QUEUE TESTS FOR RENEWAL PROCESSES

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Queuing models can be used to test whether a stochastic point process can be represented as a renewal process. The test queuing model is analyzed, perhaps by simulation, using the process of interest to generate arrivals or service times. Various congestion measures indicate departures from the renewal property.

Statistical inference, data analysis, simulation, time series, approximation

1. Introduction and summary

In many operations research applications we want to know if a stochastic point process (series of events) can be modeled as a renewal process or a Poisson process. In other words, we want to know if the intervals between points are approximately iid (independent and identically distributed) and, if so, whether they are approximately exponentially distributed. For example, we may be considering arrivals to a queue, demands for inventory, or the occurrence of failures. Whether we start with data from a physical system or a mathematical model we want to know if the point process can be modeled as a renewal process or a Poisson process. Then we can do further work with a more elementary analytic model or a more elementary simulation.

Of course, there are standard ways to analyze these problems. With data, we can apply standard statistical tests for renewal processes and Poisson processes; see Cox and Lewis [6, Chapter 6]. With mathematical models, we can do further analysis to see whether the point process of interest is actually renewal or Poisson; e.g., Disney, Farrel and DeMorais [7] and Melamed [11] have characterized when flows in a network of queues are renewal or Poisson. Alternatively, with mathematical models we can simulate, collect data and then perform the standard statistical tests.

The purpose of this paper is to suggest a different approach. We suggest using a queuing model to test whether a point process can be adequately modeled as a renewal process or a Poisson process. (This may be in addition to other statistical tests.) The idea is to analyze the queuing model using the process of interest to generate the arrivals (or the service times). If we have data, then we use that data to generate arrivals to the queue in a simulation; otherwise, we either solve the queuing model analytically or simulate it. Then we compare various congestion measures with theoretical results in the case of a renewal or Poisson arrival process.

In this approach, the queuing model is an artificial device introduced to perform the test. The actual application may have nothing to do with queues. Even though we consider only queuing models, other models (e.g., inventory and reliability) could also be used to perform the tests. Obviously, however, it should be desirable to perform the test with a model closely related to the intended application.

To make the motivation clear, consider the case of a stationary departure process from an M/G/1 queue in equilibrium. The theory may demonstrate that the process is actually not renewal [7]. However, in many applications we will be satisfied if the process is only approximately renewal. Of course, we can apply the standard statistical tests, but how should we interpret the results? With enough data, we would reject the renewal hypothesis. Obviously what matters is how the de-

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parture process is to be used. Often the departure process is of interest because it is an arrival process to another service facility. Then we want to know if the congestion at this next facility is approximately the same as if the departure process were a renewal process. We propose using tests that directly address this issue. However, if the next facility in the application is complicated, then the test queue might be more elementary. A different queue should still provide a useful test.

This work on testing the renewal and Poisson assumptions is related to previous work on approximating a point process by a renewal process in Albin [1-3], Kuczura [9], and Whitt [12-14]. In fact, the proposed approach is well illustrated by Albin [2] in which a queue is used to test whether a superposition process is nearly Poisson. In [12] test queuing models were also proposed to generate approximations. The object here in contrast is not to generate renewal process approximations but to evaluate the quality of renewal process approximations. Moreover, we do not evaluate a particular renewal process; we look for ways to determine whether any renewal process fits.

We illustrate the general approach with one specific test model: the G/M/1 queue. We define this model and the test procedure in Section 2. We apply the model to queues with superposition arrival processes and batch Poisson arrival processes in Sections 3 and 4. It remains for future research to investigate how useful tests can be for non-queuing applications.

2. The G/M/1 queue as a test model

The point process to be tested can be used as an arrival process in a single-server queue with infinite waiting room, the first-come first-served discipline and exponential service times. To show how this model can be used to test for the renewal property, we describe the GI/M/1 model having a renewal arrival process. Let λ be the arrival rate, μ the service rate, and $\rho = \lambda/\mu < 1$ the traffic intensity. Since the arrival process is renewal, the equilibrium distributions of the standard congestion measures (queue length and waiting time) depend on just two parameters: the traffic intensity ρ and the probability of delay σ (the probability that a customer will have to wait before beginning service), see Cohen [4, Section III.3]. The parameter σ is the unique root in the interval (0, 1) of the equation

$$\phi(\mu(1-\sigma)) = \sigma \tag{1}$$

where $\phi(s)$ is the Laplace–Stieltjes transform of the interarrival-time distribution.

