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QUEUES WITH SERVER VACATIONS AND LÉVY PROCESSES WITH SECONDARY JUMP INPUT

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Motivated by models of queues with server vacations, we consider a Lévy process modified to have random jumps at arbitrary stopping times. The extra jumps can counteract a drift in the Lévy process so that the overall Lévy process with secondary jump input can have a proper limiting distribution. For example, the workload process in an $M/G/1$ queue with a server vacation each time the server finds an empty system is such a Lévy process with secondary jump input. We show that a certain functional of a Lévy process with secondary jump input is a martingale and we apply this martingale to characterize the steady-state distribution. We establish stochastic decomposition results for the case in which the Lévy process has no negative jumps, which extend and unify previous decomposition results for the workload process in the $M/G/1$ queue with server vacations and Brownian motion with secondary jump input. We also apply martingales to provide a new proof of the known simple form of the steady-state distribution of the associated reflected Lévy process when the Lévy process has no negative jumps (the generalized Pollaczek–Khinchine formula).

1. Introduction. We consider a Lévy process modified to have random jumps at arbitrary stopping times. We consider this Lévy process with secondary jump input primarily because we want to extend known decomposition results for the $M/G/1$ queue with server vacations [Fuhrmann and Cooper (1985), Shanthikumar (1988, 1989), Doshi (1990a)] and jump-diffusion processes [Kella and Whitt (1990)]. These decomposition results express the steady-state distribution as the convolution of other component distributions. For surveys of vacation queueing models, see Doshi (1986, 1990b), Takagi (1987) and Teghem (1986). For background on Lévy processes, see Chapter 14 of Breiman (1968), Chapter 9 of Feller (1971), Bingham (1975) and Chapter 3 of Prabhu (1980).

The Lévy processes with secondary jump inputs, which we refer to as JLPs (defined in Sections 2 and 3), arise in these queueing vacation models in three different ways: First, the workload or virtual-waiting-time process in an $M/G/1$ queue in which the server takes a vacation each time it finds an empty system is a JLP, that is, the net input of work is a Lévy process without negative jumps (a compound Poisson process minus t) modified to have

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positive random jumps (the vacation times). Second, following Doshi (1990a), if we restrict attention to (condition upon) times at which the server is busy, then the workload process in the $M/G/1$ vacation model is a JLP. (Then, as in this paper, the jumps are not necessarily nonnegative.) Finally, as shown in Kella and Whitt (1990), special JLPs called jump-diffusion processes arise as heavy-traffic limits of (and thus approximations for) general queues with server vacations. Other JLPs may also serve as useful models for queues and related storage systems with service interruptions.

As a basis for proving our decomposition results, we prove that certain functionals associated with the Lévy process, the reflected Lévy process (RLP), the Lévy process with secondary jump input (JLP) and the reflected process associated with a Lévy process with secondary jump input (RJLP) are martingales. [Even for the $M/G/1$ queue, this martingale approach seems to be new; see Brémaud (1981), Rosenkrantz (1983) and Baccelli and Makowski (1989a, b) for other martingale results for queues. The martingale results here are analogous to previous level crossing arguments for vacation models; e.g., see Doshi (1990a) and Shanthikumar (1989).] Together with simple regenerative arguments, the first two martingales provide short proofs establishing the known simple form of the steady-state distribution of the RLP when the Lévy process has no negative jumps, that is, the generalized Pollaczek–Khinchine formula; see Section 4. See Zolotarev (1964), Bingham (1975) and Harrison (1977) for previous proofs.

In Sections 5 and 6, we characterize the steady-state distributions of JLPs and RJLPs. Under the assumption that the Lévy process has no negative jumps, we establish stochastic decompositions for the JLP and the RJLP. For example, under certain conditions, the steady-state distribution of the JLP is a convolution of three distributions: the steady-state distribution of the RLP, the stationary forward-recurrence-time distribution of the jump size and the steady-state distribution of the state of the JLP right before (not quite, see details later) a jump.

2. The Lévy process. Our basic stochastic process $X \equiv \{X_t | t \geq 0\}$ is a real-valued stochastic process with $X(0) = 0$ defined on an underlying probability space $(\Omega, \mathscr{F}, P)$ endowed with a standard filtration $\{\mathscr{F}_t | t \geq 0\}$, that is, $\{\mathscr{F}_t | t \geq 0\}$ is an increasing right-continuous family of complete sub-σ-fields of $\mathscr{F}$. We assume that X is a Lévy process with respect to the filtration $\{\mathscr{F}_t | t \geq 0\}$, that is, X_t is adapted to $\mathscr{F}_t$ and $X_u - X_t$ is independent of $\mathscr{F}_t$ and distributed as X_{u-t} for $0 \leq t < u$. Moreover, we assume that the sample paths of X are right-continuous with left limits, so that X is strong Markov. The one-dimensional marginal distributions are infinitely divisible, that is, X_t has characteristic function (cf)

$$Ee^{i\alpha X_t} = e^{\phi(\alpha)t}, \qquad t \geq 0, \tag{2.1}$$

where $\phi(\alpha)$ is the characteristic exponent; see, for example, Bingham [(1975), page 706].

