Appendix to
Stationary Birth-and-Death Processes Fit to Queues with Periodic Arrival Rate Functions

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Abstract

To better understand how to interpret birth-and-death (BD) processes fit to service system data, we investigate the consequences of fitting a BD process to a multi-server queue with a periodic time-varying arrival rate function. We study how this fitted BD process is related to the original queue-length process. If a BD process is fit to a segment of the sample path of the queue-length process, with the birth (death) rates in each state estimated by the observed number of arrivals (departures) in that state divided by the total time spent in that state, then under minor regularity conditions that BD process has the steady-state distribution of the queue length process in the original $M_t/GI/s$ queueing model as the sample size increases. The steady-state distribution can be estimated efficiently by fitting a parametric function to the observed birth and death rates.

Keywords: birth-and-death processes; grey-box stochastic models; fitting stochastic models to data; queues with time-varying arrival rate; speed ratio; transient behavior.

1 Overview

This appendix to the main paper [2] provides supplementary material. The main paper in turn is a sequel to [1], which investigates fitting general state-dependent birth-and-death (BD) processes to data, specifically, to the number in system in a service system, where arrivals and departures occur one at a time. The first paper [1] considered the $GI/GI/s$ queues, while [2] considers $M_t/GI/s$ queues with large s.

2 Additional Results from Simulation Experiments

The main paper focuses on $M_t/GI/s$ queueing models, having a nonhomogeneous Poisson process (NHPP, the M_t) as an arrival process, independent of i.i.d. service times distributed as a random variable S with mean $E[S] = 1/\mu = 1$, a large number s of servers, $s \le \infty$, and unlimited waiting space. Moreover, we consider the stylized sinusoidal arrival rate function

$$\lambda(t) \equiv \bar{\lambda}\left(1 + \beta \sin(\gamma t)\right), \quad (1)$$

where the cycle is $c = 2\pi/\gamma$. There are three parameters: (i) the average arrival rate $\bar{\lambda}$, (ii) the relative amplitude β and (iii) the time scaling factor γ or, equivalently the cycle length $c = 2\pi/\gamma$. The base model is the $M_t/M/\infty$ model, which is the special case of the $M_t/GI/s$ model in which $s = \infty$, S has an exponential distribution and $\beta = 10/35$.

2.1 The $M_t/M/s$ Model with Large Scale: $\bar{\lambda} = 100$

In §2 of the main paper we considered the base $M_t/M/s$ model above with the sinusoidal arrival rate in (1) and average arrival rate $\bar{\lambda} = 35$. We here display corresponding results for larger scale, in particular, for $\bar{\lambda} = 100$. In particular, we show results for the cases in which $\gamma = 0.1$ and 0.01. From these plots, we see that the small γ limit already holds approximately for the steady-state distribution and the fitted rates when $\gamma = 0.1$. Moreover, we see the bimodal form of the steady-state distribution as discussed in [3]. However, we see that the transient behavior becomes more different as γ decreases.

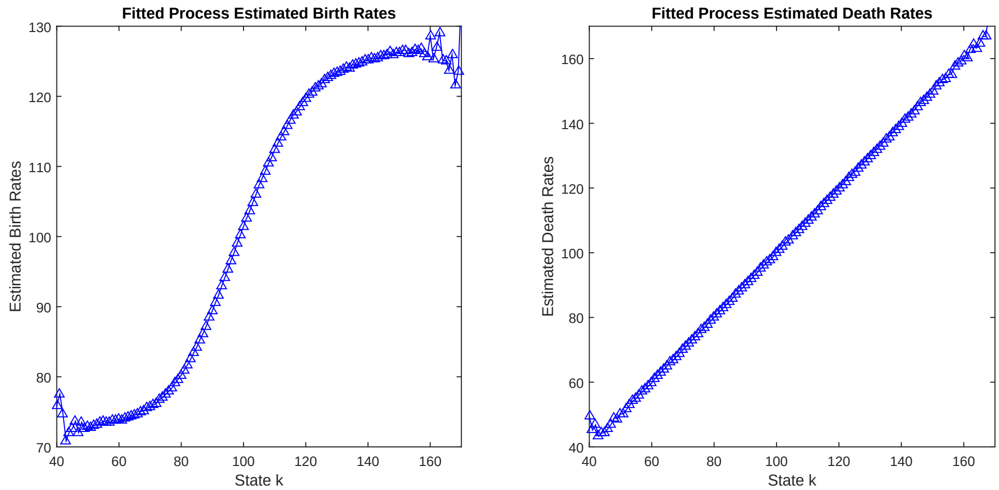

Figure 1: Fitted birth rates (left) and fitted death rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.1$

