyTer.$	$TETT$	$E[W]$	$\%Delayed$	$\%Aban.$	$\%EarlyTer.$	$TETT$
M	0	1.12 ± 0.02	67.7 ± 5.7	4.15 ± 0.78	4.85 ± 0.42	0.20 ± 0.02	1.11 ± 0.02	65.4 ± 4.4	4.97 ± 0.74	5.35 ± 0.34	0.45 ± 0.04
	1	1.04 ± 0.02	23.7 ± 4.7	0.61 ± 0.21	1.61 ± 0.32	0.06 ± 0.02	1.03 ± 0.01	21.0 ± 3.4	0.66 ± 0.19	1.68 ± 0.27	0.12 ± 0.02
	2	1.02 ± 0.02	4.6 ± 1.6	0.04 ± 0.03	0.35 ± 0.14	0.01 ± 0.01	1.02 ± 0.01	3.4 ± 1.1	0.04 ± 0.03	0.32 ± 0.10	0.02 ± 0.01
H_2	0	1.05 ± 0.04	36.3 ± 6.2	1.93 ± 0.66	3.44 ± 0.49	2.09 ± 0.76	1.04 ± 0.03	41.0 ± 5.5	3.62 ± 1.02	4.13 ± 0.50	4.38 ± 1.25
	1	1.01 ± 0.04	6.5 ± 2.3	0.13 ± 0.09	0.91 ± 0.26	1.70 ± 0.66	1.00 ± 0.03	8.8 ± 2.9	0.52 ± 0.35	1.25 ± 0.29	3.80 ± 1.18
	2	1.01 ± 0.04	0.5 ± 0.4	0.01 ± 0.01	0.37 ± 0.12	1.51 ± 0.62	0.99 ± 0.03	1.0 ± 0.8	0.04 ± 0.06	0.54 ± 0.14	3.37 ± 1.12
E_4	0	1.10 ± 0.02	66.6 ± 5.6	3.05 ± 0.63	4.76 ± 0.38	0.20 ± 0.02	1.08 ± 0.01	64.3 ± 4.4	3.84 ± 0.62	5.17 ± 0.32	0.43 ± 0.04
	1	1.02 ± 0.01	19.5 ± 4.1	0.37 ± 0.14	1.39 ± 0.30	0.05 ± 0.01	1.01 ± 0.01	17.1 ± 2.8	0.39 ± 0.11	1.42 ± 0.22	0.10 ± 0.02
	2	1.01 ± 0.01	4.0 ± 1.5	0.04 ± 0.03	0.28 ± 0.12	0.01 ± 0.00	1.00 ± 0.01	2.9 ± 0.9	0.04 ± 0.03	0.23 ± 0.08	0.01 ± 0.00

Table 25: Average performance of the 9 different models with linear decreasing arrival rate, averaged over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$; $TETT$ indicates the total early termination time.

We use three service time distributions, M, H_2 and E_4, with the same specifications given in the beginning of Section 2.2. We compute $m(t)$ and $s(t)$ as before, with $\beta = 0$, 1, and 2 and start empty at -12. This gives us nine different models. If a server is scheduled to depart when all servers are busy, then in our simulations we let that server depart immediately and force the customer with the least remaining service time to complete service at that time. In fact, we assume that the server scheduled to leave would actually depart only after that minimum remaining service time has elapsed. At that time, the server completing service can take over the service of the departing server's customer, because service switching is allowed.

To study this effect, Table 24 shows the number of staffing decreases ($\#dec$), the number of departures ($\#dep$), the number of violation ($\#v$) and the percentage of departures that are violations ($\%v$) in each case. From Table 24, we see that we could also estimate the number of violations in advance, before doing any simulation, by $\#dec \times P(W > 0)$, where $\#dec$ can be computed from the offered load and thus the staffing function given β, while $P(W > 0)$ can be estimated using the Garnett function for the stationary $M/M/s + M$ Erlang-A model in equation (11) and Figure 6 of [2]. However, in Table 24, $P(W > 0)$ is estimated from the simulation output by the number of arrivals that have to wait before starting service divided by the total number of arrivals during the interval.

Table 25 shows other key performance estimates, including the total early termination time ($TETT$), which can be divided by the number of arrivals to estimate the addition to the mean waiting time. For M service, $TETT$ can also be estimated in advance, before doing any simulation, by multiplying the number of violations (estimated as described above) by the the expected minimum remaining service time among all servers, which can be estimated by mean service time divided by the average number of service time, i.e., the expected minimum remaining service time among all servers servers. From the simulation results, we show that, for M and E_4 service, the average waiting time is consistently increased by about 0.1% for $\beta = 0$ and much less for $\beta = 1$ and 2. For H_2 service, the average waiting time is consistently increased by about 1.0% or less, which is still negligible. The H_2 case is relatively more problematic, because the remaining waiting times tend to be much longer than $E[W] \approx E[S] = 1$, usually having mean close to the larger of the two exponential means, Nevertheless, this effect is still relatively small. Figures 56 - 73 provide more information on the model performance over time.

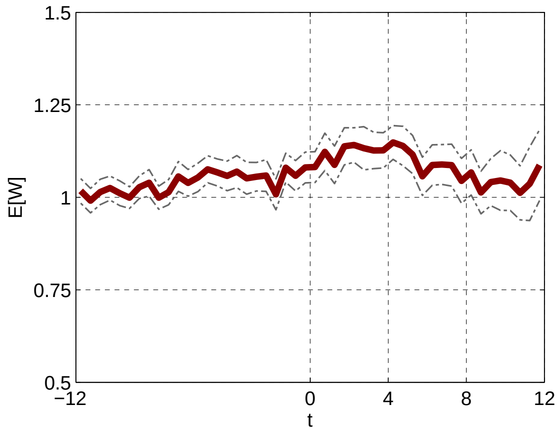

Figure 56: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

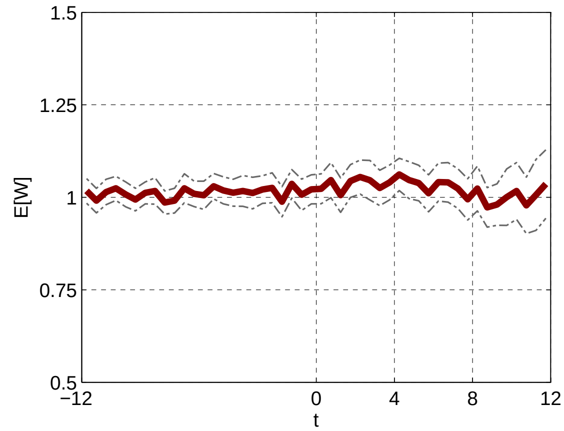

Figure 57: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

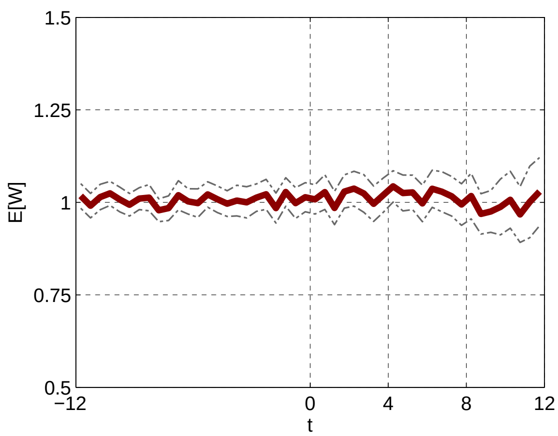

Figure 58: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

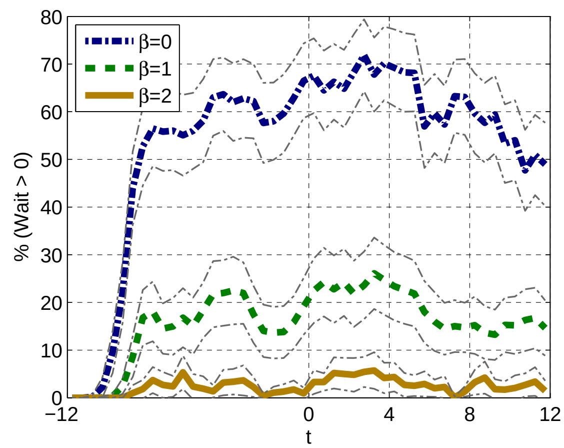

Figure 59: Average percent of arrivals delayed over periods of length 0.5.

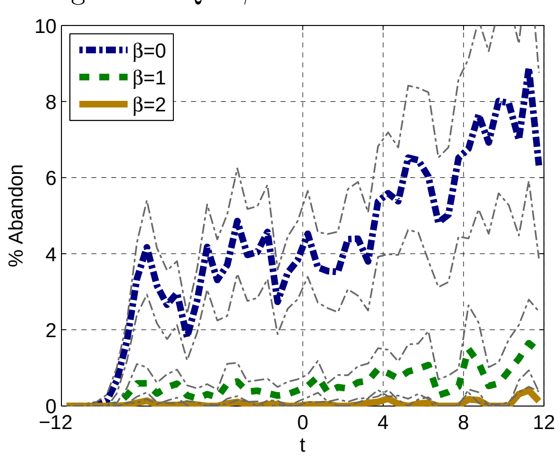

Figure 60: Average percent of abandonment over periods of length 0.5.

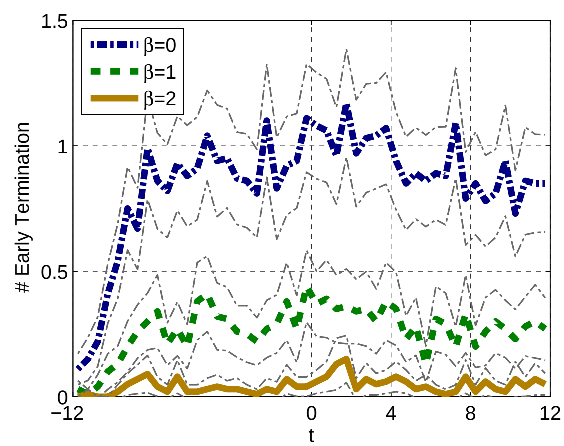

Figure 61: Number of early termination in each subinterval of length 0.5.

Figures 56-61: Linear (decreasing) arrival rate and M service time distribution

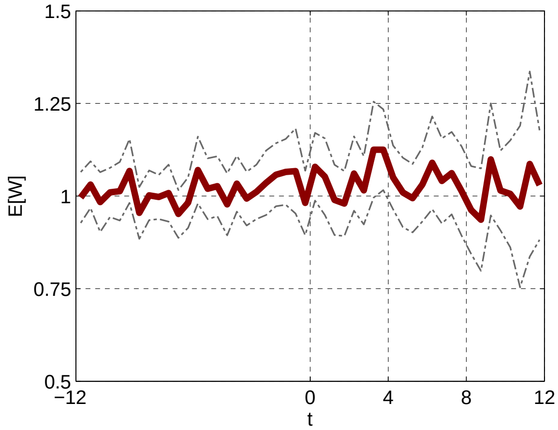

Figure 62: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

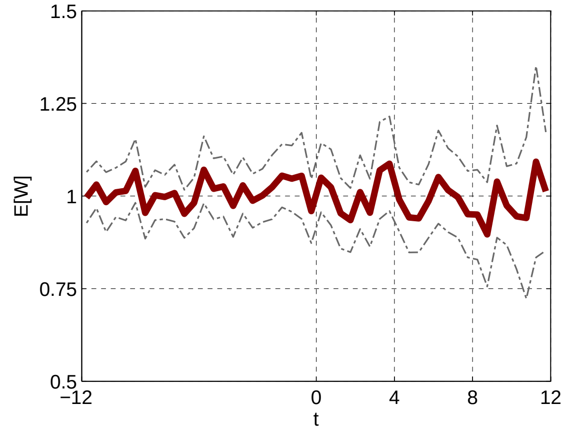

Figure 63: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

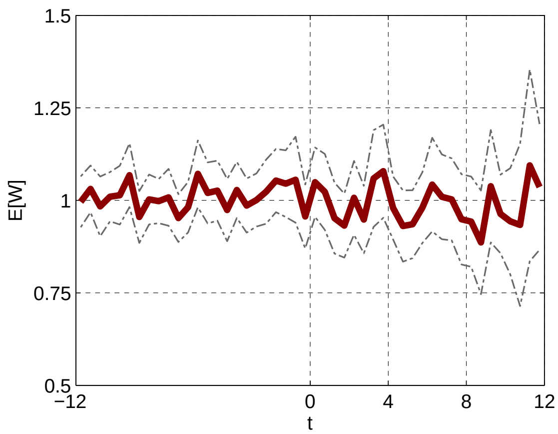

Figure 64: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

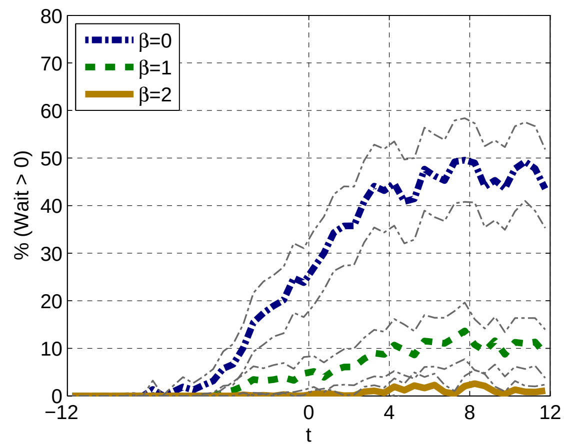

Figure 65: Average percent of arrivals delayed over periods of length 0.5.

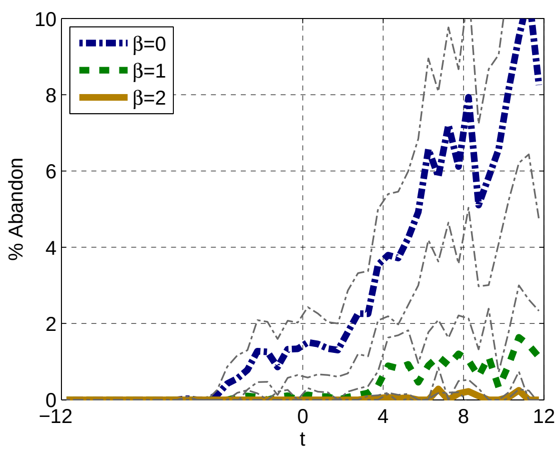

Figure 66: Average percent of abandonment over periods of length 0.5.

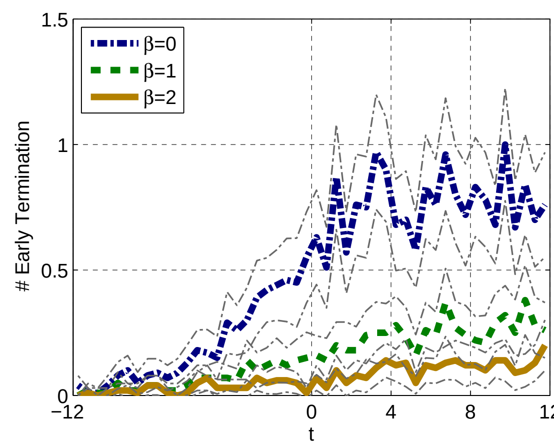

Figure 67: Number of early termination in each subinterval of length 0.5.

Figures 62-67: Linear (decreasing) arrival rate and H_2 service time distribution

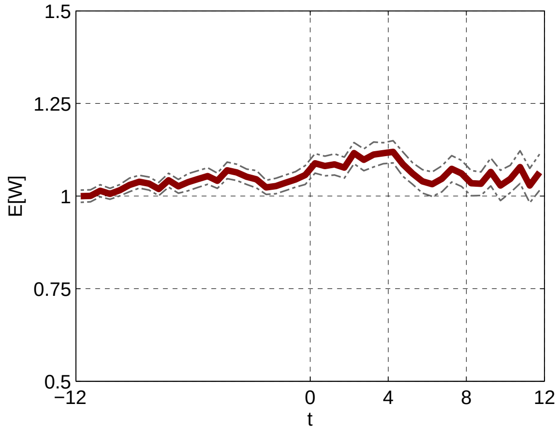

Figure 68: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

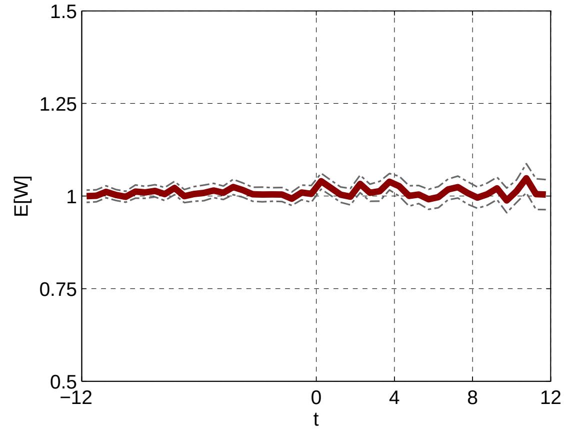

Figure 69: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

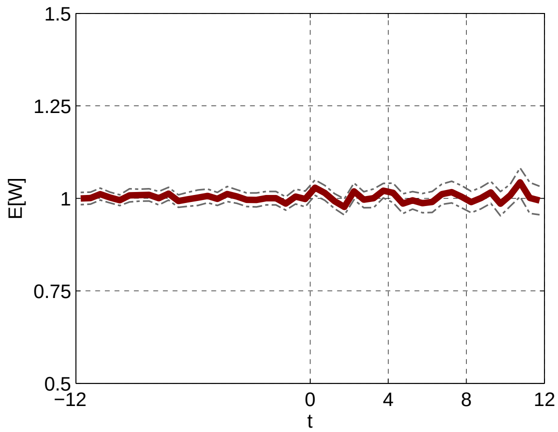

Figure 70: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

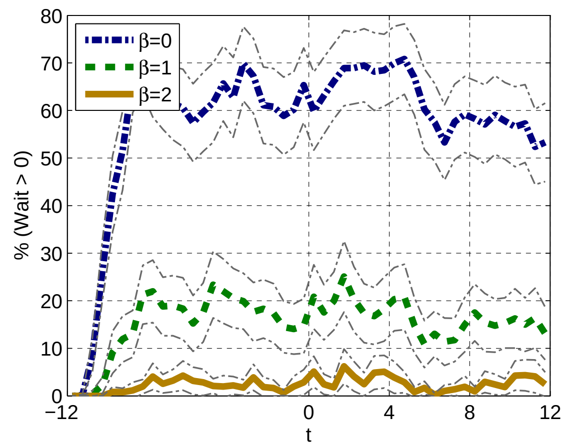

Figure 71: Average percent of arrivals delayed over periods of length 0.5.