Since all congestion measures, given ρ, depend on the single parameter σ, all the congestion measures are functions of one another. For example, the number in system, either at arrival epochs or at arbitrary times, has a geometric distribution with decay rate σ (with a different atom at 0 in the arbitrary time case). Hence, departures from the renewal property can be seen from departures from the geometric equilibrium distribution.

It is also possible to identify departures from the renewal property by examining the standard summary characteristics such as the probability of delay and the mean and squared coefficient of variation (variance divided by the square of the mean) of the number of customers in the system, say q and c^2:

$$q = \frac{\rho}{1-\sigma} \quad \text{and} \quad c^2 = \frac{1+\sigma-\rho}{\rho}; \tag{2}$$

see [4, Section III.3]. The squared coefficient of variation c^2 can be very informative by itself. Since $\sigma \in (0, 1)$,

$$c^2 \leqslant (2-\rho)/\rho. \tag{3}$$

For example $c^2 \leqslant 1.22$ for $\rho = 0.9$ and $c^2 \leqslant 1.86$ for $\rho = 0.7$. Larger observed values clearly indicate non-renewal behavior.

To perform the test, we estimate various summary congestion statistics. First, we estimate the arrival rate λ, perhaps by the total number of arrivals divided by the length of the time period, or equivalently, since the service rate μ is known, we estimate the traffic intensity ρ. Even for non-renewal (stationary) arrival processes, ρ is the long-run proportion of time that the server is busy; see Franken et al. [8, p. 107]. To specify other statistics, let $Q(t)$ be the number in system at time t and let Q_n be the number in system at the epoch of the nth arrival. Let $\hat{\sigma}$, $\hat{q}$, and $\hat{c}^2$ be the statistical estimates for σ, q, and c^2. Natural ones are:

$$\hat{\sigma} = n^{-1} \sum_{k=1}^{n} 1_{(Q_k > 0)} \tag{4}$$

where 1_A is the indicator function of the set A,

$$\hat{q} = T^{-1} \int_0^T Q(t)\,\mathrm{d}t, \tag{5}$$

and

$$\hat{c}^2 + 1 = \frac{T^{-1}\int_0^T Q(t)^2\,dt}{\hat{q}^2}. \quad (6)$$

With these statistics, the procedure, is to estimate $\hat{\sigma}$, $\hat{q}$, and $\hat{c}^2$ directly via (4), (5), and (6); then use (2) to obtain indirect estimates of q and c^2 from $\hat{\sigma}$; finally compare the direct and indirect estimates. If the predicted values for q and c^2 based on $\hat{\sigma}$ are close to the observed values $\hat{q}$ and $\hat{c}^2$, then we regard the arrival process as nearly renewal. If, in addition, $\hat{\sigma}$ is close to ρ, then we regard it as nearly Poisson. Of course, to known 'how close is close' we need some information about the distributions of the statistics, e.g., estimates of standard deviations and confidence intervals. In fact, to properly interpret the statistics, we need to understand the statistical reliability of the indirect estimates and the difference between the indirect and direct estimates. These can of course be estimated as part of the simulation. However, even if we can conclude that with very high probability the observed values could not have come from a renewal process, we still need to interpret the discrepancies. The needs of the application determine whether the errors produced by a renewal process approximation would be excessive. The test can reveal the quality possible for a renewal process approximation.