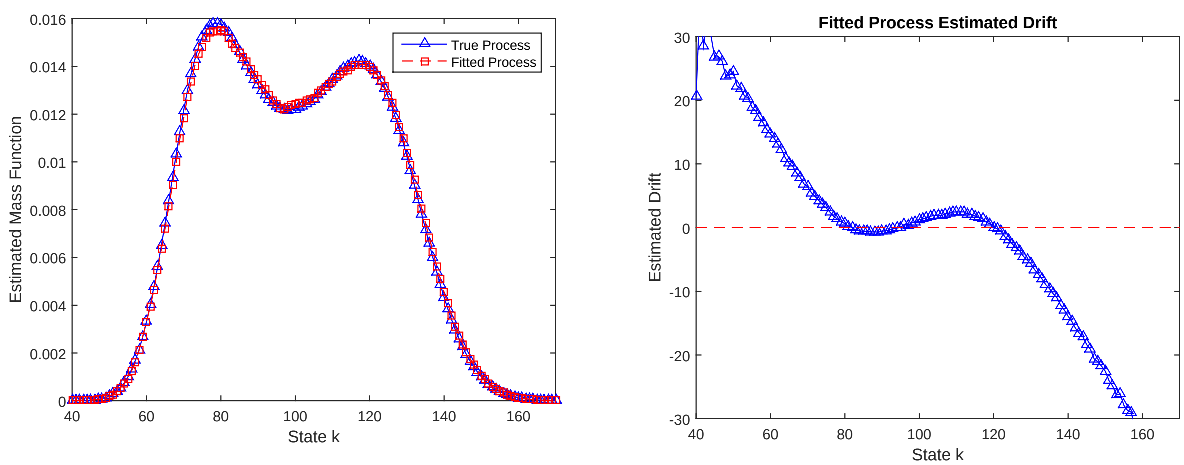

Figure 2: Estimated mass function (left) and estimated drift (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.1$

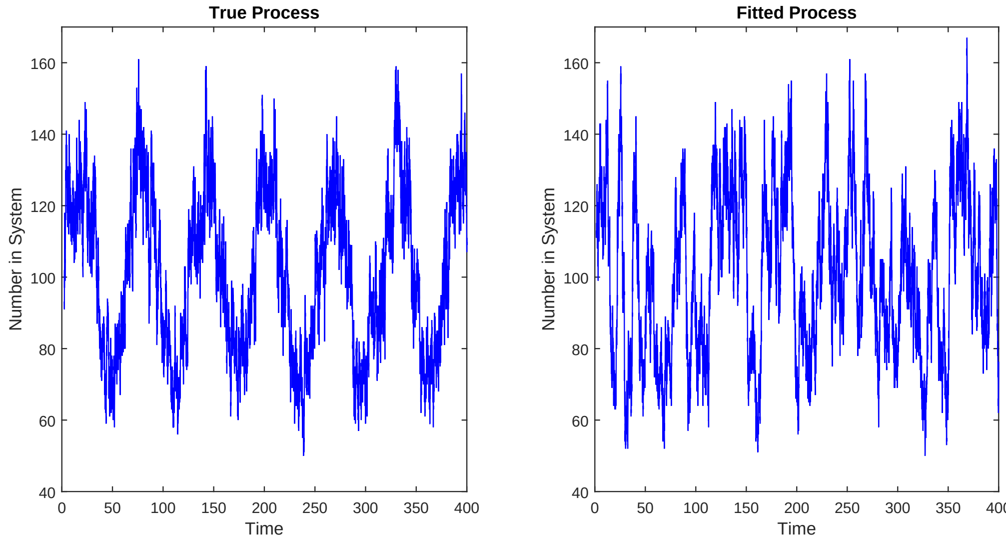

Figure 3: Sample paths of true process (left) and fitted process (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.1$