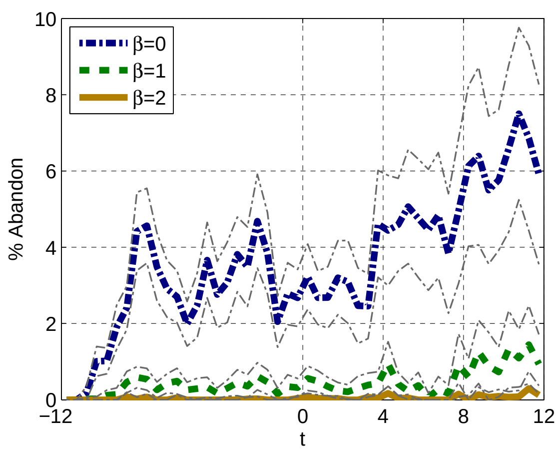

Figure 72: Average percent of arrivals abandoning over periods of length 0.5.

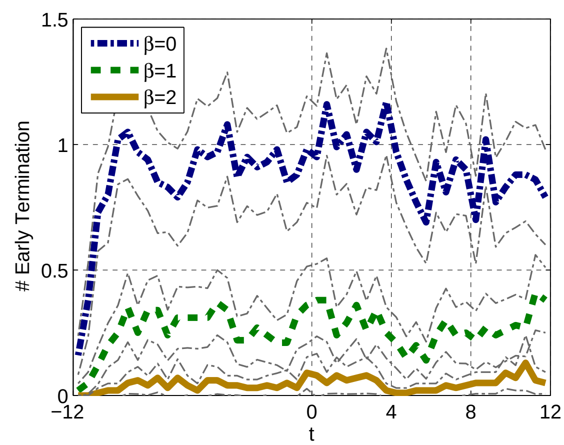

Figure 73: Number of early termination in each subinterval of length 0.5.

Figures 68-73: Linear (decreasing) arrival rate and E_4 service time distribution

We now present estimation results. Table 26 provides different estimator values and Table 27 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (19) and (32)).

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.121 ± 0.024	1.183 ± 0.026	1.109 ± 0.027	1.109 ± 0.027	1.099 ± 0.022	1.085 ± 0.022	1.085 ± 0.022	1.098 ± 0.021	1.080 ± 0.020	1.080 ± 0.020
		1	1.037 ± 0.019	1.107 ± 0.019	1.030 ± 0.020	1.030 ± 0.020	1.032 ± 0.016	1.021 ± 0.016	1.021 ± 0.016	1.031 ± 0.015	1.017 ± 0.015	1.017 ± 0.015
		2	1.016 ± 0.016	1.089 ± 0.016	1.011 ± 0.017	1.011 ± 0.017	1.017 ± 0.014	1.006 ± 0.013	1.006 ± 0.013	1.016 ± 0.013	1.002 ± 0.013	1.016 ± 0.013
	[0, 8]	0	1.113 ± 0.019	1.203 ± 0.021	1.096 ± 0.018	1.096 ± 0.018	1.101 ± 0.018	1.081 ± 0.017	1.081 ± 0.017	1.100 ± 0.017	1.078 ± 0.016	1.078 ± 0.016
		1	1.035 ± 0.014	1.122 ± 0.015	1.026 ± 0.014	1.026 ± 0.014	1.033 ± 0.013	1.016 ± 0.013	1.016 ± 0.013	1.032 ± 0.013	1.014 ± 0.012	1.014 ± 0.012
		2	1.018 ± 0.012	1.104 ± 0.013	1.011 ± 0.012	1.011 ± 0.012	1.018 ± 0.011	1.002 ± 0.011	1.002 ± 0.011	1.017 ± 0.011	1.000 ± 0.011	1.000 ± 0.011
	Avg		1.057	1.135	1.047	1.047	1.050	1.035	1.035	1.049	1.032	1.034
H_2	[0, 4]	0	1.054 ± 0.040	1.174 ± 0.029	1.122 ± 0.031	1.017 ± 0.055	0.980 ± 0.024	0.885 ± 0.026	0.980 ± 0.024	0.396 ± 0.209	0.775 ± 0.095	0.775 ± 0.095
		1	1.010 ± 0.039	1.132 ± 0.020	1.072 ± 0.021	0.951 ± 0.038	0.950 ± 0.017	0.863 ± 0.021	0.950 ± 0.017	0.417 ± 0.198	0.768 ± 0.073	0.768 ± 0.073
		2	1.006 ± 0.038	1.128 ± 0.019	1.066 ± 0.020	0.943 ± 0.035	0.947 ± 0.016	0.860 ± 0.021	0.947 ± 0.016	0.415 ± 0.198	0.771 ± 0.067	0.771 ± 0.067
	[0, 8]	0	1.049 ± 0.033	1.247 ± 0.031	1.165 ± 0.031	1.000 ± 0.041	0.995 ± 0.021	0.846 ± 0.014	0.846 ± 0.014	0.997 ± 0.022	0.839 ± 0.014	0.839 ± 0.014
		1	1.001 ± 0.029	1.194 ± 0.023	1.112 ± 0.023	0.947 ± 0.030	0.961 ± 0.015	0.828 ± 0.011	0.828 ± 0.011	0.962 ± 0.015	0.822 ± 0.011	0.822 ± 0.011
		2	0.996 ± 0.028	1.188 ± 0.021	1.105 ± 0.020	0.940 ± 0.027	0.956 ± 0.014	0.826 ± 0.010	0.826 ± 0.010	0.957 ± 0.014	0.820 ± 0.010	0.820 ± 0.010
	Avg		1.019	1.177	1.107	0.966	0.965	0.851	0.896	0.691	0.799	0.799
E_4	[0, 4]	0	1.097 ± 0.017	1.129 ± 0.019	1.055 ± 0.018	1.083 ± 0.017	1.078 ± 0.017	1.074 ± 0.017	1.078 ± 0.017	1.076 ± 0.015	1.070 ± 0.015	1.070 ± 0.015
		1	1.020 ± 0.009	1.060 ± 0.011	0.984 ± 0.012	1.012 ± 0.010	1.015 ± 0.011	1.011 ± 0.010	1.011 ± 0.010	1.014 ± 0.010	1.009 ± 0.009	1.009 ± 0.009
		2	1.006 ± 0.008	1.048 ± 0.009	0.971 ± 0.010	1.000 ± 0.008	1.004 ± 0.009	1.000 ± 0.008	1.000 ± 0.008	1.003 ± 0.008	0.998 ± 0.008	0.998 ± 0.008
	[0, 8]	0	1.087 ± 0.013	1.143 ± 0.015	1.041 ± 0.013	1.079 ± 0.013	1.082 ± 0.015	1.074 ± 0.014	1.074 ± 0.014	1.081 ± 0.014	1.072 ± 0.013	1.072 ± 0.013
		1	1.017 ± 0.007	1.069 ± 0.009	0.978 ± 0.008	1.012 ± 0.007	1.016 ± 0.009	1.009 ± 0.008	1.009 ± 0.008	1.015 ± 0.008	1.008 ± 0.007	1.008 ± 0.007
		2	1.005 ± 0.006	1.057 ± 0.007	0.967 ± 0.007	1.001 ± 0.006	1.004 ± 0.007	0.998 ± 0.006	0.998 ± 0.006	1.003 ± 0.007	0.997 ± 0.006	0.997 ± 0.006
	Avg		1.039	1.084	0.999	1.031	1.033	1.028	1.028	1.032	1.026	1.026

Table 26: LINEAR decreasing arrival rate: waiting time estimates by different methods with 95% confidence intervals.

Tables 28 and 29 quantify the performance of our estimators by two performance measures. The first one is the absolute difference between the average of the estimate in interest over the 100 replications and the average of the direct estimate of mean waiting time over the 100 replications. The second measure is the absolute relative error of the estimate in each sample path, averaged over 100 replications.

GI	*Interval*	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}'_l}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
M	[0, 4]	0	50.5 ± 1.4	$-8.16 \times 10^{-2} \pm 6.38 \times 10^{-3}$	$-3.67 \times 10^{-4} \pm 5.61 \times 10^{-3}$
		1	47.1 ± 1.0	$-7.64 \times 10^{-2} \pm 5.96 \times 10^{-3}$	$4.50 \times 10^{-5} \pm 4.93 \times 10^{-3}$
		2	46.4 ± 0.8	$-7.53 \times 10^{-2} \pm 5.90 \times 10^{-3}$	$2.90 \times 10^{-4} \pm 4.78 \times 10^{-3}$
	[0, 8]	0	43.5 ± 0.9	$-1.01 \times 10^{-1} \pm 3.19 \times 10^{-3}$	$-5.48 \times 10^{-4} \pm 1.21 \times 10^{-3}$
		1	40.5 ± 0.7	$-9.42 \times 10^{-2} \pm 2.88 \times 10^{-3}$	$-3.88 \times 10^{-4} \pm 1.07 \times 10^{-3}$
		2	39.9 ± 0.6	$-9.28 \times 10^{-2} \pm 2.86 \times 10^{-3}$	$-3.19 \times 10^{-4} \pm 1.04 \times 10^{-3}$
	Avg		44.6	-8.69×10^{-2}	-2.15×10^{-4}
H_2	[0, 4]	0	50.0 ± 1.4	$-2.42 \times 10^{-1} \pm 1.91 \times 10^{-2}$	$8.00 \times 10^{-2} \pm 9.85 \times 10^{-2}$
		1	48.1 ± 0.9	$-2.35 \times 10^{-1} \pm 1.88 \times 10^{-2}$	$7.25 \times 10^{-2} \pm 8.52 \times 10^{-2}$
		2	48.0 ± 0.9	$-2.34 \times 10^{-1} \pm 1.87 \times 10^{-2}$	$6.91 \times 10^{-2} \pm 8.14 \times 10^{-2}$
	[0, 8]	0	45.0 ± 1.2	$-3.15 \times 10^{-1} \pm 1.31 \times 10^{-2}$	$-1.35 \times 10^{-4} \pm 1.42 \times 10^{-2}$
		1	43.1 ± 0.9	$-3.02 \times 10^{-1} \pm 1.17 \times 10^{-2}$	$3.16 \times 10^{-4} \pm 1.33 \times 10^{-2}$
		2	42.8 ± 0.8	$-3.01 \times 10^{-1} \pm 1.15 \times 10^{-2}$	$2.28 \times 10^{-4} \pm 1.32 \times 10^{-2}$
	Avg		46.2	-2.72×10^{-1}	3.70×10^{-2}
E_4	[0, 4]	0	48.2 ± 1.2	$-4.86 \times 10^{-2} \pm 3.74 \times 10^{-3}$	$-5.25 \times 10^{-4} \pm 1.74 \times 10^{-3}$
		1	45.2 ± 0.8	$-4.56 \times 10^{-2} \pm 3.46 \times 10^{-3}$	$-1.86 \times 10^{-4} \pm 1.52 \times 10^{-3}$
		2	44.6 ± 0.7	$-4.51 \times 10^{-2} \pm 3.44 \times 10^{-3}$	$-1.10 \times 10^{-4} \pm 1.48 \times 10^{-3}$
	[0, 8]	0	41.3 ± 0.9	$-5.99 \times 10^{-2} \pm 1.71 \times 10^{-3}$	$-2.19 \times 10^{-4} \pm 3.66 \times 10^{-4}$
		1	38.6 ± 0.6	$-5.61 \times 10^{-2} \pm 1.62 \times 10^{-3}$	$-1.48 \times 10^{-4} \pm 3.20 \times 10^{-4}$
		2	38.2 ± 0.5	$-5.55 \times 10^{-2} \pm 1.63 \times 10^{-3}$	$-1.29 \times 10^{-4} \pm 3.12 \times 10^{-4}$
	Avg		42.7	-5.18×10^{-2}	-2.19×10^{-4}

Table 27: LINEAR decreasing arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

GI	*Interval*	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	[0, 4]	0	6.2	1.2	1.2	2.2	3.6	3.6	2.3	4.1	4.1
		1	7.0	0.7	0.7	0.4	1.6	1.6	0.5	2.0	2.0
		2	7.3	0.6	0.6	0.1	1.0	1.0	0.0	1.4	0.0
	[0, 8]	0	9.1	1.7	1.7	1.1	3.2	3.2	1.3	3.4	3.4
		1	8.7	0.9	0.9	0.3	1.9	1.9	0.4	2.1	2.1
		2	8.6	0.8	0.8	0.1	1.7	1.7	0.2	1.9	1.9
	Avg		7.8	1.0	1.0	0.7	2.2	2.2	0.8	2.5	2.3
H_2	[0, 4]	0	12.0	6.8	3.7	7.4	16.9	7.4	65.8	27.9	27.9
		1	12.2	6.2	5.9	6.1	14.8	6.1	59.3	24.2	24.2
		2	12.2	6.1	6.3	5.9	14.5	5.9	59.1	23.4	23.4
	[0, 8]	0	19.7	11.5	4.9	5.4	20.4	20.4	5.2	21.0	21.0
		1	19.3	11.1	5.3	4.0	17.3	17.3	3.9	17.9	17.9
		2	19.2	10.9	5.6	3.9	17.0	17.0	3.8	17.6	17.6
	Avg		15.8	8.8	5.3	5.4	16.8	12.3	32.8	22.0	22.0
E_4	[0, 4]	0	3.2	4.2	1.4	1.9	2.3	1.9	2.1	2.7	2.7
		1	4.0	3.6	0.8	0.5	0.9	0.9	0.6	1.1	1.1
		2	4.1	3.5	0.7	0.3	0.6	0.6	0.3	0.8	0.8
	[0, 8]	0	5.6	4.6	0.8	0.5	1.2	1.2	0.6	1.4	1.4
		1	5.3	3.8	0.4	0.1	0.7	0.7	0.2	0.9	0.9
		2	5.2	3.7	0.4	0.1	0.7	0.7	0.1	0.8	0.8
	Avg		4.6	3.9	0.7	0.5	1.1	1.0	0.7	1.3	1.3

Table 28: LINEAR decreasing arrival rate: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0, 4]$	0	7.2 ± 1.1	4.0 ± 0.5	4.0 ± 0.5	5.6 ± 0.7	5.8 ± 0.8	5.2 ± 0.7	5.8 ± 0.8
		1	8.1 ± 1.2	3.9 ± 0.5	3.9 ± 0.5	5.5 ± 0.7	5.6 ± 0.8	5.2 ± 0.8	5.4 ± 0.8
		2	8.4 ± 1.2	4.0 ± 0.5	4.0 ± 0.5	5.6 ± 0.8	5.6 ± 0.8	5.2 ± 0.8	5.3 ± 0.8
	$[0, 8]$	0	8.3 ± 0.8	2.4 ± 0.3	2.4 ± 0.3	3.1 ± 0.5	3.7 ± 0.6	3.3 ± 0.4	3.9 ± 0.5
		1	8.6 ± 0.8	2.3 ± 0.3	2.3 ± 0.3	3.0 ± 0.5	3.2 ± 0.5	3.1 ± 0.4	3.3 ± 0.5
		2	8.6 ± 0.8	2.2 ± 0.3	2.2 ± 0.3	3.1 ± 0.5	3.3 ± 0.5	3.2 ± 0.4	3.3 ± 0.5
	Avg		8.2	3.1	3.1	4.3	4.5	4.2	4.5
H_2	$[0, 4]$	0	19.3 ± 3.4	14.6 ± 2.6	16.0 ± 2.6	14.2 ± 1.9	17.8 ± 2.2	69.0 ± 20.0	31.5 ± 7.2
		1	20.9 ± 3.8	16.0 ± 2.9	13.8 ± 2.4	15.2 ± 2.2	18.1 ± 2.3	68.7 ± 20.2	29.9 ± 6.0
		2	21.0 ± 3.8	16.2 ± 2.9	14.1 ± 2.4	15.2 ± 2.2	18.1 ± 2.3	68.8 ± 20.3	29.6 ± 5.7
	$[0, 8]$	0	21.0 ± 2.6	13.8 ± 1.8	10.8 ± 1.5	9.8 ± 1.3	18.3 ± 2.0	9.9 ± 1.3	18.8 ± 2.1
		1	21.8 ± 2.8	14.4 ± 2.0	10.1 ± 1.5	10.1 ± 1.4	16.9 ± 1.9	10.2 ± 1.4	17.3 ± 2.0
		2	21.9 ± 2.8	14.5 ± 2.0	10.1 ± 1.5	10.1 ± 1.4	16.8 ± 1.9	10.2 ± 1.4	17.2 ± 2.0
	Avg		21.0	14.9	12.5	12.4	17.6	39.5	24.0
E_4	$[0, 4]$	0	5.2 ± 0.8	4.1 ± 0.5	2.4 ± 0.3	3.8 ± 0.5	4.1 ± 0.6	3.5 ± 0.5	3.9 ± 0.6
		1	5.0 ± 0.8	4.0 ± 0.5	2.3 ± 0.3	3.6 ± 0.5	3.6 ± 0.5	3.1 ± 0.4	3.1 ± 0.5
		2	5.1 ± 0.8	4.1 ± 0.5	2.3 ± 0.3	3.6 ± 0.5	3.5 ± 0.5	3.2 ± 0.4	3.1 ± 0.5
	$[0, 8]$	0	5.3 ± 0.6	4.2 ± 0.3	1.4 ± 0.2	2.2 ± 0.3	2.5 ± 0.4	2.2 ± 0.3	2.4 ± 0.3
		1	5.2 ± 0.6	3.9 ± 0.4	1.4 ± 0.2	2.2 ± 0.3	2.3 ± 0.3	2.3 ± 0.3	2.0 ± 0.3
		2	5.2 ± 0.6	3.9 ± 0.4	1.3 ± 0.2	2.3 ± 0.3	2.3 ± 0.3	2.3 ± 0.3	2.0 ± 0.3
	Avg		5.2	4.0	1.9	3.0	3.1	2.8	2.8

Table 29: LINEAR decreasing arrival rate: average of the absolute relative error of the estimate from the direct estimate in each sample path.