The Lévy process X can be represented as the independent sum of a Brownian motion and another Lévy process $\tilde{X}$. If X has no negative jumps and the paths of $\tilde{X}$ are of bounded variation, then without loss of generality, $\tilde{X}$ can be a subordinator (a Lévy process with nondecreasing sample paths). The subordinator in turn can be represented as a nonnegative compound Poisson process or as the limit of a sequence of nonnegative compound Poisson processes; see Feller [(1971), page 303]. The process depicting the net input of work in an $M/G/1$ queue is a Lévy process without negative jumps, having a degenerate Brownian motion component (with drift coefficient -1 and diffusion coefficient 0) and a subordinator which is a compound Poisson process with Poisson rate equal to the arrival rate and jumps equal to the service times.

We conclude this section by identifying a martingale associated with X that we will apply; it is similar to the familiar Wald martingale $W_t = \exp\{i\alpha X_t - \phi(\alpha)t\}$, $t \geq 0$ [see page 7 of Harrison (1985) and page 243 of Karlin and Taylor (1975)]. In particular, let

$$Z_t = \phi(\alpha)\int_0^t e^{i\alpha X_s}\,ds - e^{i\alpha X_t}, \qquad t \geq 0. \tag{2.2}$$

Since we work with cf's, we work with complex-valued martingales. As usual, if $z = u + iv$, then $|z| = (u^2 + v^2)^{1/2}$.

PROPOSITION 2.1. *For all real α, Z is a complex-valued martingale with respect to $\{\mathscr{F}_t | t \geq 0\}$.*

PROOF. First, suppose that $\phi(\alpha) \neq 0$. The finiteness of $E|Z_t|$ is a consequence of the finiteness of $\phi(\alpha)$ and Fubini's theorem. For $0 \leq s < t$,

$$\begin{aligned}\phi(\alpha)E\left[\int_s^t e^{i\alpha(X_u - X_s)}\,du \,|\, \mathscr{F}_s\right] &= e^{\phi(\alpha)(t-s)} - 1 \\ &= E\left[e^{i\alpha(X_t - X_s)} | \mathscr{F}_s\right] - 1 \quad \text{w.p.1},\end{aligned} \tag{2.3}$$

where the first equality follows from the independent increments property, Fubini's theorem, and integrating $e^{\phi(\alpha)(u-s)}$ from s to t. The result now follows by multiplying the left and right sides by $e^{i\alpha X_s}$ and adding $\phi(\alpha)\int_0^s e^{i\alpha X_s}\,du$ (both of which are $\mathscr{F}_s$-measurable). □

3. The Lévy process with secondary jump input (JLP). Let $\{T_n | n \geq 0\}$ be a strictly increasing sequence of stopping times with respect to the filtration $\{\mathscr{F}_t | t \geq 0\}$, with $T_0 = 0$. Let $\{N_t | t \geq 0\}$ be the associated counting process, that is,

$$N_t = \sup\{n | T_n \leq t\}, \qquad t \geq 0. \tag{3.1}$$

Let $\{U_n | n \geq 0\}$ be a sequence of random variables and assume that U_n is $\mathscr{F}_{T_n}$-measurable for $n \geq 0$. Then the Lévy process with secondary jump input

(JLP) is $\{Y_t|t \geq 0\}$, where

$$Y_t = X_t + \sum_{k=0}^{N_t} U_k, \qquad t \geq 0. \tag{3.2}$$

An example of interest is the special case in which X has no negative jumps, $U_n > 0$ for all n and

$$T_n = \inf\left\{t \geq 0 | X_t + \sum_{k=0}^{n-1} U_k = 0\right\}, \qquad n \geq 0, \tag{3.3}$$

but in general we do not restrict attention to this case.

The following is our main tool. The random variable $Y_{T_n} - U_n$ can be thought of as the value of Y just prior to the nth jump, but note that $Y_{T_n} - U_n = Y_{T_n-}$ only if X is continuous at T_n.

THEOREM 3.1. (a) *If $T_n \to \infty$ w.p.1 as $n \to \infty$, then $\{M_t|t \geq 0\}$ is a local martingale with respect to $\{\mathscr{F}_t|t \geq 0\}$ with localizing sequence $\{T_n\}$, where*

$$M_t \equiv \phi(\alpha)\int_0^t e^{i\alpha Y_s}\,ds + 1 - e^{i\alpha Y_t} - \sum_{k=0}^{N_t} \left(e^{i\alpha(Y_{T_k} - U_k)} - e^{i\alpha Y_{T_k}}\right), \qquad t \geq 0. \tag{3.4}$$

(b) *If, in addition, $EN_t < \infty$ for all t, then $\{M_t|t \geq 0\}$ in (3.4) is a zero-mean complex-valued martingale with respect to $\{\mathscr{F}_t|t \geq 0\}$.*