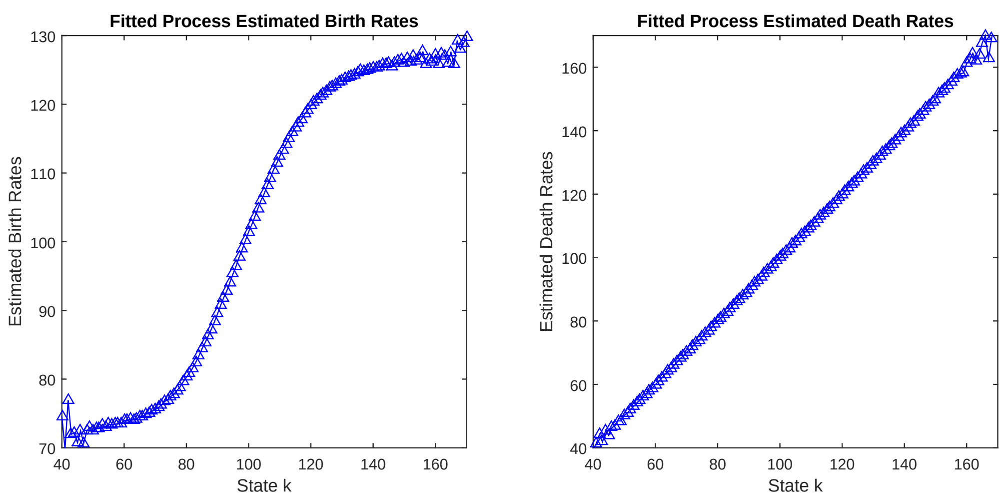

Figure 4: Fitted birth rates (left) and fitted death rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.01$

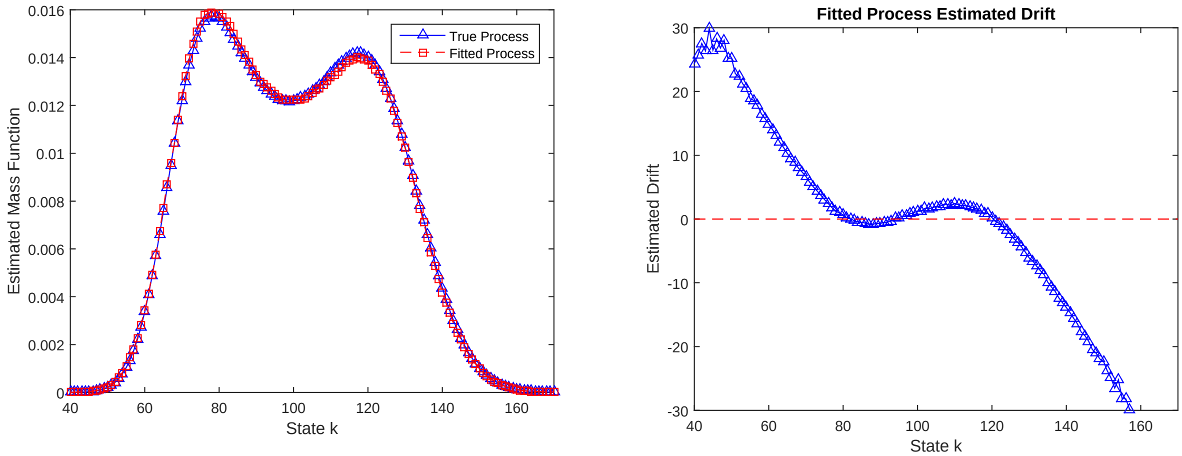

Figure 5: Estimated mass function (left) and estimated drift (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.01$

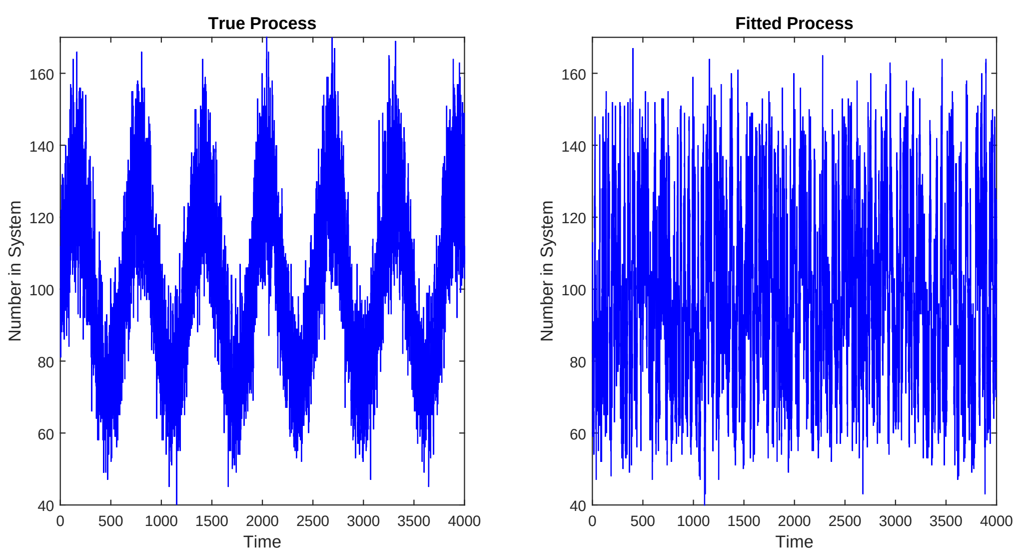

Figure 6: Sample paths of true process (left) and fitted process (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 100$, $\beta = 10/35$ and $\gamma = 0.01$