2.7 Sinusoidal Arrival Rate Function

In Section 8.6 of the main paper, we consider a sinusoidal arrival rate function, in order to illustrate how the estimation procedure should apply for a realistic arrival rate function arising in applications, which is not exactly linear or quadratic. In this section, we provide additional information about the simulation experiments and results summarized in Section 8.6.

The arrival rate function is now $\lambda(t) = 40 + 25\sin(t/2)$ over the intervals $[0,4]$ and $[0,8]$, starting empty at time -36. Following previous experiments, we let $E[S] = 1$ and consider two service time distributions: exponential and hyperexponential. Assuming that the system starts empty in the infinite past, as in (10) of the main paper, an exact expression for the offered load for exponential service time is $m(t) = 40 + 20(\sin(t/2) - (1/2)\cos(t/2))$ by (15) of [1]. If we let the service time by hyperexponential H_2 with $c_W^2 = 5$ and balanced means, an exact expression for the offered load is $m(t) = 40 + 25(0.5242\sin(t/2) - 0.2897\cos(t/2))$ by (29) of [1] after correcting an error in (29); see the short appendix in the main paper.

Based on this explicit offered load formula, we consider three levels of staffing, according to (33) with QoS parameter $\beta = 0$, 1 and 2 as before. The arrival rate, offered load and staffing with $\beta = 1$ are shown in Figures 74 and 75 for the exponential and H_2 service time distribution, respectively.

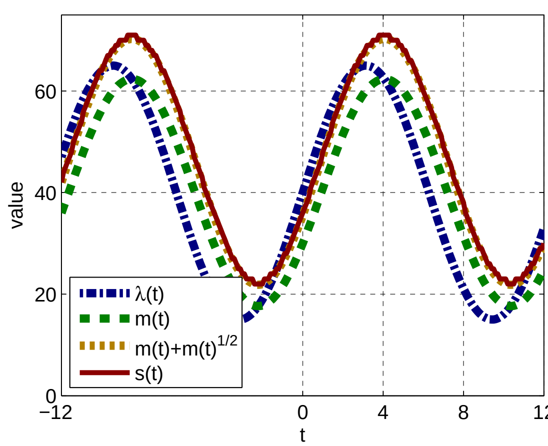

Figure 74: The sinusoidal arrival rate function, offered load function and staffing for exponential service time distribution according to (33) with $\beta = 1$ based on starting empty in the distant past. These will be applied over the intervals $[0,4]$ and $[0,8]$ to the system starting empty at time -36.

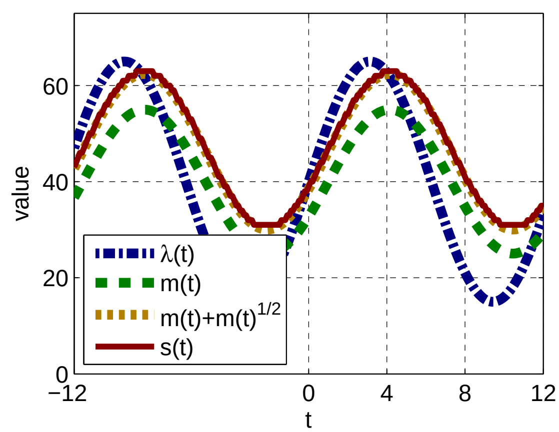

Figure 75: The sinusoidal arrival rate function, offered load function and staffing for H_2 service time distribution according to (33) with $\beta = 1$ based on starting empty in the distant past. These will be applied over the intervals $[0,4]$ and $[0,8]$ to the system starting empty at time -36.

Figures 74 and 75 show that a quadratic approximation to the arrival rate function should be appropriate over the target intervals $[0,4]$ and $[0,8]$. The arrival rate can be regarded as approximately nondecreasing over $[0,4]$, but it is decreasing over a significant portion of $[0,8]$, so that we also study the effect of server release when all servers are busy.

The experiment involves fitting a quadratic function to simulation data for the arrival process. For the target intervals $[0,4]$ and $[0,8]$, we base the estimation on data from the intervals $[-2,4]$ and $[-2,8]$, respectively, which supports a reasonable quadratic approximation. We simulate the $M_t/M/s_t+M$ and $M_t/H_2/s_t+M$ systems over the interval $[-36,12]$, starting empty at time -36.

Figure 30 illustrates the average of the estimated parameters for the sinusoidal arrival rate function over 100 replications and Figures 76 and 77 show plots of the approximations. We approximate the arrival rate by constant function, we consider intervals $[0,4]$ and $[0,8]$ instead of $[-2,4]$ and $[-2,8]$. The halfwidths of 95% confidence intervals for all estimates are also reported.

	Constant	Linear		Quadratic		
Int.	$\bar{\lambda}(t)$	a	b	a	b	c
$[-2,4]$	57.5 ± 0.7	39.0 ± 0.5	8.671 ± 0.312	41.7 ± 0.8	11.377 ± 0.433	-1.353 ± 0.195
$[-2,8]$	50.2 ± 0.5	44.4 ± 0.5	0.471 ± 0.116	43.1 ± 0.5	12.121 ± 0.326	-1.942 ± 0.051

Table 30: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,4]$ and $[-2,8]$ to the arrival data for sinusoidal arrival process; estimates and halfwidths of 95% confidence intervals over 100 replications.

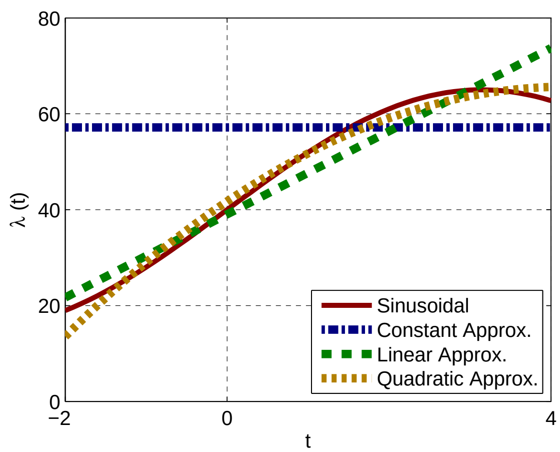

Figure 76: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,4]$ to the arrival data for sinusoidal arrival process.

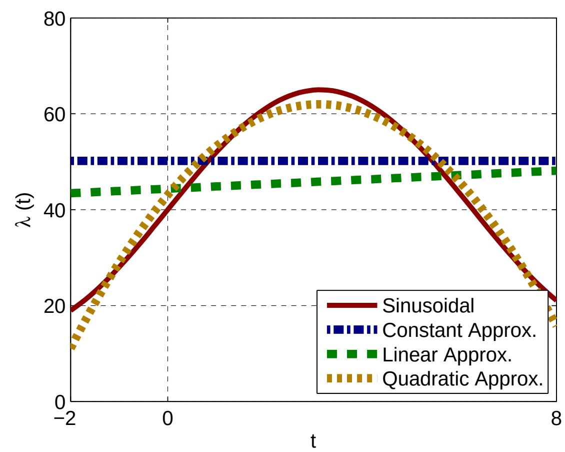

Figure 77: Fitting constant, linear and quadratic arrival rate functions over the intervals $[-2,8]$ to the arrival data for sinusoidal arrival process.

Following previous examples, we now examine average performance of $M_t/M/s_t+M$ and

$M_t/H_2/s_t + M$ systems used in our sinusoidal example. Table 32 illustrates $E[W]$, percent delayed, and percent abandoned. In each replication, we average the performance measures over periods of length 0.5 in the intervals $[0, 4]$ and $[0, 8]$. The reported numbers are averages over the 100 replications and the halfwidths of 95% confidence intervals.

Int.		[0, 4]						[0, 8]					
GI	β	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$	$\%v$	$\#dec$	$Pr(Delay)$	$E[\#v]$	$\#dep$	$\#v$	$\%v$
M	0	0	0.46 ± 0.05	0.00 ± 0.00	197.7 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	31	0.48 ± 0.04	14.84 ± 1.19	400.7 ± 3.7	0.00 ± 0.00	0.00 ± 0.00
	1	0	0.16 ± 0.03	0.00 ± 0.00	197.9 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	33	0.15 ± 0.03	4.93 ± 0.89	400.0 ± 3.6	0.00 ± 0.00	0.00 ± 0.00
	2	0	0.02 ± 0.01	0.00 ± 0.00	197.8 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	36	0.02 ± 0.01	0.79 ± 0.33	399.9 ± 3.6	0.00 ± 0.00	0.00 ± 0.00
H_2	0	0	0.48 ± 0.06	0.00 ± 0.00	207.0 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	8	0.49 ± 0.05	3.89 ± 0.44	400.3 ± 3.9	0.00 ± 0.00	0.00 ± 0.00
	1	0	0.16 ± 0.04	0.00 ± 0.00	207.7 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	8	0.17 ± 0.03	1.32 ± 0.28	399.3 ± 3.7	0.00 ± 0.00	0.00 ± 0.00
	2	0	0.02 ± 0.01	0.00 ± 0.00	207.8 ± 2.6	0.00 ± 0.00	0.00 ± 0.00	9	0.02 ± 0.01	0.19 ± 0.08	399.1 ± 3.7	0.00 ± 0.00	0.00 ± 0.00

Table 31: Early service termination in the 6 different models with sinusoidal arrival rate; $\#dec$ indicates the number of staffing decreases, $\#dep$ indicates the number of departures and v indicates violations.

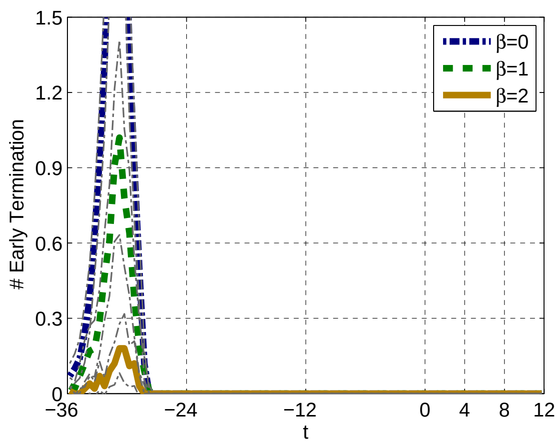

Figure 78: Number of early termination in each subinterval of length 0.5. Exponential service time with sinusoidal arrival rate.

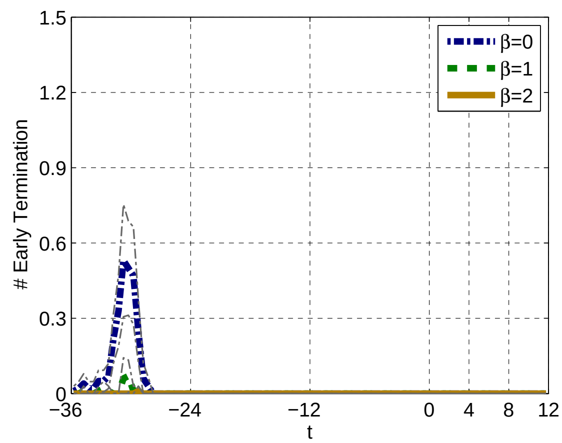

Figure 79: Number of early termination in each subinterval of length 0.5. Hyperexponential service time with sinusoidal arrival rate.

Figures 80 - 91 provide more information on the model performance over time. Performance of models with exponential service time distribution is shown in Figures 80 - 85. Model performance with hyperexponential service time distribution is shown in Figures 86 - 91. These plots suggest that the performance is stabilized approximately by time $t = 0$.

Performance Int.		$[0,4]$			$[0,8]$		
GI	β	$E[W]$	$\%Delayed$	$\%Aban.$	$E[W]$	$\%Delayed$	$\%Aban.$
M	0	1.03 ± 0.01	44.4 ± 4.5	3.66 ± 0.58	0.00 ± 0.00	0.0 ± 0.0	1.02 ± 0.01
	1	1.01 ± 0.01	14.5 ± 3.0	0.69 ± 0.23	0.00 ± 0.00	0.0 ± 0.0	1.00 ± 0.01
	2	1.01 ± 0.01	2.0 ± 1.0	0.05 ± 0.04	0.00 ± 0.00	0.0 ± 0.0	1.00 ± 0.01
H_2	0	1.04 ± 0.03	46.0 ± 5.9	4.26 ± 0.85	0.00 ± 0.00	0.0 ± 0.0	1.02 ± 0.02
	1	1.00 ± 0.03	14.9 ± 3.6	0.79 ± 0.27	0.00 ± 0.00	0.0 ± 0.0	0.99 ± 0.02
	2	1.00 ± 0.03	1.9 ± 0.9	0.04 ± 0.03	0.00 ± 0.00	0.0 ± 0.0	0.99 ± 0.02

Table 32: Average performance of the 6 different models with sinusoidal arrival rate, averaged over periods of length 0.5 in the intervals $[0,4]$ and $[0,8]$; $TETT$ indicates the total early termination time.

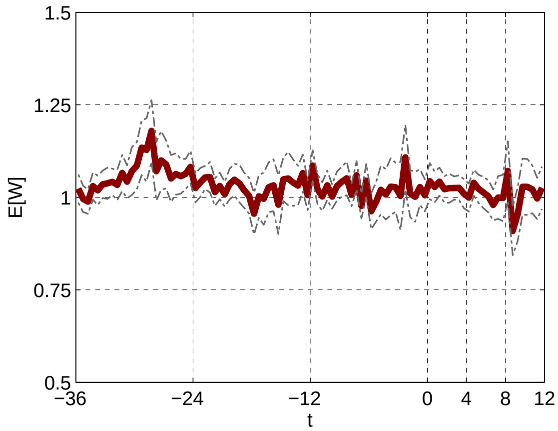

Figure 80: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

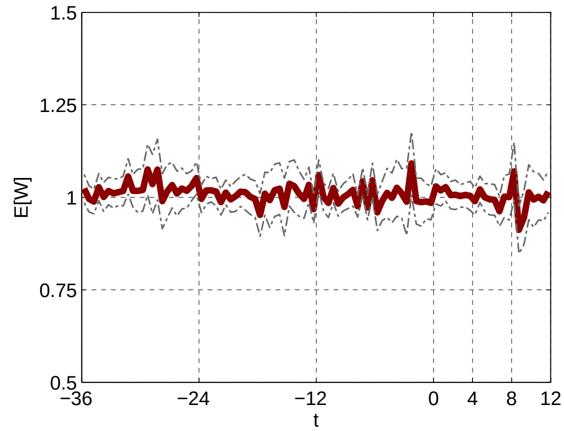

Figure 81: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

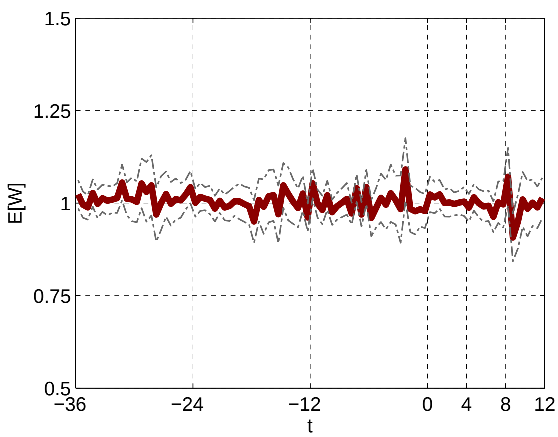

Figure 82: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

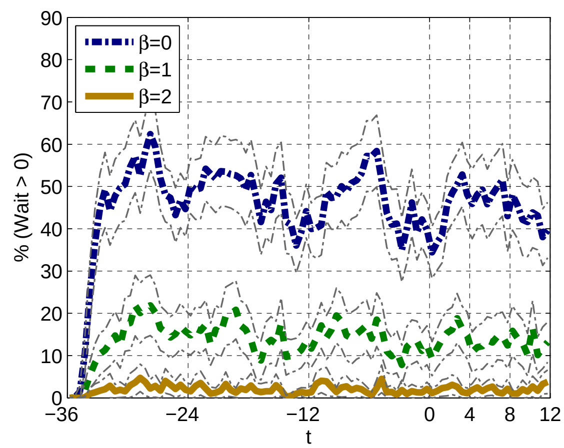

Figure 83: Average percent of arrivals delayed over periods of length 0.5.