3. Superposition processes

In this section we apply the G/M/1 test to superpositions of independent and identically distributed stationary renewal processes, which are known to be non-renewal unless all component processes are Poisson. We use simulation results for the resulting $\Sigma GI_i/M/1$ queue in Albin [1, Chapter 4 and Appendix 14]. In each case there are n component processes each with rate n^{-1}. We consider $n = 2$ and 16. The traffic intensity ρ is either 0.7 and 0.9. There are three interarrival-time distributions for the component processes: Erlang (E_2: the convolution of two exponential distributions), hyperexponential (H_2: the mixture of two exponential distributions) and lognormal. An Erlang distribution has squared coefficient of variation 0.5. The hyperexponential and lognormal distributions are given squared coefficient of variation 5.0. The third parameter of the hyperexponential distribution is specified by assuming balanced means: $p_1/\lambda_1 = p_2/\lambda_2$, where p_i is the mixing probability attached to the exponential distribution with mean λ_i^{-1}.

Albin's simulation program was written in FORTRAN and used the Super-Duper program for generating uniform random numbers [10]. A different random number seed was used for each simulation. The simulations began with an empty system and 1000 customer were processed before data collection began to allow the system to approach steady-state. Each simulation consists of 20 batches with 15000 customers per batch for $\rho = 0.7$ and 50000 for $\rho = 0.9$. The simulation estimate is the average of the observed values X_j:

$$\bar{X} = \sum_{j=1}^{20} X_j/20 \quad (7)$$

and the sample standard deviation is

$$\mathrm{SD}(\bar{X}) = \left(\sum_{j=1}^{20} (X_j - \bar{X})^2/380\right)^{1/2}. \quad (8)$$

The results for the twelve cases ($2 \times 2 \times 3$) are given in Table 1. Departures from the renewal property are evident. Since we are using Albin's previous simulation results, we do not have standard deviation estimates for the predicted values and the difference between the observed and predicted values. However, the statistical reliability of the observed values is very good, so that we can be confident that the larger differences between the observed and predicted values reflect departures from the renewal property.

As should be expected, the discrepancy is much greater for the hyperexponential and lognormal distributions than the Erlang distributions, indicating that the Erlang renewal process is 'closer' to a Poisson process for which superpositions are Poisson and thus renewal. (Compare the squared coefficients of variation.) For the Erlang case, the errors range from 3–13 percent, so a single renewal process approximation might be reasonable there. The Erlang observed values are also smaller than predicted, suggesting negative correlations in the superposition process and indicating that a renewal process approximation should be conservative in a similar queuing context.

The departure from the renewal property is more significant for the hyperexponential and lognormal cases. Moreover, the observed values

Table 1
A comparison of observed queue characteristics for superposition arrival processes (the $\Sigma GI_i/M/1$ model) with predictions based on the renewal property

Arrival process			Probability of delay σ	Mean number in the system q		Squared coefficient of variation of number in system c^2	
Distribution	ρ	n	Observed	Observed	Predicted	Observed	Predicted
Erlang E_2 $c_a^2 = 0.5$	0.7	2	0.636 (0.002)	1.85 (0.01)	1.92 –	1.28 (0.11)	1.33 –
	0.7	16	0.674 (0.002)	2.00 (0.01)	2.14 –	1.24 (0.09)	1.39 –
	0.9	2	0.872 (0.001)	6.68 (0.08)	7.03 –	1.02 (1.61)	1.08 –
	0.9	16	0.890 (0.001)	7.27 (0.11)	8.18 –	1.04 (2.79)	1.10 –
Hyperexponential $c_a^2 = 5.0$ balanced means	0.7	2	0.812 (0.002)	4.69 (0.07)	3.72 –	1.87 (1.54)	1.59 –
	0.7	16	0.732 (0.003)	3.25 (0.09)	2.61 –	2.08 (1.70)	1.47 –
	0.9	2	0.948 (0.001)	24.05 (0.46)	17.31 –	1.23 (42.30)	1.16 –
	0.9	16	0.913 (0.002)	18.87 (0.47)	10.34 –	1.54 (28.63)	1.13 –
Lognormal $c_a^2 = 5.0$	0.7	2	0.800 (0.002)	4.16 (0.07)	3.50 –	1.78 (1.17)	1.57 –
	0.7	16	0.727 (0.002)	3.05 (0.04)	2.56 –	1.87 (0.66)	1.47 –
	0.9	2	0.942 (0.001)	21.00 (0.60)	15.51 –	1.30 (42.67)	1.16 –
	0.9	16	0.914 (0.002)	17.20 (0.42)	10.47 –	1.51 (27.96)	1.13 –