2.2 Estimating the Steady-State Distribution

In this section we investigate how we may efficiently estimate the steady-state distribution by fitting parametric models to the estimated birth and death rates and then solve the local balance equation

$$\bar{\alpha}_k^e \bar{\lambda}_k = \bar{\alpha}_{k+1}^e \bar{\mu}_{k+1}, \quad k \geq 0. \tag{2}$$

First, for IS model we do not need to consider the death rates, because we have $\bar{\mu}_k = k$ throughout. Hence, we concentrate on the birth rates. As our parametric function, we choose the function

$$\lambda_k^p = a \arctan b(k - c) + d, \tag{3}$$

which is nondecreasing with finite limits as k increases and decreases, and has the parameter four-tuple (a, b, c, d). We let $c = d = \bar{\lambda}$, so that leaves only the two parameters a and b.

Figures 7-12 show the fitted mass function and birth rates for the six gamma values: $\gamma = 1/8$, $1/4$, $1/2$, 1, 2 and 4, respectively. The figures show that the special arctangent function in (3) doees much better than a linear fit for small γ, but a simple linear fit works well for large γ. The parameter pairs in the six cases were $(a, b) = (7.541, 0.125)$, $(7.392, 0.1262)$, $(6.682, 0.1253)$, $(5.333, 0.1114)$, $(3.577, 0.0744)$ and $(6.629, 0.1186)$, respectively. The main point is that a parametric fit based on only two parameters yields an accurate fit to a mass function that can be quite complicated. The anomalous value of a in the final case of $\gamma = 4$ evidently occurs because the arctangent fit is less robust. For larger γ values a simple linear fit seems to be better.

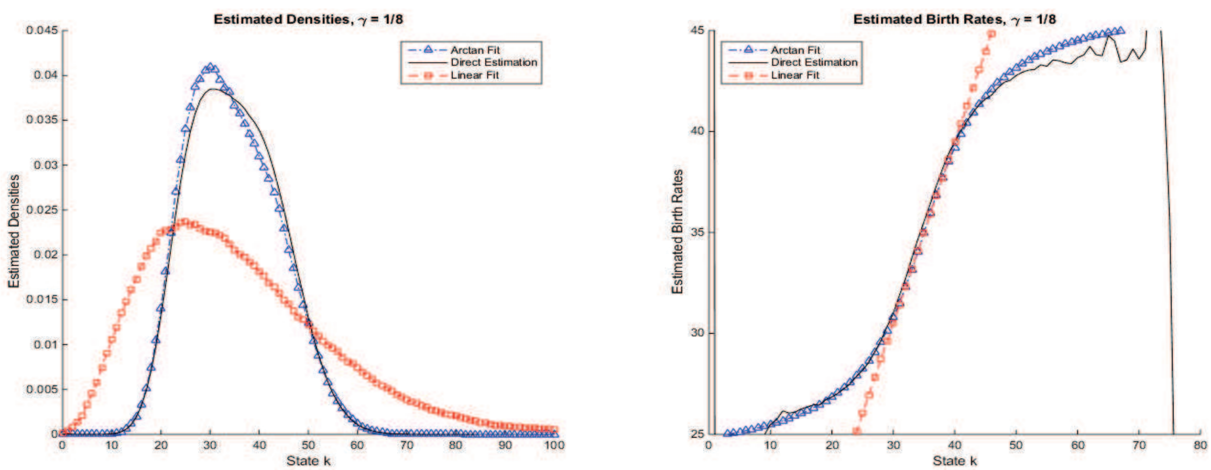

Figure 7: Fitted mass function (left) and birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 0.125$