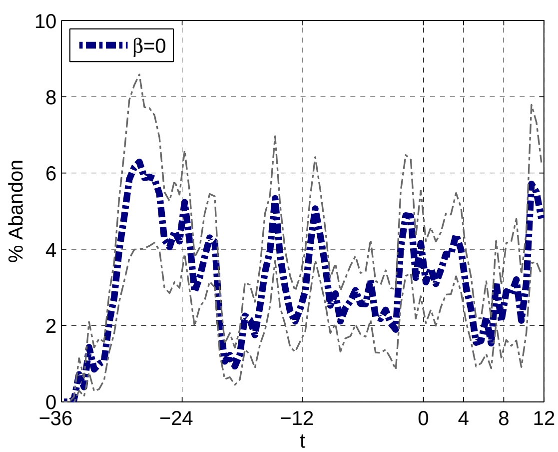

Figure 84: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

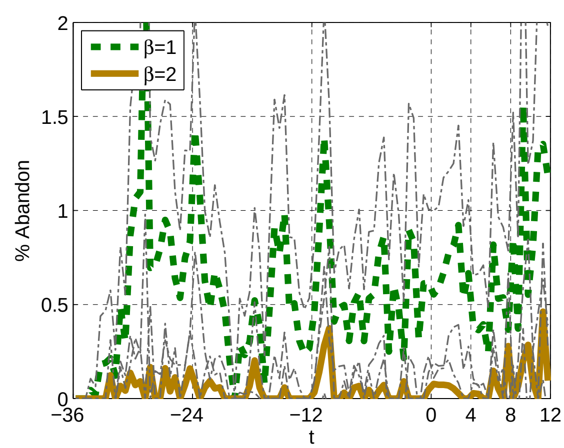

Figure 85: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 80-85: Sinusoidal arrival rate and M service time distribution.

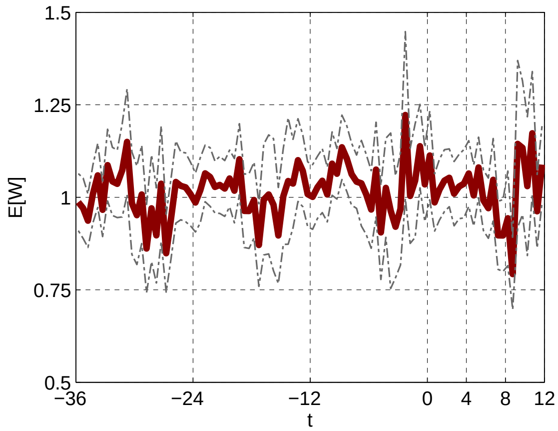

Figure 86: Average waiting time over periods of length 0.5: QoS $\beta = 0$.

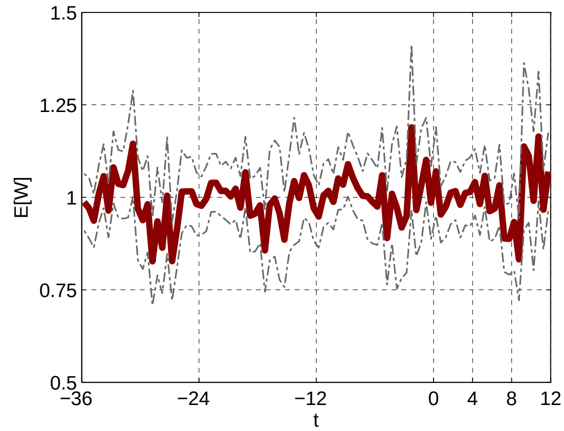

Figure 87: Average waiting time over periods of length 0.5: QoS $\beta = 1$.

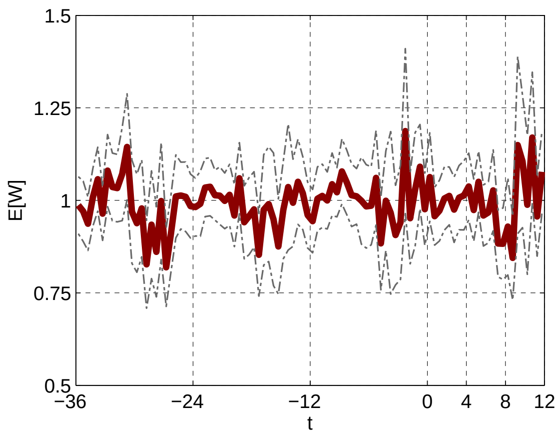

Figure 88: Average waiting time over periods of length 0.5: QoS $\beta = 2$.

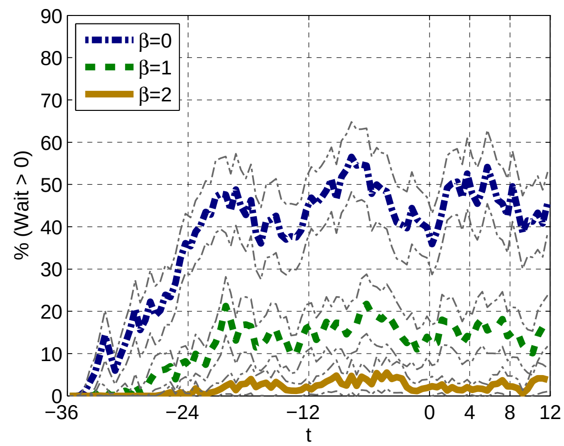

Figure 89: Average percent of arrivals delayed over periods of length 0.5.

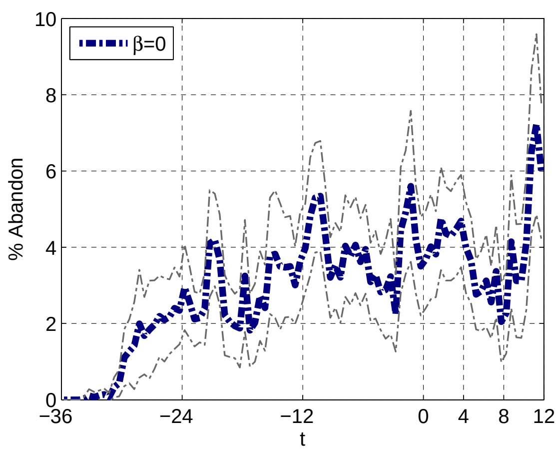

Figure 90: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 0$.

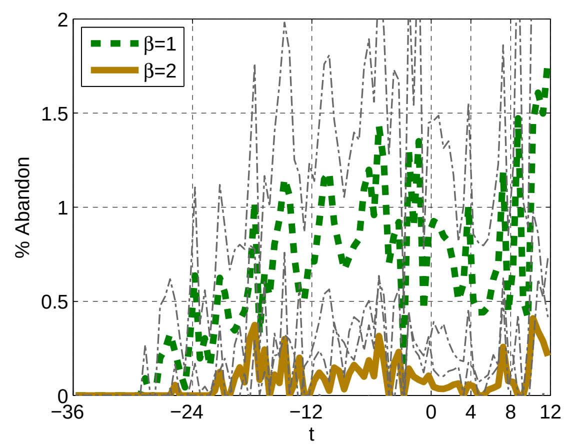

Figure 91: Average percent of arrivals abandoning over periods of length 0.5: QoS $\beta = 1, 2$.

Figures 86-91: Sinusoidal arrival rate and H_2 service time distribution.

We now present estimation results for sinusoidal arrival rate case. Table 33 provides different estimator values and Table 34 gives the value of $\bar{L}(t)$ and parameters for the perturbation analysis (equations (19) and (32)).

GI	Int	β	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,b}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,b}(t)$
M	[0, 4]	0	1.026 ± 0.014	0.886 ± 0.013	1.009 ± 0.014	1.009 ± 0.014	1.084 ± 0.018	1.006 ± 0.015	1.006 ± 0.015	1.067 ± 0.019	1.005 ± 0.015	1.005 ± 0.015
		1	1.011 ± 0.014	0.870 ± 0.011	0.990 ± 0.013	0.990 ± 0.013	1.060 ± 0.016	0.985 ± 0.013	0.985 ± 0.013	1.040 ± 0.016	0.984 ± 0.013	0.984 ± 0.013
		2	1.006 ± 0.013	0.865 ± 0.011	0.985 ± 0.012	0.985 ± 0.012	1.053 ± 0.015	0.979 ± 0.012	0.979 ± 0.012	1.033 ± 0.015	0.978 ± 0.012	0.978 ± 0.012
	Avg		1.015	0.874	0.995	0.995	1.066	0.990	0.990	1.047	0.989	0.989
	[0, 8]	0	1.021 ± 0.011	1.017 ± 0.010	1.019 ± 0.011	1.019 ± 0.011	1.116 ± 0.012	1.028 ± 0.010	1.028 ± 0.010	1.031 ± 0.012	1.018 ± 0.011	1.018 ± 0.011
		1	1.006 ± 0.010	1.001 ± 0.009	1.005 ± 0.010	1.005 ± 0.010	1.098 ± 0.010	1.011 ± 0.009	1.011 ± 0.009	1.013 ± 0.010	1.001 ± 0.010	1.001 ± 0.010
		2	1.002 ± 0.009	0.997 ± 0.009	1.001 ± 0.009	1.001 ± 0.009	1.094 ± 0.010	1.007 ± 0.009	1.007 ± 0.009	1.009 ± 0.010	0.997 ± 0.009	0.997 ± 0.009
	Avg		1.010	1.005	1.008	1.008	1.103	1.015	1.015	1.018	1.005	1.005
H_2	[0, 4]	0	1.035 ± 0.031	0.831 ± 0.021	0.912 ± 0.023	1.075 ± 0.032	1.104 ± 0.034	1.152 ± 0.038	1.104 ± 0.034	-1.598 ± 1.687	2.037 ± 0.215	2.037 ± 0.215
		1	1.003 ± 0.030	0.798 ± 0.015	0.874 ± 0.016	1.026 ± 0.024	1.104 ± 0.035	1.091 ± 0.025	1.091 ± 0.025	-1.599 ± 1.687	1.913 ± 0.384	1.913 ± 0.384
		2	0.999 ± 0.030	0.793 ± 0.014	0.869 ± 0.015	1.019 ± 0.022	1.104 ± 0.035	1.083 ± 0.023	1.083 ± 0.023	-1.597 ± 1.687	1.930 ± 0.458	1.930 ± 0.458
	Avg		1.012	0.808	0.885	1.040	1.104	1.109	1.093	-1.598	1.960	1.960
	[0, 8]	0	1.025 ± 0.024	0.957 ± 0.020	0.960 ± 0.019	0.965 ± 0.022	1.077 ± 0.024	0.985 ± 0.022	0.985 ± 0.022	-1.173 ± 0.013	1.517 ± 0.056	-1.173 ± 0.013
		1	0.999 ± 0.022	0.928 ± 0.015	0.933 ± 0.016	0.943 ± 0.019	1.042 ± 0.019	0.953 ± 0.017	0.953 ± 0.017	-1.166 ± 0.013	1.432 ± 0.041	-1.166 ± 0.013
		2	0.994 ± 0.021	0.924 ± 0.014	0.929 ± 0.015	0.940 ± 0.018	1.037 ± 0.018	0.949 ± 0.016	0.949 ± 0.016	-1.165 ± 0.013	1.420 ± 0.038	-1.165 ± 0.013
	Avg		1.006	0.936	0.941	0.949	1.052	0.962	0.962	-1.168	1.456	-1.168

Table 33: SINUSOIDAL arrival rate: waiting time estimates by different methods with 95% confidence intervals.

GI	$Interval$	β	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\bar{\lambda}(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
M	[0, 4]	0	50.9 ± 1.0	$1.35 \times 10^{-1} \pm 3.61 \times 10^{-3}$	$-4.40 \times 10^{-2} \pm 6.28 \times 10^{-3}$
		1	50.0 ± 0.9	$1.33 \times 10^{-1} \pm 3.64 \times 10^{-3}$	$-4.20 \times 10^{-2} \pm 5.94 \times 10^{-3}$
		2	49.7 ± 0.8	$1.32 \times 10^{-1} \pm 3.64 \times 10^{-3}$	$-4.14 \times 10^{-2} \pm 5.83 \times 10^{-3}$
	Avg		50.2	1.33×10^{-1}	-4.25×10^{-2}
	[0, 8]	0	51.1 ± 0.7	$1.01 \times 10^{-2} \pm 2.52 \times 10^{-3}$	$-7.03 \times 10^{-2} \pm 1.90 \times 10^{-3}$
		1	50.2 ± 0.7	$9.94 \times 10^{-3} \pm 2.48 \times 10^{-3}$	$-6.82 \times 10^{-2} \pm 1.75 \times 10^{-3}$
		2	50.0 ± 0.6	$9.91 \times 10^{-3} \pm 2.46 \times 10^{-3}$	$-6.77 \times 10^{-2} \pm 1.70 \times 10^{-3}$
	Avg		50.5	9.98×10^{-3}	-6.87×10^{-2}
H_2	[0, 4]	0	47.8 ± 1.4	$3.81 \times 10^{-1} \pm 1.34 \times 10^{-2}$	$-1.11 \times 10^{0} \pm 1.87 \times 10^{-1}$
		1	45.9 ± 1.0	$3.66 \times 10^{-1} \pm 1.15 \times 10^{-2}$	$-1.12 \times 10^{0} \pm 4.56 \times 10^{-1}$
		2	45.6 ± 1.0	$3.64 \times 10^{-1} \pm 1.12 \times 10^{-2}$	$-1.16 \times 10^{0} \pm 5.49 \times 10^{-1}$
	Avg		46.4	3.70×10^{-1}	-1.13×10^{0}
	[0, 8]	0	48.1 ± 1.2	$2.84 \times 10^{-2} \pm 6.96 \times 10^{-3}$	$-7.68 \times 10^{-1} \pm 3.09 \times 10^{-2}$
		1	46.6 ± 0.9	$2.76 \times 10^{-2} \pm 6.75 \times 10^{-3}$	$-7.27 \times 10^{-1} \pm 2.44 \times 10^{-2}$
		2	46.4 ± 0.9	$2.75 \times 10^{-2} \pm 6.71 \times 10^{-3}$	$-7.20 \times 10^{-1} \pm 2.31 \times 10^{-2}$
	Avg		47.0	2.78×10^{-2}	-7.38×10^{-1}

Table 34: SINUSOIDAL arrival rate: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Tables 35 and 36 quantify the performance of our estimators by two performance measures.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,l,best}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$	$\bar{W}_{L,\lambda,q,best}(t)$
M	$[0,4]$	0	14.0	1.8	1.8	5.8	2.1	2.1	4.1	2.1	2.1
		1	14.1	2.1	2.1	4.9	2.6	2.6	3.0	2.7	2.7
		2	14.1	2.1	2.1	4.6	2.7	2.7	2.6	2.8	2.8
	Avg		14.1	2.0	2.0	5.1	2.4	2.4	3.2	2.5	2.5
	$[0,8]$	0	0.3	0.1	0.1	9.6	0.7	0.7	1.1	0.2	0.2
		1	0.5	0.1	0.1	9.3	0.5	0.5	0.7	0.5	0.5
		2	0.5	0.1	0.1	9.2	0.5	0.5	0.7	0.5	0.5
	Avg		0.4	0.1	0.1	9.3	0.6	0.6	0.8	0.4	0.4
H_2	$[0,4]$	0	20.4	12.2	4.0	6.9	11.7	6.9	263.2	100.2	100.2
		1	20.5	12.9	2.2	10.1	8.8	8.8	260.2	91.0	91.0
		2	20.6	13.0	2.0	10.5	8.4	8.4	259.7	93.1	93.1
	Avg		20.5	12.7	2.8	9.2	9.6	8.1	261.0	94.7	94.7
	$[0,8]$	0	6.8	6.5	6.0	5.1	4.1	4.1	219.8	49.2	219.8
		1	7.1	6.6	5.6	4.3	4.6	4.6	216.5	43.3	216.5
		2	7.1	6.5	5.4	4.3	4.5	4.5	215.9	42.6	215.9
	Avg		7.0	6.6	5.7	4.6	4.4	4.4	217.4	45.0	217.4

Table 35: SINUSOIDAL arrival rate: absolute difference of the estimates from the direct estimate, in units of 10^{-2}.

GI	$Interval$	β	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,r,\gamma}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
M	$[0,4]$	0	13.6 ± 1.0	3.7 ± 0.5	3.7 ± 0.5	6.6 ± 1.1	4.7 ± 0.6	5.8 ± 1.0	4.8 ± 0.6
		1	13.8 ± 0.9	3.7 ± 0.5	3.7 ± 0.5	6.2 ± 1.0	4.7 ± 0.6	5.2 ± 0.8	4.8 ± 0.6
		2	13.9 ± 0.9	3.6 ± 0.5	3.6 ± 0.5	6.1 ± 1.0	4.8 ± 0.6	5.1 ± 0.8	4.9 ± 0.6
	Avg		13.7 ± 0.9	3.7 ± 0.5	3.7 ± 0.5	6.3 ± 1.0	4.8 ± 0.6	5.4 ± 0.9	4.8 ± 0.6
	$[0,8]$	0	2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.5 ± 0.7	2.2 ± 0.3	2.5 ± 0.4	2.2 ± 0.3
		1	2.3 ± 0.3	1.5 ± 0.2	1.5 ± 0.2	9.3 ± 0.7	2.1 ± 0.3	2.4 ± 0.3	2.2 ± 0.3
		2	2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.2 ± 0.7	2.1 ± 0.3	2.4 ± 0.3	2.2 ± 0.3
	Avg		2.3 ± 0.3	1.6 ± 0.2	1.6 ± 0.2	9.3 ± 0.7	2.2 ± 0.3	2.4 ± 0.4	2.2 ± 0.3
H_2	$[0,4]$	0	19.4 ± 2.1	13.6 ± 1.7	12.4 ± 1.9	19.1 ± 3.6	17.4 ± 2.8	397.1 ± 139.4	99.9 ± 19.2
		1	19.9 ± 2.1	14.2 ± 1.8	11.8 ± 1.9	20.6 ± 4.0	16.1 ± 2.6	403.3 ± 140.7	97.3 ± 39.0
		2	20.0 ± 2.1	14.4 ± 1.8	12.0 ± 1.9	20.7 ± 4.0	15.8 ± 2.5	403.5 ± 140.4	101.1 ± 47.0
	Avg		19.8 ± 2.1	14.1 ± 1.8	12.0 ± 1.9	20.2 ± 3.9	16.4 ± 2.6	401.3 ± 140.2	99.4 ± 35.1
	$[0,8]$	0	8.6 ± 1.1	8.0 ± 1.1	7.8 ± 1.1	9.0 ± 1.3	7.6 ± 1.1	215.8 ± 2.7	47.8 ± 3.9
		1	9.0 ± 1.1	8.4 ± 1.1	7.9 ± 1.1	8.8 ± 1.3	7.8 ± 1.1	218.0 ± 2.7	43.8 ± 3.4
		2	9.0 ± 1.1	8.4 ± 1.1	8.0 ± 1.1	8.7 ± 1.3	7.8 ± 1.1	218.5 ± 2.7	43.3 ± 3.3
	Avg		8.9 ± 1.1	8.3 ± 1.1	7.9 ± 1.1	8.8 ± 1.3	7.7 ± 1.1	217.5 ± 2.7	44.9 ± 3.5

Table 36: SINUSOIDAL arrival rate: average of the absolute relative error of the estimate from the direct estimate in each sample path.