[1] In each case there are n independent and identically distributed component renewal processes each with rate n^{-1} and the indicated interarrival-time distribution.
[2] The sample standard deviations in the simulation are given in parentheses below the observed values. For c^2, the sample standard deviations given are for the estimates of the variance of the number in the system.
[3] The predicted values are obtained from the estimated probability of delay $\hat{\sigma}$ in (4) and the exact formulas (2).
[4] c_a^2 denotes the squared coefficient of variation of the interarrival-time distribution in a component renewal process.

are larger than predicted suggesting positive correlations and indicating that a renewal approximation might underestimate congestion.

For the mean number in system, q, it is apparent that the departure from the renewal property increases in both n and ρ. For c^2, the error increases in n, but ρ does not seem to matter.

4. Batch point processes

We now consider a point process for which we can evaluate the quality of the G/M/1 queue as a test of the renewal property. In particular, we consider a batch Poisson process in which points occur in iid batches of random size B. A batch Poisson process is a renewal process if and only if the batch size distribution is geometric, i.e., if

$$P(B=k)=(1-p)p^{k-1}, \quad k=1,2,\ldots \tag{9}$$

so that $EB = 1/p$ and $c_B^2 = 1-p$.

We focus on batch Poisson processes because it is easy to solve the $M^B/M/1$ queue obtained by using the batch Poisson process as the arrival process. For this special case we can analytically determine the effect of departures from the renewal property. Thus this system may serve as a useful theoretical reference point.

As with any stationary arrival process, the server is busy at an arbitrary time with probability $\rho = \lambda EB/\mu$, where λ is the arrival rate of batches. Since Poisson arrivals see time averages [15], the first customer in a batch thus finds the server busy with probability ρ; all others find the server busy with probability one. The proportion of customers that are first in a batch is $1/EB$. Hence, the probability of delay for an arbitrary customer, σ, is just

$$\sigma = 1 - (1 - \rho)/EB, \tag{10}$$

which depends on the batch-size distribution only through its mean. From Cooper [5, Section 5.10] and Little's formula, the mean number of customers in the system is

$$q_1 = \frac{(EB)\rho}{1-\rho}\left(1 + \frac{\delta}{2}\right), \tag{11}$$

where

$$\delta = c_B^2 - (EB - 1)/EB; \tag{12}$$

i.e., δ is the difference between the actual c_B^2 and what it would be with geometric batches.

If we know ρ, σ and q, for an $M^B/M/1$ queue, then q provides the only indication of a departure from the renewal property: We can determine if c_B^2 is consistent with a geometric distribution, i.e., if $c_B^2 = (EB - 1)/EB$. Other batch-size distributions with the same c_B^2 cannot be detected with these statistics. When $c_B^2 \neq (EB - 1)/EB$, then δ in (12) provides a quantitative measure of departure from the renewal property. This example can be used to calibrate departures from the renewal property in more general batchy point processes. We can relate an observed departure from the renewal property with the value of c_B^2 or δ required to produce an equal change in the $M^B/M/1$ queue.

5. Concluding remarks

We have shown how the G/M/1 queue can be used to test whether a point process can be represented as a renewal process. In some cases the analysis clearly demonstrates that no renewal process can adequately represent the point process as an arrival process to the queue. In particular, the superposition arrival processes having many components studied by Albin [2] that were found to be not approximately Poisson are also not approximately renewal. It may seem paradoxical, therefore, to consider renewal process approximations for queues with such point processes as arrival processes. Since a single renewal process does not fit, we use different renewal processes to approximate different characteristics of the queue at different traffic intensities. For example, in [1] and [3] three different renewal processes are used to approximate the probability of delay, the mean number in system and the standard deviation of the number in system. Moreover, for the last two characteristics, different renewal processes are used for different traffic intensities. In this way, renewal process approximations can be useful even if the process of interest is significantly different from a renewal process. On the other hand, it is desirable to know if a single renewal process can be used as an approximation in a wide range of circumstances.

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