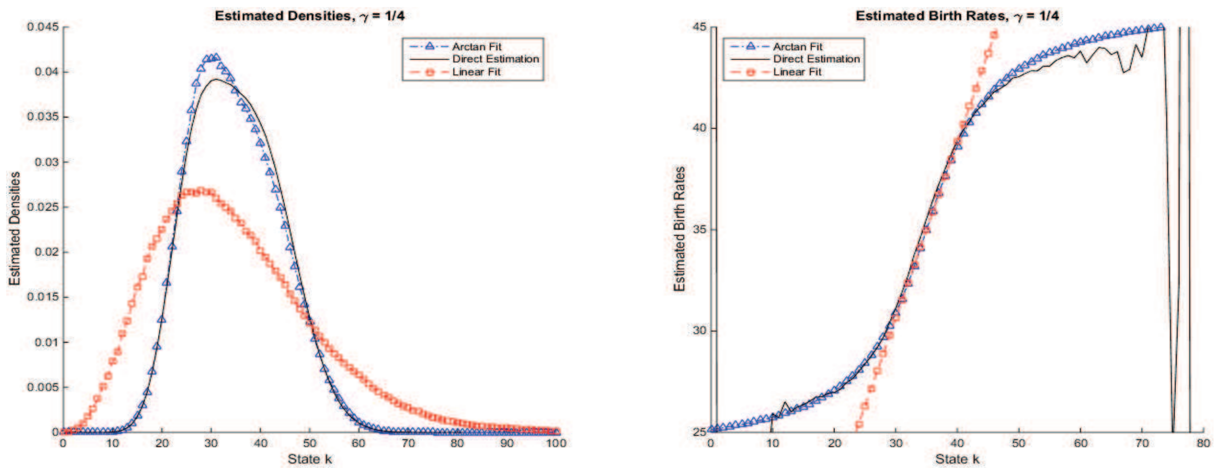

Figure 8: Fitted mass function (left) and birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 0.25$

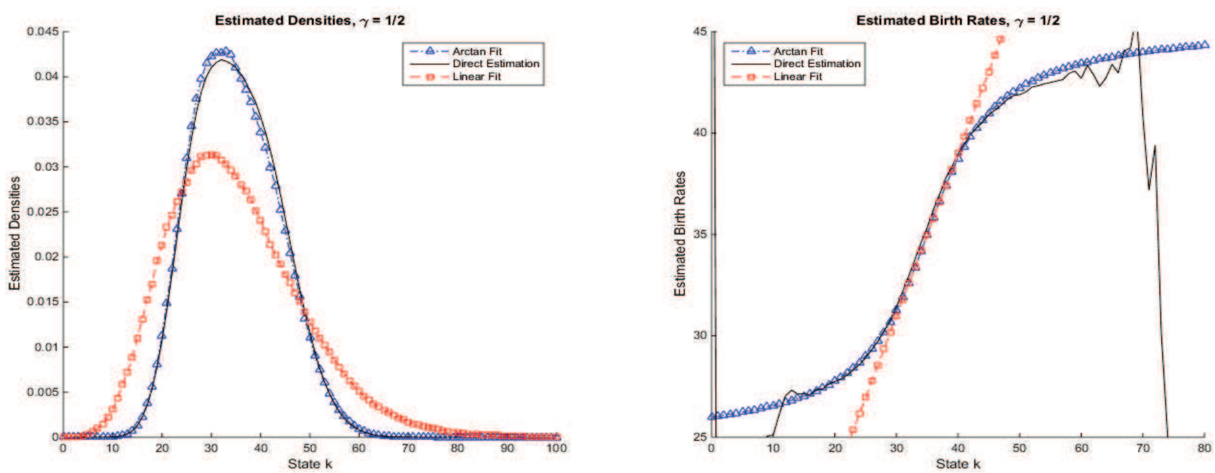

Figure 9: Fitted mass function (left) and birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 0.5$

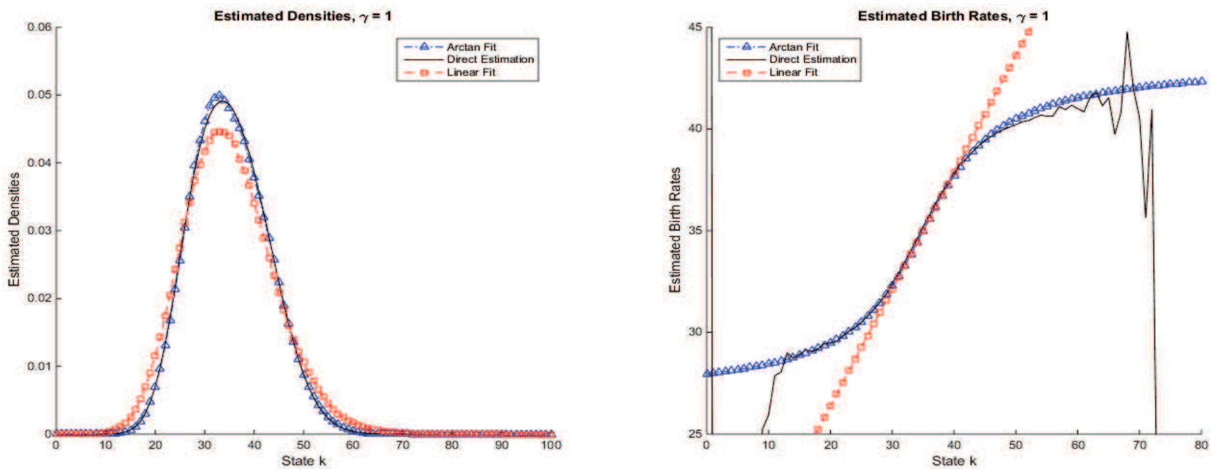

Figure 10: Fitted mass function (left) and fitted birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 1.0$