3 Additional Results for the Call Center Data in Section 9

In this section, we compare the performance of the different estimators using the same call center data as in [4]. The data are for one class of customers from an American bank on 18 weekdays in May 2001. In this case, we have data for waiting times as well as arrivals and the number in the system, so that we can compare all the indirect estimators for $E[W]$ based on $\bar{L}(t)$ and the estimated arrival rate to the direct sample mean $\bar{W}(t)$ in (1).

We plot the finite averages $\bar{\lambda}(t)$ and $\bar{W}_{L,\lambda}(t)$ in (1) and (2) in Section 3.1 for 18 weekdays in May. We divide each day into 5 intervals, $[7,10]$, $[10,13]$, $[13,16]$, $[16,18]$ and $[18,22]$, so that the arrival rate is increasing in $[7,10]$, approximately stationary in $[10,13]$ and $[13,16]$, decreasing in $[16,18]$ and again decreasing in $[18,22]$ but with less steep slope.

These plots suggest that Assumption 1 is often reasonable for this data. Figures 3,5 and 6 of [4] show that the waiting times are relatively stationary over the day, unlike the arrival rate and the number in the system. Nevertheless, we observe that the waiting times do fluctuate over time substantially for some days, especially outside of normal business hours ($[9,17]$, i.e., nine to five). Possible reasons are inappropriate time-varying staffing and the lower call volumes outside of normal business hours.

We now describe how the arrival rate approximations were done. We used an iterated least squares fit, as specified in [5], to fit a linear arrival rate function for each interval of each day. It minimizes the sum of squared deviations of the data from the model with a constraint requiring that the estimated rate function be nonnegative throughout the interval. We counted the number of arrivals in each one-minute subinterval and used that as a point in the least square fit. Similarly, a least-square fit was used to fit a quadratic arrival rate function.

We provide two sets of estimation results: for the selected 3 days in §3.2 and all 18 days in §3.3.

3.1 Plots for Arrival Rates and Time Spent in the System

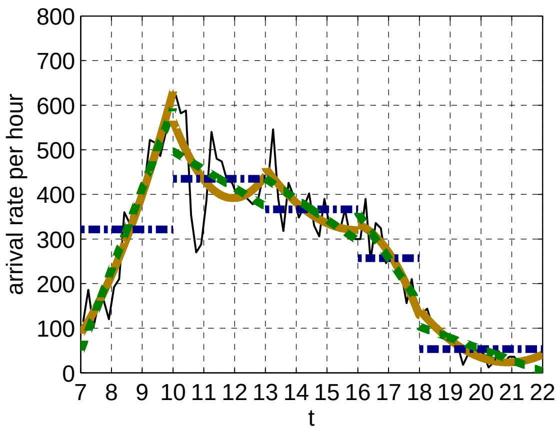

Figure 92: Arrival rate and its approximations by constant, linear and quadratic functions on May 1.

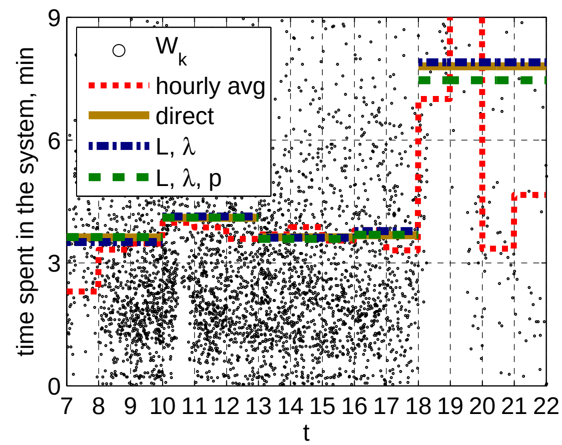

Figure 93: W_{sys} and its estimators on May 1.

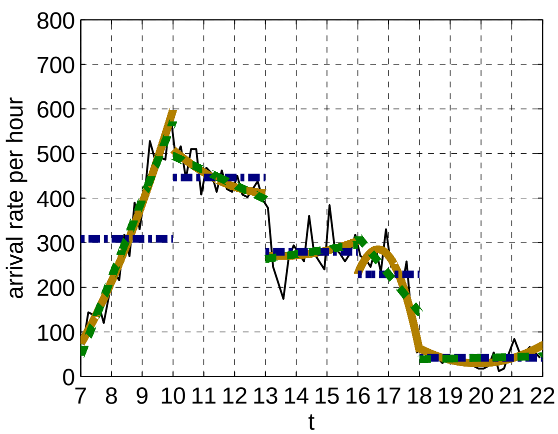

Figure 94: Arrival rate and its approximations by constant, linear and quadratic functions on May 2.

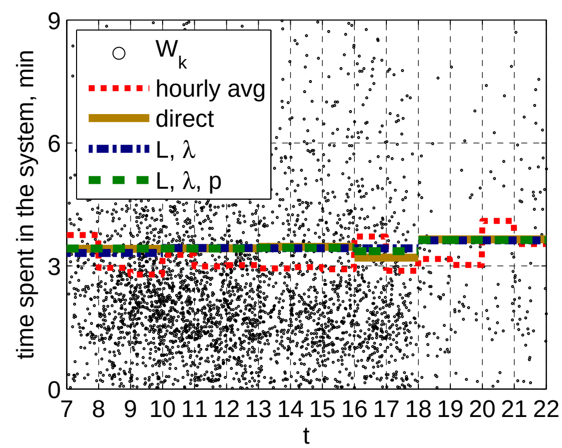

Figure 95: W_{sys} and its estimators on May 2.

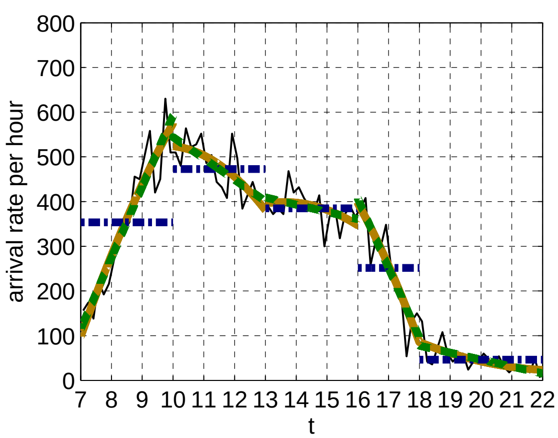

Figure 96: Arrival rate and its approximations by constant, linear and quadratic functions on May 4.

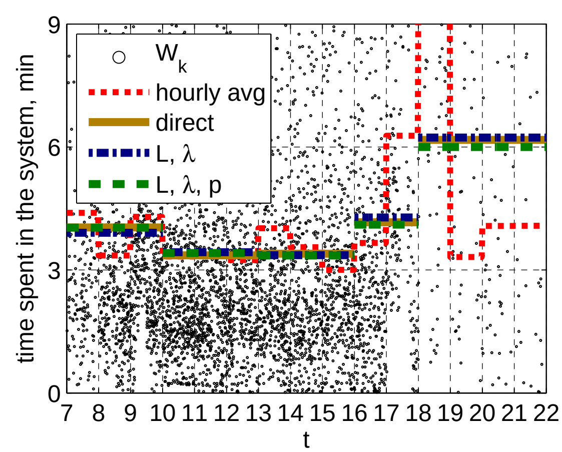

Figure 97: W_{sys} and its estimators on May 4.

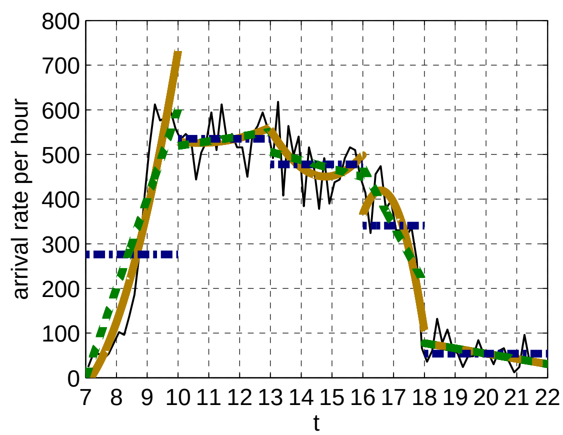

Figure 98: Arrival rate and its approximations by constant, linear and quadratic functions on May 7.

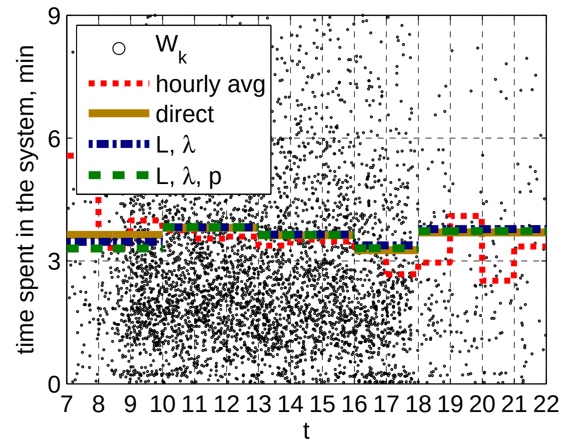

Figure 99: W_{sys} and its estimators on May 7.

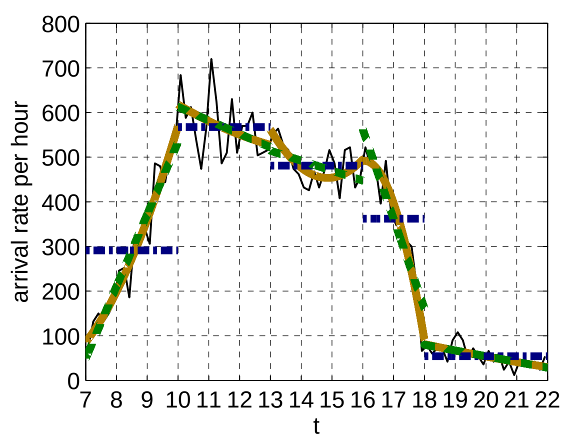

Figure 100: Arrival rate and its approximations by constant, linear and quadratic functions on May 8.

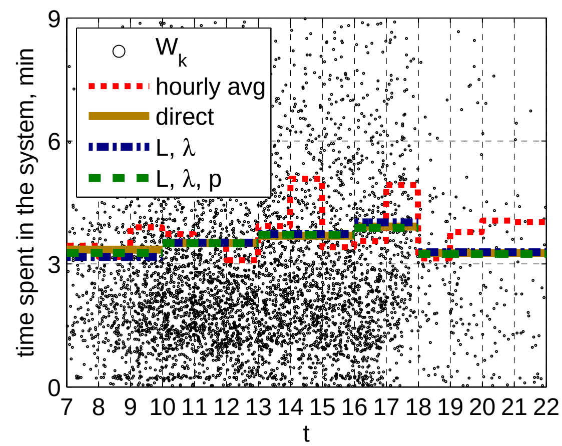

Figure 101: W_{sys} and its estimators on May 8.

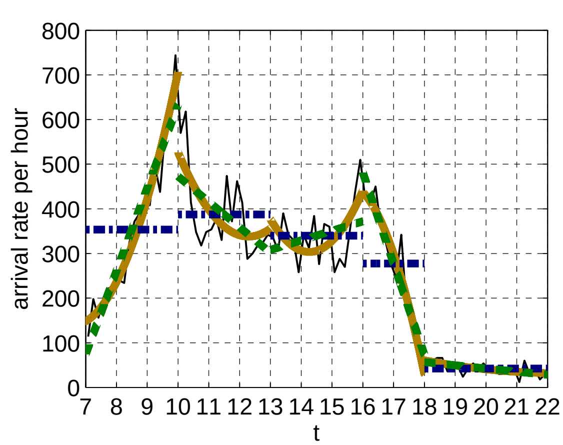

Figure 102: Arrival rate and its approximations by constant, linear and quadratic functions on May 11.

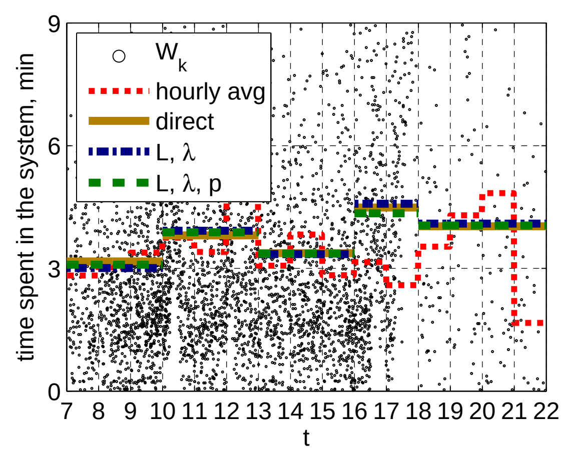

Figure 103: W_{sys} and its estimators on May 11.

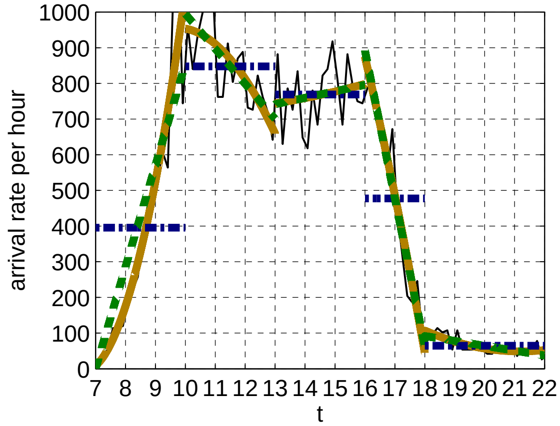

Figure 104: Arrival rate and its approximations by constant, linear and quadratic functions on May 14.

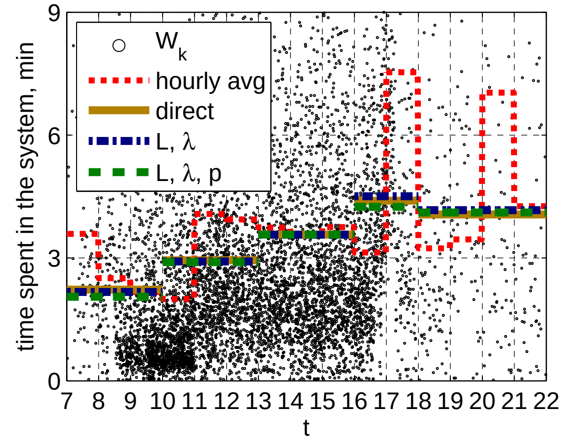

Figure 105: W_{sys} and its estimators on May 14.

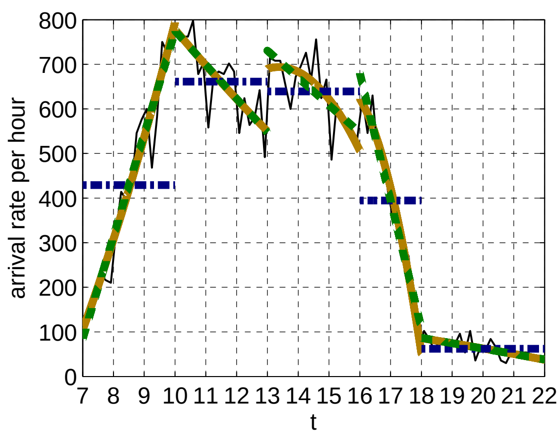

Figure 106: Arrival rate and its approximations by constant, linear and quadratic functions on May 15.

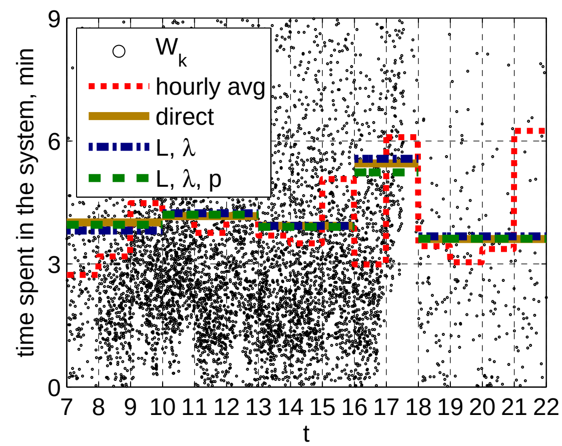

Figure 107: W_{sys} and its estimators on May 15.