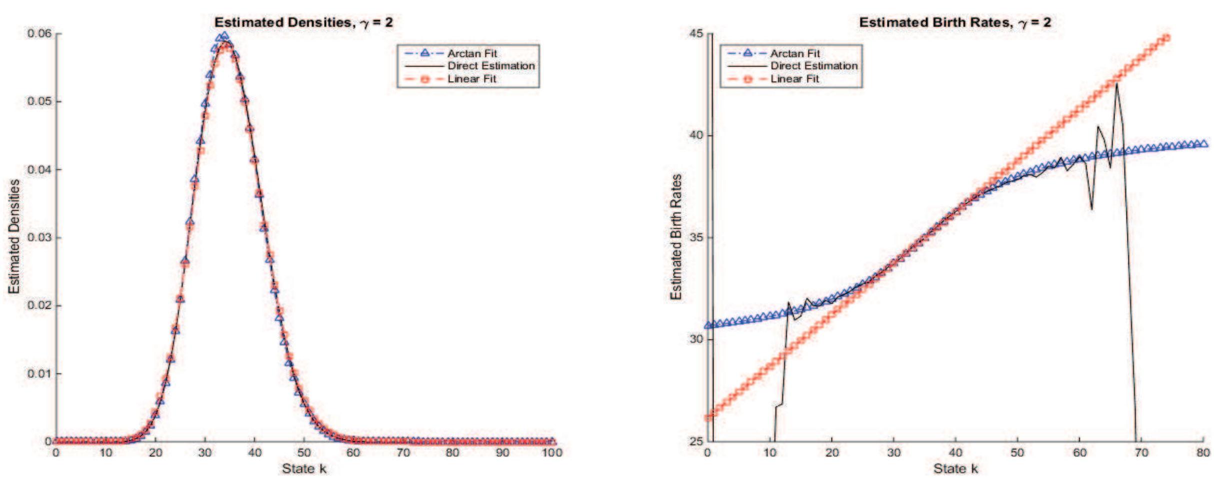

Figure 11: Fitted mass function (left) and fitted birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 2.0$

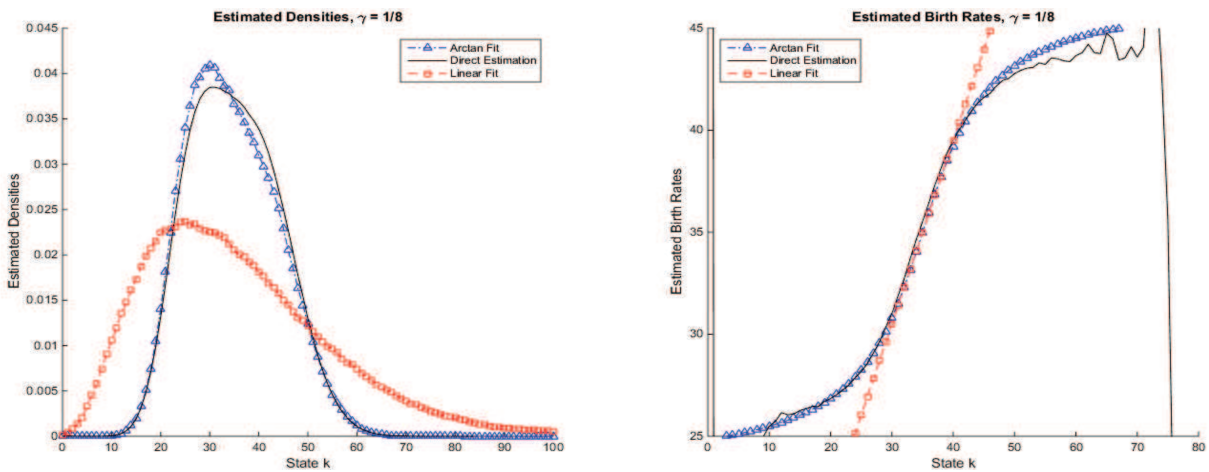

Figure 12: Fitted mass function (left) and fitted birth rates (right) for the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$, $\beta\bar{\lambda} = 10$ and $\gamma = 4.0$

2.3 Different Service Distributions

We have also conducted corresponding simulation experiments for the $M_t/GI/\infty$ model with non-exponential service-time distributions. Figures 13 and 14 show the fitted rates for H_2 and E_2 service distributions, while Figure 15 shows the associated steady-state mass functions. The H_2 distribution is just as in §2 of [1] with scv $c^2 = 2$.

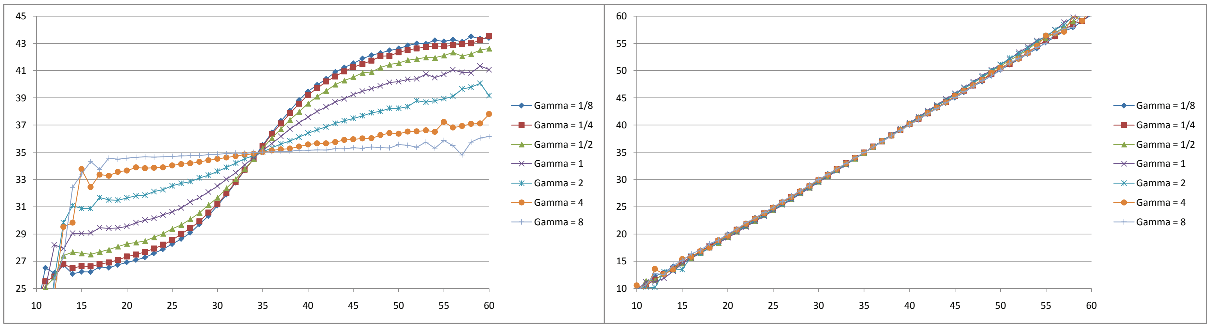

Figure 13: Fitted birth rates (left) and fitted death rates (right) for the $M_t/H_2/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$ and $\beta\bar{\lambda} = 10$ and 7 values of γ ranging from 1/8 to 8.

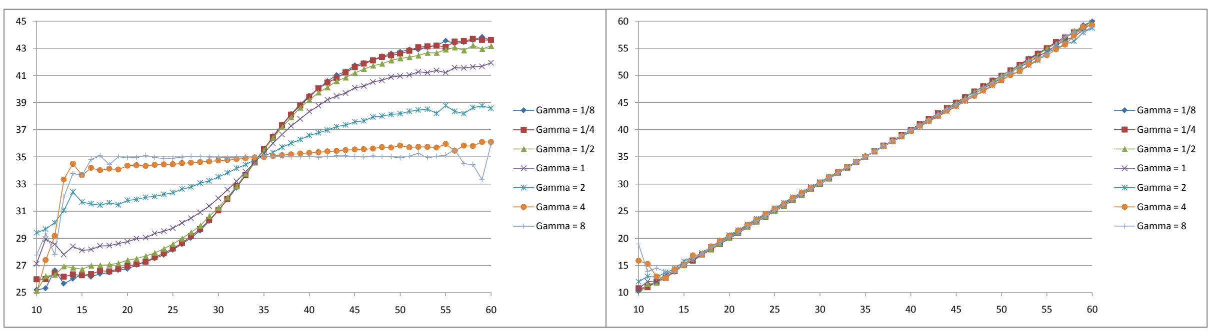

Figure 14: Fitted birth rates (left) and fitted death rates (right) for the $M_t/E_2/\infty$ model with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$ and $\beta\bar{\lambda} = 10$ and 7 values of γ ranging from 1/8 to 8.

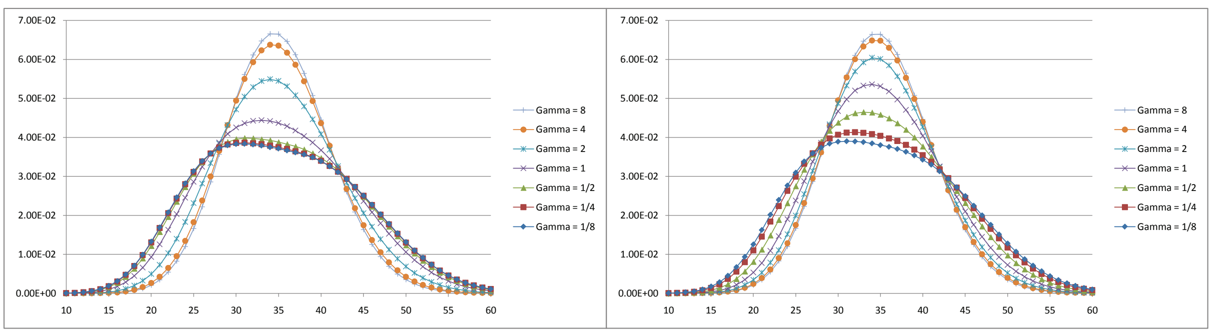

Figure 15: Fitted steady-state mass functions for the $M_t/H_2/\infty$ model (left) and the $M_t/E_2/\infty$ model (right) for with the sinusoidal arrival rate function in (1) having parameters $\bar{\lambda} = 35$ and $\beta\bar{\lambda} = 10$ and 7 values of γ ranging from 1/8 to 8.

2.4 More on Speed Ratios

We now present more results about the speed ratios in $M_t/GI/\infty$ and $M_t/M/40$ models. As before, we use the speed ratio $\omega(p)$ defined in (16) of the main paper [1] for $p = 0.1$. We show estimates for each of five independent replications of our experiment. The high accuracy shows that the statistical precision is excellent for our very large simulation sample sizes.