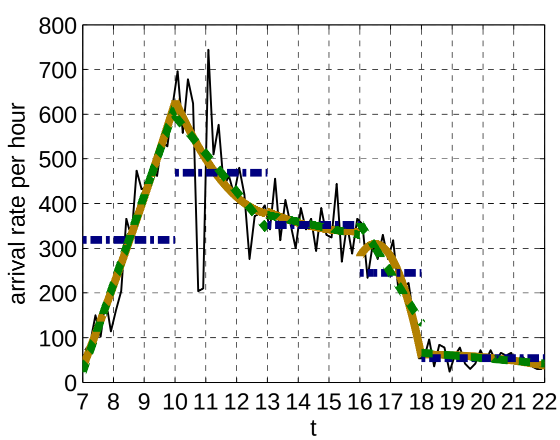

Figure 108: Arrival rate and its approximations by constant, linear and quadratic functions on May 16.

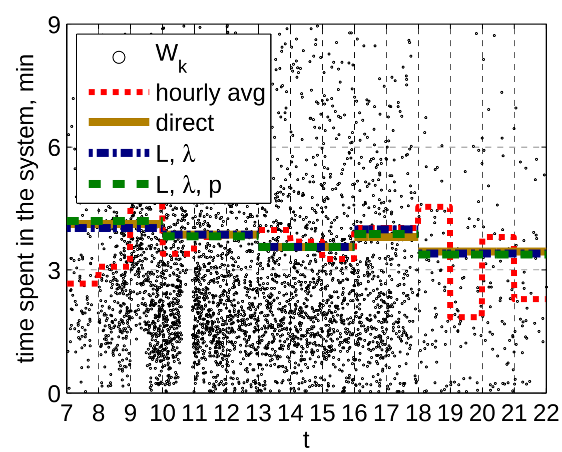

Figure 109: W_{sys} and its estimators on May 16.

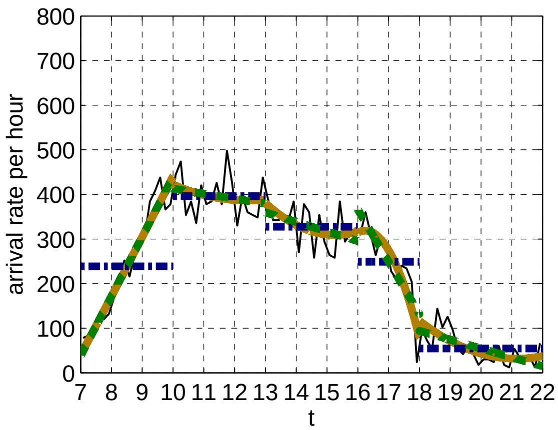

Figure 110: Arrival rate and its approximations by constant, linear and quadratic functions on May 17.

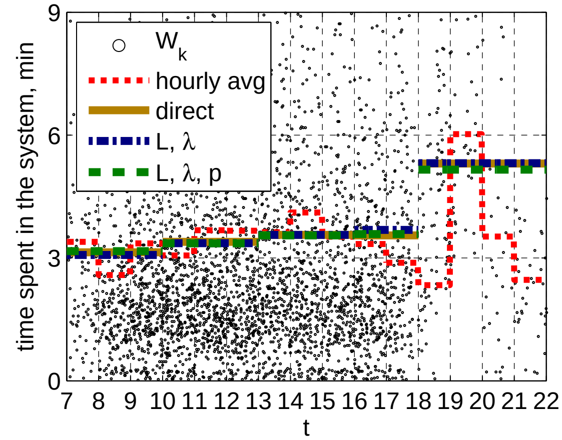

Figure 111: W_{sys} and its estimators on May 17.

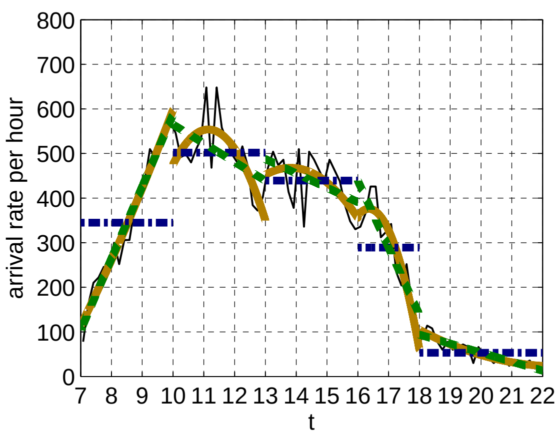

Figure 112: Arrival rate and its approximations by constant, linear and quadratic functions on May 18.

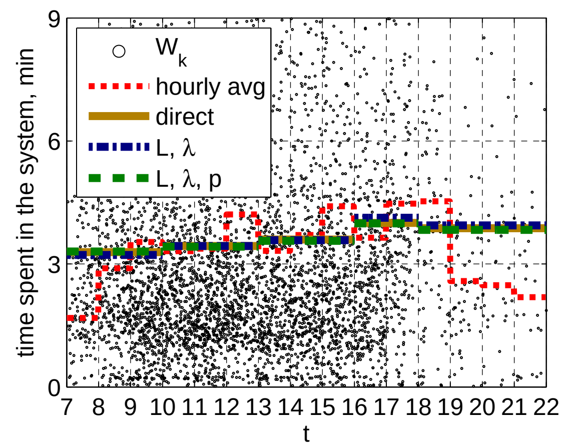

Figure 113: W_{sys} and its estimators on May 18.

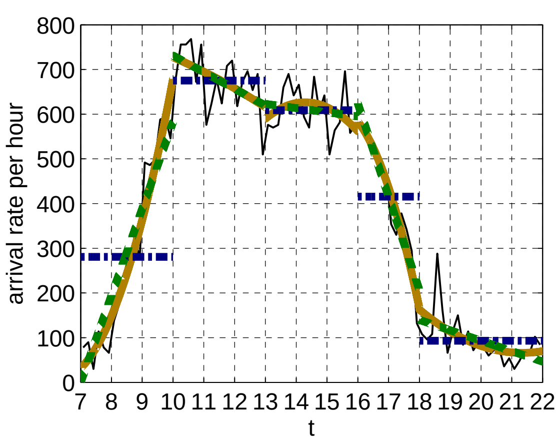

Figure 114: Arrival rate and its approximations by constant, linear and quadratic functions on May 21.

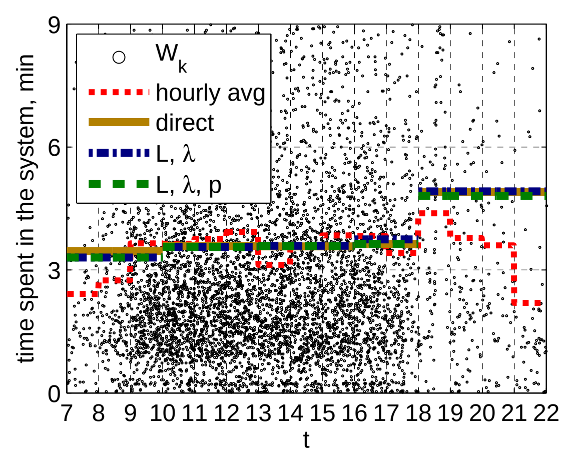

Figure 115: W_{sys} and its estimators on May 21.

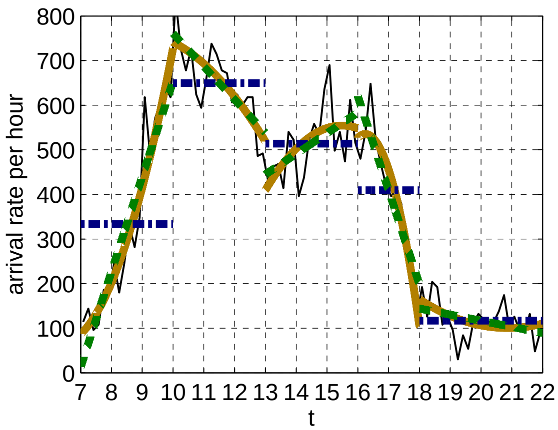

Figure 116: Arrival rate and its approximations by constant, linear and quadratic functions on May 22.

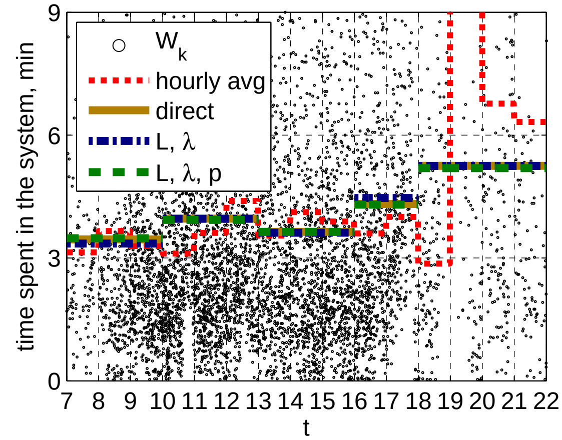

Figure 117: W_{sys} and its estimators on May 22.

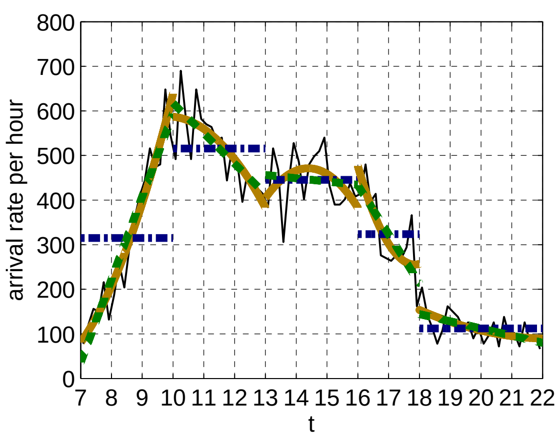

Figure 118: Arrival rate and its approximations by constant, linear and quadratic functions on May 23.

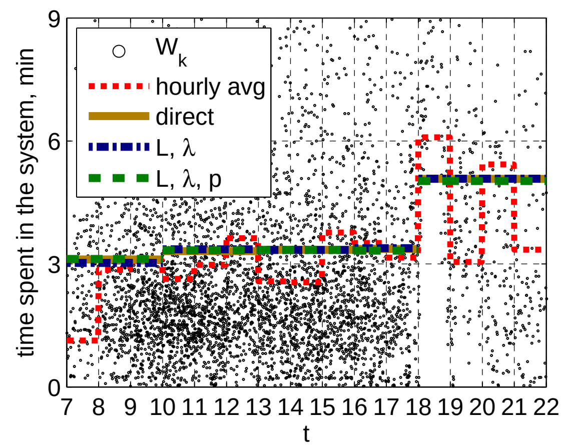

Figure 119: W_{sys} and its estimators on May 23.

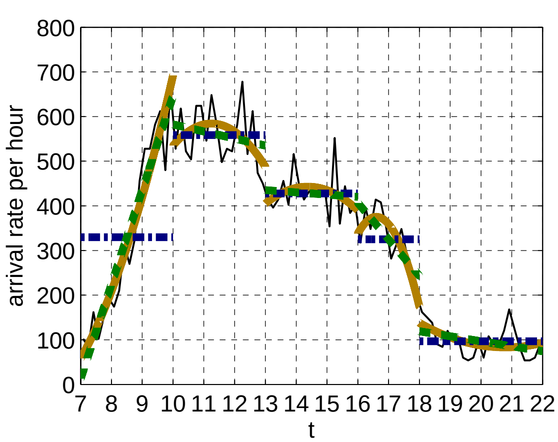

Figure 120: Arrival rate and its approximations by constant, linear and quadratic functions on May 24.

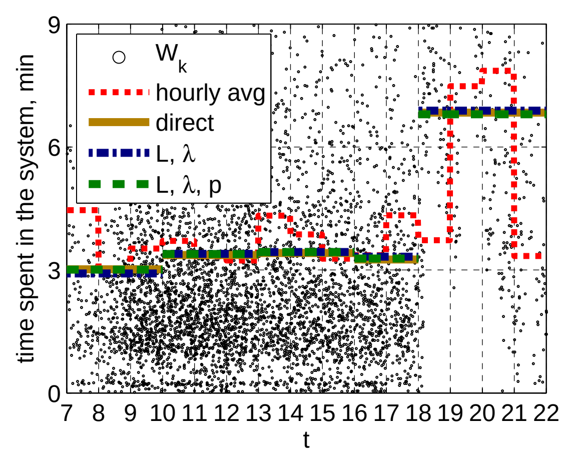

Figure 121: W_{sys} and its estimators on May 24.

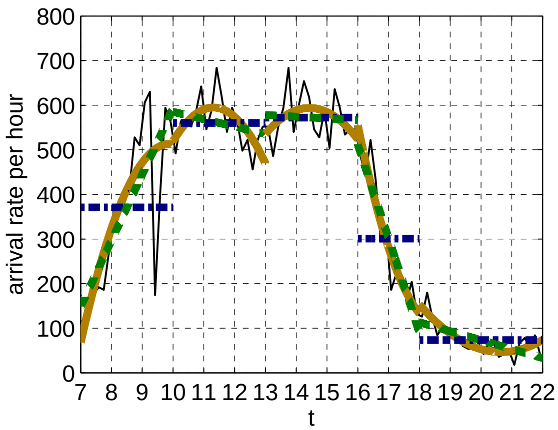

Figure 122: Arrival rate and its approximations by constant, linear and quadratic functions on May 25.

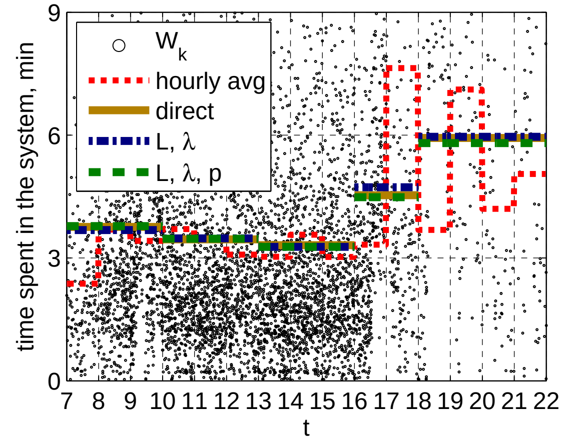

Figure 123: W_{sys} and its estimators on May 25.

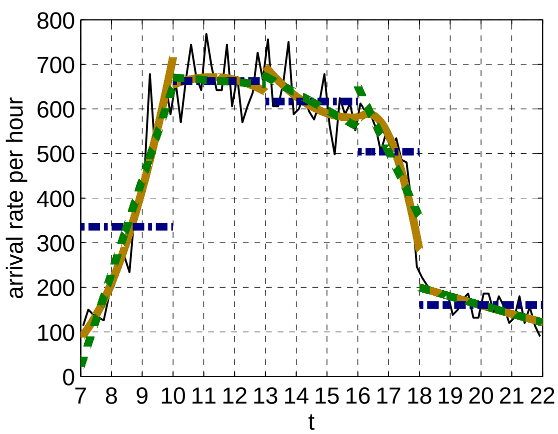

Figure 124: Arrival rate and its approximations by constant, linear and quadratic functions on May 29.

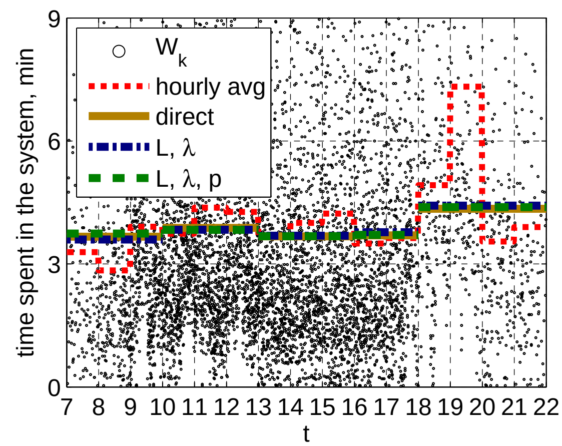

Figure 125: W_{sys} and its estimators on May 29.

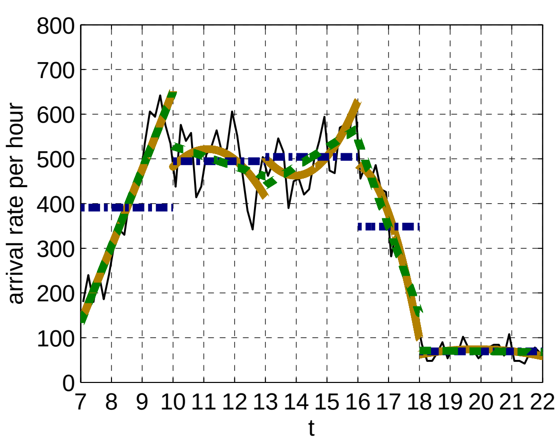

Figure 126: Arrival rate and its approximations by constant, linear and quadratic functions on May 30.

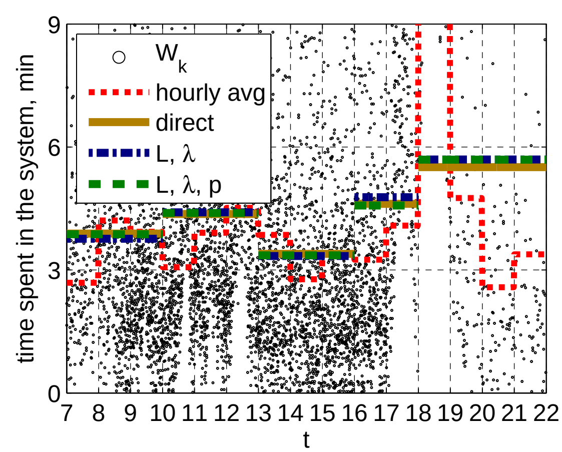

Figure 127: W_{sys} and its estimators on May 30.