Table 1: Estimated speed ratios in each of five independent replications for the $M_t/GI/\infty$ model with sinusoidal arrival rate having $\bar{\lambda} = 35, \bar{\lambda}\beta = 10$ for Erlang (E_2), exponential (M), and hyper-exponential (H_2) service for several values of γ.

γ	$M_t/E_2/\infty$					$M_t/M/\infty$					$M_t/H_2(2)/\infty$				
0.001	-	-	-	-	-	0.26	0.27	0.27	0.27	0.27	-	-	-	-	-
0.01	-	-	-	-	-	0.27	0.27	0.27	0.27	0.26	-	-	-	-	-
0.1	-	-	-	-	-	0.43	0.43	0.44	0.44	0.43	-	-	-	-	-
0.125	0.53	0.53	0.52	0.53	0.52	0.51	0.50	0.51	0.50	0.50	0.49	0.48	0.48	0.48	0.48
0.25	0.89	0.90	0.89	0.90	0.89	0.87	0.83	0.87	0.84	0.87	0.79	0.79	0.80	0.80	0.79
0.5	1.51	1.52	1.51	1.51	1.51	1.42	1.42	1.40	1.41	1.40	1.16	1.15	1.16	1.15	1.15
1	2.10	2.10	2.08	2.08	2.09	1.65	1.66	1.67	1.65	1.64	1.33	1.32	1.32	1.33	1.32
2	1.89	1.90	1.89	1.92	1.89	1.46	1.44	1.46	1.44	1.45	1.23	1.24	1.23	1.23	1.22
4	1.44	1.44	1.43	1.43	1.43	1.17	1.18	1.18	1.18	1.17	1.03	1.03	1.03	1.02	1.03
8	1.24	1.23	1.25	1.25	1.25	1.07	1.06	1.05	1.05	1.06	0.92	0.91	0.92	0.92	0.92

Table 2: Estimated speed ratios summary in 5 independent trials for the $M_t/GI/\infty$ model with sinusoidal arrival rate and $\bar{\lambda} = 35, \bar{\lambda}\beta = 10$ for Erlang, Exponential, and Hyperexponential service for several values of γ

	$M_t/E_2/\infty$			$M_t/M/\infty$			$M_t/H_2(2)/\infty$		
γ	$E[T(p)]$	$E[T_f(p)]$	$\omega(0.1)$	$E[T(p)]$	$E[T_f(p)]$	$\omega(0.1)$	$E[T(p)]$	$E[T_f(p)]$	$\omega(0.1)$
0.001	-	-	-	80.55	21.40	0.27	-	-	-
0.01	-	-	-	80.01	21.48	0.27	-	-	-
0.1	-	-	-	49.42	21.43	0.43	-	-	-
0.125	40.67	21.46	0.53	42.61	21.37	0.50	44.46	21.39	0.48
0.25	23.90	21.39	0.90	24.17	20.67	0.86	24.53	19.47	0.79
0.5	13.00	19.62	1.51	13.75	19.38	1.41	14.26	16.45	1.15
1	8.59	17.94	2.09	9.11	15.09	1.66	10.20	13.51	1.32
2	6.47	12.28	1.90	7.55	10.95	1.45	8.84	10.87	1.23
4	6.77	9.70	1.43	8.21	9.65	1.18	9.40	9.67	1.03
8	7.69	9.57	1.24	9.05	9.57	1.06	10.41	9.57	0.92

Table 3: Estimated speed ratios in each of five independent replications and summary statistics for the $M_t/M/40$ model with sinusoidal arrival rate and $\bar{\lambda} = 35, \bar{\lambda}\beta = 10$ for exponential service for several values of γ.

$M_t/M/\infty$								
γ	$\omega_1(0.1)$	$\omega_2(0.1)$	$\omega_3(0.1)$	$\omega_4(0.1)$	$\omega_5(0.1)$	$E[T(p)]$	$E[T_f(p)]$	$\bar{\omega}(0.1)$
0.125	4.27	4.17	4.13	4.08	4.43	83.90	353.74	4.22
0.25	3.02	3.07	3.06	3.05	3.17	45.02	138.53	3.08
0.5	2.37	2.42	2.39	2.32	2.40	26.19	62.27	2.38
1	1.78	1.80	1.79	1.76	1.81	18.73	33.49	1.79
2	1.36	1.38	1.38	1.37	1.39	15.52	21.37	1.38
4	1.15	1.14	1.13	1.13	1.13	16.38	18.59	1.14
8	1.07	1.04	1.04	1.05	1.05	16.00	16.80	1.05

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