3.2 Selected 3 days

		Constant	Linear		Quadratic		
Int.	*Day*	$\bar{\lambda}(t)$	a	b	a	b	c
[7, 10]	507	276.0	0.0	200.3	2050.4	−676.5	54.5
	518	345.0	103.3	161.4	−528.4	43.0	7.0
	521	280.7	0.0	194.3	2006.8	−630.7	49.8
	Avg	300.6	34.4	185.3	1176.3	−421.4	37.1
[10, 13]	507	535.0	519.1	10.6	1265.2	−138.4	6.5
	518	502.0	565.0	−42.0	−6847.3	1327.9	−59.6
	521	675.3	731.2	−37.2	782.8	18.9	−2.4
	Avg	570.8	605.1	−22.9	−1599.8	402.8	−18.5
[13, 16]	507	477.7	506.0	−18.9	7637.2	−972.0	32.9
	518	439.3	487.6	−32.2	−4044.4	653.1	−23.6
	521	609.0	623.0	−9.3	−3905.4	634.3	−22.2
	Avg	508.7	538.9	−20.1	−104.2	105.1	−4.3
[16, 18]	507	340.5	470.5	−130.0	−42905.0	5223.9	−157.5
	518	289.0	439.3	−150.3	−31042.0	3840.9	−117.4
	521	415.5	624.3	−208.8	−13647.0	1865.6	−61.0
	Avg	348.3	511.4	−163.1	−29198.0	3643.5	−112.0
[18, 22]	507	54.0	77.5	−11.7	288.8	−11.7	0.0
	518	53.3	93.9	−20.2	2062.3	−181.2	4.0
	521	93.0	139.9	−23.3	4016.0	−370.1	8.7
	Avg	66.8	103.7	−18.4	2122.4	−187.7	4.2

Table 37: Fitting constant, linear and quadratic arrival rate functions for 3 days. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Int.	*Day*	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
[7, 10]	507	3.64	3.47	3.62	3.30	3.60	0.79	3.38
	518	3.29	3.22	3.31	3.30	3.30	−2.52	3.20
	521	3.46	3.30	3.42	3.30	3.42	0.76	3.23
	Avg	3.46	3.33	3.45	3.30	3.44	−0.32	3.27
[10, 13]	507	3.81	3.82	3.81	3.83	3.83	1.89	3.80
	518	3.44	3.44	3.43	3.42	3.42	−0.34	3.39
	521	3.58	3.57	3.58	3.55	3.55	3.00	3.57
	Avg	3.61	3.61	3.60	3.60	3.60	1.52	3.59
[13, 16]	507	3.63	3.65	3.64	3.64	3.64	0.28	3.61
	518	3.60	3.58	3.57	3.56	3.56	−0.50	3.54
	521	3.58	3.59	3.60	3.59	3.59	−0.73	3.55
	Avg	3.60	3.61	3.60	3.60	3.60	−0.32	3.57
[16, 18]	507	3.26	3.38	3.25	3.31	3.31	−0.03	3.36
	518	4.00	4.13	4.03	3.99	3.98	−0.04	4.09
	521	3.63	3.75	3.62	3.64	3.63	−0.13	3.71
	Avg	3.63	3.75	3.63	3.65	3.64	−0.07	3.72
[18, 22]	507	3.70	3.78	3.73	3.73	3.73	0.77	3.77
	518	3.87	3.94	3.85	3.83	3.84	0.12	3.92
	521	4.91	4.92	4.85	4.81	4.82	0.14	4.88
	Avg	4.16	4.21	4.14	4.12	4.13	0.34	4.19

Table 38: Call center example: waiting time estimates by different methods for the 3 days.

Int.	*Day*	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
[7, 10]	507	15.94	3.85×10^{-2}	2.89×10^{-4}
	518	18.52	2.51×10^{-2}	-8.92×10^{-5}
	521	15.45	3.67×10^{-2}	2.39×10^{-4}
	Avg	16.64	3.34×10^{-2}	1.46×10^{-4}
[10, 13]	507	34.09	1.26×10^{-3}	4.82×10^{-5}
	518	28.78	-4.79×10^{-3}	7.58×10^{-5}
	521	40.13	-3.28×10^{-3}	-2.15×10^{-5}
	Avg	34.33	-2.27×10^{-3}	3.42×10^{-5}
[13, 16]	507	29.02	-2.41×10^{-3}	3.80×10^{-5}
	518	26.20	-4.37×10^{-3}	5.24×10^{-5}
	521	36.47	-9.17×10^{-4}	5.15×10^{-5}
	Avg	30.57	-2.56×10^{-3}	4.73×10^{-5}
[16, 18]	507	19.19	-2.15×10^{-2}	2.60×10^{-5}
	518	19.88	-3.58×10^{-2}	3.99×10^{-5}
	521	25.94	-3.14×10^{-2}	3.94×10^{-5}
	Avg	21.67	-2.96×10^{-2}	3.51×10^{-5}
[18, 22]	507	3.40	-1.37×10^{-2}	4.48×10^{-18}
	518	3.50	-2.48×10^{-2}	1.99×10^{-5}
	521	7.62	-2.05×10^{-2}	3.45×10^{-5}
	Avg	4.84	-1.97×10^{-2}	1.82×10^{-5}

Table 39: Call center example: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32) for the 3 days.

Int.	*Day*	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
[7, 10]	507	4.76	0.62	9.18	1.09	78.20	7.14
	518	2.08	0.48	0.41	0.38	176.72	2.83
	521	4.46	1.17	4.47	0.96	77.95	6.58
	Avg	3.77	0.76	4.69	0.81	110.96	5.51
[10, 13]	507	0.30	0.08	0.42	0.42	50.35	0.41
	518	0.00	0.33	0.47	0.48	109.98	1.32
	521	0.44	0.15	0.77	0.77	16.29	0.36
	Avg	0.25	0.19	0.56	0.56	58.88	0.70
[13, 16]	507	0.51	0.44	0.27	0.27	92.36	0.34
	518	0.52	0.67	0.95	0.95	113.96	1.62
	521	0.48	0.54	0.39	0.39	120.31	0.66
	Avg	0.50	0.55	0.54	0.54	108.87	0.87
[16, 18]	507	3.59	0.51	1.46	1.36	100.93	2.83
	518	3.13	0.63	0.32	0.56	101.09	2.19
	521	3.19	0.17	0.14	0.05	103.62	2.23
	Avg	3.30	0.44	0.64	0.66	101.88	2.42
[18, 22]	507	2.18	0.76	0.82	0.78	79.21	1.89
	518	1.86	0.53	1.00	0.67	96.85	1.22
	521	0.29	1.06	1.95	1.77	97.19	0.55
	Avg	1.44	0.78	1.26	1.07	91.09	1.22

Table 40: Call center example: absolute relative error of the estimates from the direct estimate for the 3 days, in units of 10^{-2}.

3.3 All 18 days

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	321.3	49.6	181.2	576.6	−245.6	25.1
502	309.0	46.2	175.2	63.3	−120.4	17.4
504	353.7	115.0	159.1	−1819.7	354.3	−11.5
507	276.0	0.0	200.3	2050.4	−676.5	54.5
508	291.3	48.8	161.7	649.3	−250.2	24.2
511	353.7	74.2	186.3	2282.5	−648.8	49.1
514	395.3	0.0	285.0	3937.7	−1199.1	91.1
515	429.0	84.2	229.9	−556.5	−0.4	13.5
516	319.0	22.6	197.6	−879.5	83.2	6.7
517	238.7	38.4	133.5	−814.6	114.1	1.1
518	345.0	103.3	161.4	−528.4	43.0	7.0
521	280.7	0.0	194.3	2006.8	−630.7	49.8
522	333.3	12.8	213.7	2027.4	−620.9	49.1
523	315.0	36.0	186.0	855.7	−318.5	29.7
524	329.7	12.7	211.3	704.4	−304.8	30.4
525	370.7	148.4	148.2	−4730.9	1061.7	−53.7
529	336.0	21.6	209.6	1662.3	−529.3	43.5
530	391.0	132.6	172.3	−983.7	151.0	1.3
Avg	332.7	52.6	189.3	361.3	−207.7	23.8

Table 41: $[7, 10]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.62	3.51	3.63	3.63	3.63	3.83	3.39
502	3.42	3.32	3.42	3.43	3.42	51.85	3.51
504	4.04	3.91	4.05	4.03	4.02	−1.05	3.85
507	3.64	3.47	3.62	3.30	3.60	0.79	3.38
508	3.35	3.17	3.31	3.27	3.26	2.61	3.08
511	3.17	3.01	3.14	3.09	3.09	0.73	2.95
514	2.23	2.17	2.22	2.05	2.22	0.35	2.14
515	4.01	3.82	4.00	3.96	3.95	−3.18	3.80
516	4.12	4.02	4.14	4.20	4.18	−1.75	3.98
517	3.14	3.07	3.17	3.17	3.16	−1.15	3.05
518	3.29	3.22	3.31	3.30	3.30	−2.52	3.20
521	3.46	3.30	3.42	3.30	3.42	0.76	3.23
522	3.46	3.35	3.47	3.48	3.47	0.89	3.28
523	3.11	3.03	3.09	3.12	3.12	2.01	2.95
524	3.02	2.91	3.02	3.01	3.00	2.76	2.82
525	3.76	3.68	3.80	3.78	3.77	−0.41	3.62
529	3.67	3.60	3.69	3.74	3.73	1.20	3.51
530	3.89	3.76	3.86	3.87	3.87	−1.97	3.71
Avg	3.47	3.35	3.46	3.43	3.46	3.10	3.30

Table 42: $[7, 10]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2 \bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
501	18.80	3.30×10^{-2}	5.67×10^{-4}
502	17.07	3.13×10^{-2}	-1.84×10^{-3}
504	23.05	2.93×10^{-2}	7.15×10^{-5}
507	15.94	3.85×10^{-2}	2.89×10^{-4}
508	15.40	2.93×10^{-2}	3.70×10^{-4}
511	17.73	2.64×10^{-2}	1.64×10^{-4}
514	14.29	2.41×10^{-2}	9.60×10^{-5}
515	27.29	3.41×10^{-2}	-2.10×10^{-4}
516	21.35	4.15×10^{-2}	-8.06×10^{-5}
517	12.23	2.87×10^{-2}	-9.20×10^{-6}
518	18.52	2.51×10^{-2}	-8.92×10^{-5}
521	15.45	3.67×10^{-2}	2.39×10^{-4}
522	18.62	3.58×10^{-2}	2.36×10^{-4}
523	15.88	2.98×10^{-2}	3.08×10^{-4}
524	16.01	3.11×10^{-2}	3.99×10^{-4}
525	22.75	2.45×10^{-2}	1.19×10^{-4}
529	20.15	3.74×10^{-2}	2.99×10^{-4}
530	24.52	2.76×10^{-2}	-1.28×10^{-5}
Avg	18.61	3.13×10^{-2}	5.07×10^{-5}

Table 43: $[7, 10]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.11	0.10	0.31	0.08	5.69	6.57
502	3.00	0.07	0.25	0.04	1417.20	2.79
504	3.21	0.35	0.19	0.37	125.99	4.74
507	4.76	0.62	9.18	1.09	78.20	7.14
508	5.27	1.04	2.31	2.49	22.16	7.87
511	5.11	0.81	2.46	2.60	77.06	6.76
514	2.56	0.42	7.79	0.21	84.07	3.92
515	4.92	0.27	1.44	1.68	179.32	5.38
516	2.48	0.47	1.93	1.56	142.53	3.40
517	1.98	1.03	1.00	0.83	136.68	2.90
518	2.08	0.48	0.41	0.38	176.72	2.83
521	4.46	1.17	4.47	0.96	77.95	6.58
522	3.00	0.30	0.75	0.47	74.14	5.09
523	2.66	0.49	0.43	0.24	35.41	5.10
524	3.42	0.00	0.21	0.41	8.57	6.42
525	2.06	1.11	0.47	0.35	111.02	3.71
529	1.85	0.69	2.13	1.83	67.24	4.23
530	3.19	0.80	0.36	0.52	150.58	4.44
	3.28	0.57	2.01	0.89	165.03	4.99

Table 44: $[7, 10]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	435.0	496.7	−41.1	6909.8	−1090.9	45.6
502	446.0	494.8	−32.5	2023.9	−243.1	9.2
504	472.7	544.3	−47.8	−767.8	265.3	−13.6
507	535.0	519.1	10.6	1265.2	−138.4	6.5
508	567.3	612.6	−30.2	1379.6	−111.6	3.5
511	387.3	472.8	−57.0	5778.6	−885.3	36.0
514	847.7	995.0	−98.2	−1643.3	535.0	−27.5
515	661.0	775.0	−76.0	1804.0	−123.1	2.0
516	469.0	596.0	−84.7	4515.1	−622.0	23.4
517	396.0	412.8	−11.2	1322.0	−150.6	6.1
518	502.0	565.0	−42.0	−6847.3	1327.9	−59.6
521	675.3	731.2	−37.2	782.8	18.9	−2.4
522	649.7	761.1	−74.3	−236.7	230.2	−13.2
523	515.7	617.6	−68.0	−1383.2	400.8	−20.4
524	558.7	582.4	−15.8	−3423.6	712.5	−31.7
525	560.3	585.1	−16.5	−5033.4	995.1	−44.0
529	662.7	670.5	−5.2	−704.7	244.4	−10.9
530	494.7	528.6	−22.6	−3310.9	688.5	−30.9
Avg	546.4	608.9	−41.6	135.0	114.1	−6.8

Table 45: $[10, 13]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	4.10	4.14	4.12	4.11	4.11	0.33	4.09
502	3.43	3.44	3.43	3.43	3.43	0.91	3.41
504	3.36	3.43	3.37	3.41	3.41	−4.10	3.32
507	3.81	3.82	3.81	3.83	3.83	1.89	3.80
508	3.51	3.52	3.52	3.51	3.51	1.63	3.50
511	3.81	3.92	3.81	3.88	3.88	0.33	3.88
514	2.94	2.92	2.93	2.90	2.90	−2.74	2.85
515	4.17	4.24	4.16	4.20	4.20	1.72	4.21
516	3.86	3.86	3.86	3.82	3.82	0.50	3.82
517	3.39	3.36	3.37	3.36	3.36	1.19	3.34
518	3.44	3.44	3.43	3.42	3.42	−0.34	3.39
521	3.58	3.57	3.58	3.55	3.55	3.00	3.57
522	3.96	3.96	3.95	3.93	3.93	−453.96	4.69
523	3.31	3.36	3.34	3.33	3.33	−2.08	3.28
524	3.36	3.40	3.37	3.39	3.39	−0.78	3.35
525	3.45	3.48	3.45	3.47	3.47	−0.53	3.43
529	3.88	3.83	3.86	3.83	3.83	−7.37	3.69
530	4.37	4.41	4.41	4.39	4.39	−0.92	4.32
Avg	3.65	3.67	3.65	3.65	3.65	−25.63	3.66

Table 46: $[10, 13]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
501	29.99	-6.52×10^{-3}	7.83×10^{-5}
502	25.57	-4.18×10^{-3}	3.51×10^{-5}
504	27.04	-5.78×10^{-3}	2.04×10^{-4}
507	34.09	1.26×10^{-3}	4.82×10^{-5}
508	33.26	-3.12×10^{-3}	1.97×10^{-5}
511	25.31	-9.61×10^{-3}	6.60×10^{-5}
514	41.20	-5.63×10^{-3}	1.34×10^{-4}
515	46.67	-8.12×10^{-3}	1.24×10^{-5}
516	30.19	-1.16×10^{-2}	5.20×10^{-5}
517	22.19	-1.59×10^{-3}	3.37×10^{-5}
518	28.78	-4.79×10^{-3}	7.58×10^{-5}
521	40.13	-3.28×10^{-3}	-2.15×10^{-5}
522	42.85	-7.54×10^{-3}	-2.64×10^{-3}
523	28.87	-7.38×10^{-3}	1.45×10^{-4}
524	31.61	-1.60×10^{-3}	8.05×10^{-5}
525	32.48	-1.71×10^{-3}	7.83×10^{-5}
529	42.35	-5.02×10^{-4}	2.23×10^{-4}
530	36.33	-3.35×10^{-3}	1.36×10^{-4}
Avg	33.27	-4.73×10^{-3}	-6.87×10^{-5}

Table 47: $[10, 13]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	0.84	0.38	0.19	0.19	91.90	0.39
502	0.22	0.16	0.20	0.20	73.55	0.52
504	2.22	0.42	1.63	1.63	222.16	1.00
507	0.30	0.08	0.42	0.42	50.35	0.41
508	0.17	0.11	0.14	0.14	53.63	0.32
511	2.94	0.11	1.97	1.95	91.26	1.79
514	0.89	0.42	1.44	1.44	193.05	3.26
515	1.69	0.06	0.87	0.86	58.74	1.17
516	0.04	0.04	1.18	1.20	87.18	1.02
517	0.87	0.79	1.03	1.03	64.85	1.53
518	0.00	0.33	0.47	0.48	109.98	1.32
521	0.44	0.15	0.77	0.77	16.29	0.36
522	0.07	0.32	0.81	0.82	11564.00	18.45
523	1.35	0.69	0.61	0.60	162.86	0.95
524	0.92	0.26	0.76	0.76	123.09	0.53
525	0.70	0.14	0.53	0.53	115.40	0.68
529	1.17	0.62	1.22	1.22	289.86	4.80
530	0.92	0.99	0.58	0.58	121.14	0.96
	0.87	0.34	0.82	0.82	749.41	2.19

Table 48: $[10, 13]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	366.3	435.9	−46.4	4302.3	−498.1	15.6
502	279.7	263.3	10.9	1263.1	−147.1	5.4
504	385.0	409.4	−16.2	−1288.5	248.0	−9.1
507	477.7	506.0	−18.9	7637.2	−972.0	32.9
508	481.0	514.5	−22.3	7389.5	−933.8	31.4
511	340.0	308.0	21.3	9592.7	−1302.3	45.6
514	768.7	741.5	18.1	850.2	−29.5	1.6
515	639.0	730.9	−61.3	−4164.3	726.6	−27.2
516	352.0	374.4	−14.9	1645.6	−164.0	5.1
517	327.7	359.7	−21.4	3591.2	−430.2	14.1
518	439.3	487.6	−32.2	−4044.4	653.1	−23.6
521	609.0	623.0	−9.3	−3905.4	634.3	−22.2
522	514.0	444.7	46.2	−5042.2	722.6	−23.3
523	445.3	456.4	−7.4	−6622.3	985.7	−34.2
524	427.7	435.2	−5.0	−3699.7	576.4	−20.0
525	572.3	578.2	−3.9	−5354.6	824.4	−28.6
529	616.3	673.2	−37.9	4193.1	−456.9	14.4
530	504.3	440.3	42.7	8463.6	−1144.7	40.9
Avg	474.7	487.9	−8.8	822.6	−39.3	1.1

Table 49: $[13, 16]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.63	3.62	3.58	3.59	3.59	0.37	3.59
502	3.47	3.43	3.43	3.44	3.44	0.91	3.41
504	3.40	3.36	3.37	3.36	3.36	−1.38	3.32
507	3.63	3.65	3.64	3.64	3.64	0.28	3.61
508	3.69	3.72	3.70	3.71	3.71	0.29	3.69
511	3.38	3.35	3.37	3.36	3.36	0.15	3.32
514	3.57	3.57	3.58	3.57	3.57	3.38	3.56
515	3.93	3.93	3.96	3.90	3.90	−0.80	3.88
516	3.56	3.57	3.54	3.56	3.56	0.89	3.54
517	3.56	3.56	3.57	3.55	3.55	0.39	3.54
518	3.60	3.58	3.57	3.56	3.56	−0.50	3.54
521	3.58	3.59	3.60	3.59	3.59	−0.73	3.55
522	3.64	3.62	3.62	3.64	3.64	−0.46	3.58
523	3.34	3.34	3.34	3.34	3.34	−0.28	3.31
524	3.42	3.44	3.41	3.44	3.44	−0.51	3.41
525	3.31	3.27	3.29	3.27	3.27	−0.45	3.24
529	3.67	3.69	3.67	3.68	3.68	0.64	3.66
530	3.39	3.34	3.36	3.36	3.36	0.25	3.31
Avg	3.54	3.54	3.53	3.53	3.53	0.13	3.50

Table 50: $[13, 16]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
501	22.08	-7.64×10^{-3}	3.10×10^{-5}
502	16.01	2.23×10^{-3}	3.32×10^{-5}
504	21.58	-2.36×10^{-3}	5.91×10^{-5}
507	29.02	-2.41×10^{-3}	3.80×10^{-5}
508	29.86	-2.88×10^{-3}	3.91×10^{-5}
511	18.99	3.50×10^{-3}	3.60×10^{-5}
514	45.72	1.40×10^{-3}	1.43×10^{-5}
515	41.84	-6.28×10^{-3}	7.19×10^{-5}
516	20.93	-2.52×10^{-3}	2.54×10^{-5}
517	19.47	-3.87×10^{-3}	3.28×10^{-5}
518	26.20	-4.37×10^{-3}	5.24×10^{-5}
521	36.47	-9.17×10^{-4}	5.15×10^{-5}
522	30.99	5.42×10^{-3}	4.13×10^{-5}
523	24.81	-9.20×10^{-4}	3.98×10^{-5}
524	24.52	-6.72×10^{-4}	4.46×10^{-5}
525	31.22	-3.72×10^{-4}	3.97×10^{-5}
529	37.91	-3.79×10^{-3}	3.04×10^{-5}
530	28.07	4.71×10^{-3}	3.63×10^{-5}
Avg	28.09	-1.21×10^{-3}	3.98×10^{-5}

Table 51: $[13, 16]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	0.36	1.26	1.11	1.12	89.87	1.11
502	0.98	1.09	0.76	0.76	73.89	1.68
504	0.99	0.65	1.22	1.22	140.61	2.29
507	0.51	0.44	0.27	0.27	92.36	0.34
508	0.98	0.29	0.70	0.69	92.02	0.12
511	0.73	0.24	0.38	0.38	95.66	1.56
514	0.07	0.15	0.07	0.07	5.42	0.25
515	0.10	0.68	0.72	0.73	120.28	1.44
516	0.24	0.52	0.01	0.01	75.10	0.38
517	0.02	0.08	0.40	0.40	89.05	0.79
518	0.52	0.67	0.95	0.95	113.96	1.62
521	0.48	0.54	0.39	0.39	120.31	0.66
522	0.54	0.47	0.01	0.00	112.71	1.51
523	0.05	0.20	0.14	0.14	108.49	0.99
524	0.72	0.06	0.66	0.66	114.90	0.31
525	1.17	0.76	1.20	1.20	113.47	2.12
529	0.50	0.06	0.12	0.12	82.58	0.23
530	1.57	0.99	1.10	1.11	92.78	2.39
	0.58	0.51	0.57	0.57	96.30	1.10

Table 52: $[13, 16]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	257.0	359.0	−102.0	−9346.5	1233.3	−39.3
502	229.0	314.6	−85.6	−34631.0	4191.7	−125.8
504	251.5	407.0	−155.5	−1699.7	385.6	−15.9
507	340.5	470.5	−130.0	−42905.0	5223.9	−157.5
508	362.0	562.4	−200.4	−26816.0	3401.9	−106.0
511	277.5	481.5	−204.0	−15081.0	2013.5	−65.2
514	477.5	892.0	−414.5	−402.4	519.1	−27.5
515	394.5	680.4	−285.9	−19460.0	2625.0	−85.6
516	245.0	356.5	−111.5	−30032.0	3677.9	−111.5
517	249.0	366.3	−117.3	−20275.0	2534.9	−78.0
518	289.0	439.3	−150.3	−31042.0	3840.9	−117.4
521	415.5	624.3	−208.8	−13647.0	1865.6	−61.0
522	409.5	620.3	−210.8	−36324.0	4537.8	−139.7
523	323.5	433.4	−109.9	20863.0	−2309.1	64.7
524	325.0	406.8	−81.8	−28059.0	3425.1	−103.1
525	301.0	512.3	−210.8	22246.0	−2373.3	63.6
529	504.0	650.3	−146.3	−28964.0	3617.4	−110.7
530	348.5	543.4	−194.4	−21076.0	2718.3	−85.7
Avg	333.3	506.7	−173.3	−17592.0	2285.0	−72.3

Table 53: [16, 18] in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, [7, 10] is treated as [0, 3]).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	3.68	3.78	3.74	3.69	3.68	−0.12	3.75
502	3.20	3.43	3.29	3.36	3.36	−0.03	3.40
504	4.16	4.28	4.17	4.11	4.09	−0.81	4.20
507	3.26	3.38	3.25	3.31	3.31	−0.03	3.36
508	3.91	4.02	3.89	3.88	3.87	−0.06	3.98
511	4.49	4.58	4.47	4.35	4.32	−0.10	4.53
514	4.40	4.51	4.33	4.25	4.22	−517.80	6.61
515	5.45	5.57	5.38	5.23	5.19	−0.13	5.49
516	3.81	3.99	3.85	3.87	3.87	−0.04	3.95
517	3.55	3.69	3.53	3.58	3.58	−0.05	3.66
518	4.00	4.13	4.03	3.99	3.98	−0.04	4.09
521	3.63	3.75	3.62	3.64	3.63	−0.13	3.71
522	4.32	4.46	4.34	4.30	4.29	−0.06	4.42
523	3.36	3.39	3.34	3.32	3.32	0.06	3.36
524	3.25	3.33	3.25	3.28	3.28	−0.04	3.30
525	4.53	4.73	4.51	4.48	4.47	0.07	4.68
529	3.70	3.78	3.70	3.71	3.71	−0.07	3.74
530	4.61	4.77	4.58	4.57	4.56	−0.09	4.72
Avg	3.96	4.09	3.96	3.94	3.93	−28.86	4.16

Table 54: [16, 18] in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
501	16.19	-2.50×10^{-2}	3.75×10^{-5}
502	13.09	-2.14×10^{-2}	2.65×10^{-5}
504	17.95	-4.41×10^{-2}	1.17×10^{-4}
507	19.19	-2.15×10^{-2}	2.60×10^{-5}
508	24.23	-3.70×10^{-2}	3.96×10^{-5}
511	21.18	-5.61×10^{-2}	5.65×10^{-5}
514	35.89	-6.53×10^{-2}	-3.01×10^{-2}
515	36.60	-6.72×10^{-2}	8.47×10^{-5}
516	16.29	-3.03×10^{-2}	3.65×10^{-5}
517	15.30	-2.89×10^{-2}	3.25×10^{-5}
518	19.88	-3.58×10^{-2}	3.99×10^{-5}
521	25.94	-3.14×10^{-2}	3.94×10^{-5}
522	30.45	-3.83×10^{-2}	4.74×10^{-5}
523	18.25	-1.92×10^{-2}	2.18×10^{-5}
524	18.01	-1.40×10^{-2}	2.52×10^{-5}
525	23.71	-5.51×10^{-2}	3.89×10^{-5}
529	31.71	-1.83×10^{-2}	3.38×10^{-5}
530	27.73	-4.43×10^{-2}	5.75×10^{-5}
Avg	22.87	-3.63×10^{-2}	-1.63×10^{-3}

Table 55: $[16, 18]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	2.72	1.72	0.27	0.15	103.23	1.80
502	7.12	2.68	4.93	4.83	100.80	6.33
504	2.87	0.21	1.31	1.67	119.44	0.91
507	3.59	0.51	1.46	1.36	100.93	2.83
508	2.69	0.43	0.85	1.11	101.58	1.76
511	2.01	0.38	3.15	3.72	102.15	0.89
514	2.57	1.51	3.37	4.12	11876.00	50.33
515	2.10	1.40	3.97	4.77	102.38	0.72
516	4.75	1.12	1.76	1.58	100.97	3.84
517	3.79	0.59	0.95	0.79	101.45	2.94
518	3.13	0.63	0.32	0.56	101.09	2.19
521	3.19	0.17	0.14	0.05	103.62	2.23
522	3.36	0.59	0.32	0.59	101.32	2.34
523	0.66	0.74	1.20	1.27	98.25	0.01
524	2.19	0.01	0.80	0.76	101.34	1.45
525	4.23	0.45	1.11	1.51	98.43	3.30
529	2.06	0.14	0.27	0.20	102.02	1.21
530	3.45	0.71	0.90	1.13	101.95	2.31
	3.14	0.78	1.50	1.68	756.52	4.85

Table 56: $[16, 18]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

	Constant	Linear		Quadratic		
Day	$\bar{\lambda}(t)$	a	b	a	b	c
501	53.3	103.5	−25.1	6220.8	−593.5	14.2
502	42.0	38.5	1.8	3714.3	−370.2	9.3
504	46.3	77.6	−15.5	1554.9	−135.7	3.0
507	54.0	77.5	−11.7	288.8	−11.7	0.0
508	54.3	80.0	−12.9	489.3	−30.7	0.4
511	42.5	56.8	−7.1	771.7	−66.0	1.5
514	64.5	91.9	−13.7	2631.6	−243.8	5.8
515	62.3	86.7	−12.1	181.7	0.2	−0.3
516	54.3	66.6	−6.2	−509.4	62.7	−1.7
517	54.5	94.1	−19.8	3825.0	−358.4	8.5
518	53.3	93.9	−20.2	2062.3	−181.2	4.0
521	93.0	139.9	−23.3	4016.0	−370.1	8.7
522	116.8	144.1	−13.7	3499.8	−325.7	7.8
523	112.3	144.4	−16.1	1846.9	−157.9	3.5
524	96.8	119.1	−11.2	3118.1	−291.9	7.0
525	73.3	113.3	−20.0	6524.9	−627.2	15.2
529	160.3	199.5	−19.6	661.8	−30.6	0.3
530	69.5	71.3	−0.9	−1285.6	136.9	−3.4
Avg	72.4	99.9	−13.7	2200.7	−199.7	4.6

Table 57: $[18, 22]$ in 18 days in May: Fitting constant, linear and quadratic arrival rate functions. In arrival rate approximation, each interval is time-shifted to start at 0 (for instance, $[7, 10]$ is treated as $[0, 3]$).

Day	$\bar{W}(t)$	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	7.80	7.90	7.57	7.46	7.41	0.08	7.79
502	3.65	3.63	3.65	3.63	3.63	0.05	3.60
504	6.17	6.23	6.06	6.00	6.02	0.22	6.17
507	3.70	3.78	3.73	3.73	3.73	0.77	3.77
508	3.27	3.29	3.26	3.24	3.24	0.41	3.27
511	4.03	4.10	3.95	4.05	4.05	0.27	4.07
514	4.06	4.17	4.12	4.11	4.11	0.12	4.14
515	3.61	3.67	3.50	3.62	3.63	1.27	3.67
516	3.45	3.40	3.38	3.38	3.38	−0.47	3.37
517	5.31	5.32	5.32	5.16	5.15	0.09	5.27
518	3.87	3.94	3.85	3.83	3.84	0.12	3.92
521	4.91	4.92	4.85	4.81	4.82	0.14	4.88
522	5.25	5.24	5.26	5.19	5.19	0.21	5.20
523	5.07	5.08	5.02	5.02	5.02	0.37	5.04
524	6.84	6.89	6.83	6.80	6.79	0.26	6.80
525	5.93	5.96	5.88	5.81	5.80	0.08	5.90
529	4.35	4.42	4.36	4.39	4.38	1.18	4.41
530	5.51	5.70	5.49	5.69	5.69	−0.38	5.63
Avg	4.82	4.87	4.78	4.77	4.77	0.27	4.83

Table 58: $[18, 22]$ in 18 days in May: waiting time estimates by different methods.

Day	$\bar{L}(t)$	$\bar{W}_{L,\lambda}(t)\left(\frac{\gamma_W^2\bar{\lambda}_l'}{\lambda(t)}\right)$ in (19)	$w\delta - w^2\epsilon\left(\frac{1}{1-2w\delta}\right)$ in (32)
501	7.01	-6.21×10^{-2}	9.39×10^{-5}
502	2.54	2.52×10^{-3}	2.22×10^{-5}
504	4.80	-3.47×10^{-2}	4.89×10^{-5}
507	3.40	-1.37×10^{-2}	$4.48 \times 10^{-1}8$
508	2.97	-1.30×10^{-2}	6.16×10^{-6}
511	2.90	-1.15×10^{-2}	2.09×10^{-5}
514	4.48	-1.47×10^{-2}	2.52×10^{-5}
515	3.81	-1.19×10^{-2}	-1.28×10^{-5}
516	3.07	-6.43×10^{-3}	2.77×10^{-5}
517	4.83	-3.22×10^{-2}	4.14×10^{-5}
518	3.50	-2.48×10^{-2}	1.99×10^{-5}
521	7.62	-2.05×10^{-2}	3.45×10^{-5}
522	10.20	-1.02×10^{-2}	4.05×10^{-5}
523	9.51	-1.21×10^{-2}	3.23×10^{-5}
524	11.10	-1.32×10^{-2}	7.02×10^{-5}
525	7.27	-2.71×10^{-2}	5.48×10^{-5}
529	11.82	-9.03×10^{-3}	4.92×10^{-6}
530	6.60	-1.24×10^{-3}	5.90×10^{-5}
Avg	5.97	-1.76×10^{-2}	3.28×10^{-5}

Table 59: $[18, 22]$ in 18 days in May: $\bar{L}(t)$ and parameters for perturbation analysis in equations (19) and (32).

Day	$\bar{W}_{L,\lambda}(t)$	$\bar{W}_{L,\lambda,r}(t)$	$\bar{W}_{L,\lambda,l}(t)$	$\bar{W}_{L,\lambda,l,p}(t)$	$\bar{W}_{L,\lambda,q}(t)$	$\bar{W}_{L,\lambda,q,p}(t)$
501	1.24	3.03	4.37	5.05	98.95	0.17
502	0.68	0.09	0.43	0.43	98.62	1.35
504	0.93	1.80	2.86	2.57	96.41	0.07
507	2.18	0.76	0.82	0.78	79.21	1.89
508	0.51	0.42	0.76	0.80	87.33	0.14
511	1.58	2.00	0.44	0.42	93.34	0.94
514	2.65	1.46	1.18	1.14	96.96	1.92
515	1.84	3.06	0.27	0.64	64.89	1.81
516	1.46	1.92	2.09	2.10	113.62	2.26
517	0.16	0.16	2.87	3.06	98.27	0.75
518	1.86	0.53	1.00	0.67	96.85	1.22
521	0.29	1.06	1.95	1.77	97.19	0.55
522	0.09	0.12	1.09	1.11	95.97	0.99
523	0.13	0.98	1.05	1.08	92.75	0.66
524	0.71	0.07	0.58	0.62	96.21	0.48
525	0.49	0.88	2.10	2.24	98.63	0.57
529	1.81	0.22	0.91	0.89	72.93	1.44
530	3.43	0.29	3.30	3.30	106.98	2.25
	1.23	1.05	1.56	1.59	93.62	1.08

Table 60: $[18, 22]$ in 18 days in May: absolute relative error of the estimates from the direct estimate, in units of 10^{-2}.

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