PROOF. (a) From (3.4), by considering the three cases $t \leq T_{n-1}$, $T_{n-1} < t \leq T_n$ and $t > T_n$, we see that

$$M_{T_n \wedge t} - M_{T_{n-1} \wedge t} = \left(Z_{T_n \wedge t} - Z_{T_{n-1} \wedge t}\right)e^{i\alpha \sum_{j=0}^{n-1} U_j}, \tag{3.5}$$

where $x \wedge y = \min\{x, y\}$. Since $\{Z_t|t \geq 0\}$ is a right-continuous martingale with respect to the standard filtration $\{F_t|t \geq 0\}$ by Proposition 2.1 and T_n is a stopping time, $\{Z_{T_n \wedge t}|t \geq 0\}$ is a martingale; see, for example, Karatzas and Shreve [(1988), page 20]. Moreover, since U_k is $\mathscr{F}_{T_k}$-measurable for all k and since $e^{i\alpha \sum_{j=0}^{k-1} U_j}$ is bounded, $\{M_{T_n \wedge t} - M_{T_{n-1} \wedge t}|t \geq 0\}$ and thus $\{M_{T_n \wedge t}|t \geq 0\}$ are martingales with respect to $\{\mathscr{F}_t|t \geq 0\}$. Since $T_n \to \infty$ w.p.1, $\{M_t|t \geq 0\}$ is a local martingale with localizing sequence $\{T_n\}$.

(b) From (a), we have

$$E\left(M_{T_n \wedge t}|\mathscr{F}_s\right) = M_{T_n \wedge s} \quad \text{w.p.1.} \tag{3.6}$$

Since $\sup_{0 \leq s \leq t}|M_s| \leq |\phi(\alpha)|t + 2(N(t) + 1)$ and $EN(t) < \infty$, the result follows from the dominated convergence theorem for conditional expectations [Chung (1974), page 301], letting $n \to \infty$ in (3.6). □

4. The reflected Lévy process (RLP). Let

$$I_t = -\inf_{0 \leq s \leq t} X_s \quad \text{and} \quad R_t = X_t + I_t, \qquad t \geq 0. \tag{4.1}$$

We call $R \equiv \{R_t|t \geq 0\}$ the *reflected* or *regulated* Lévy process (RLP) associated with the Lévy process X. The process I in (4.1) can also be defined as the minimal right-continuous nondecreasing process such that $X_t + I_t \geq 0$ for all t; then I increases only when $R = 0$ [see page 19 of Harrison (1985)].

We now characterize the RLP R for the special case in which the Lévy process X has no negative jumps as the limit of JLPs Y^a for which $U_n = a$ w.p.1 for all n. We first characterize the approximating JLPs. Let Y^a be the JLP associated with X, $U_n = a$ w.p.1 for all n and $T_n^a = \inf\{t \geq 0, X_t \leq -na\}$, $n \geq 1$, as in (3.3). Then $Y_t^a = X_t + I_t^a$, where $I_t^a = a(N_t^a + 1)$ and N_t^a is the renewal counting process associated with $\{T_n^a\}$. It is immediate that

$$0 \leq I_t^a - I_t = Y_t^a - R_t \leq a \quad \text{for all } t \text{ w.p.1.} \tag{4.2}$$

LEMMA 4.1. *If X has no negative jumps, then $EI_t^a < \infty$ and*

$$M_t^a = \phi(\alpha)\int_0^t e^{i\alpha Y_s^a}\,ds + 1 - e^{i\alpha Y_t^a} - \alpha I_t^a \frac{(1 - e^{i\alpha a})}{\alpha a} \tag{4.3}$$

is a zero-mean complex-valued martingale with respect to $\{\mathscr{F}_t | t \geq 0\}$.

PROOF. We first show that $EI_t^a < \infty$. Since X has no negative jumps, $\{T_n^a | n \geq 1\}$ is a random walk with $T_n^a < T_{n+1}^a$ w.p.1 and N^a is the associated renewal counting process. Hence, $EN_t^a < \infty$; see Karlin and Taylor [(1975), page 182]. By (4.2), $EI_t \leq EI_t^a \leq a(EN_t^a + 1)$. Since $I_t^a = a(N_t^a + 1)$ and (4.2) holds, $EN_t^a < \infty$ for each $a > 0$ and $t > 0$. Hence, we can apply Theorem 3.1. □

From (4.2), we see that $M_t^a \to M_t^0$ as $a \to 0$ uniformly on $\Omega \times [0, t_0]$ for all $t_0 > 0$, where M_t^a is in (4.3) and

$$M_t^0 = \phi(\alpha)\int_0^t e^{i\alpha R_s}\,ds + 1 - e^{i\alpha R_s} + i\alpha I_t, \qquad t \geq 0. \tag{4.4}$$

As an immediate consequence, we obtain the martingale property for M^0.

THEOREM 4.1. *If X has no negative jumps, then M_t^0 in (4.4) is a zero-mean complex-valued martingale with respect to $\{\mathscr{F}_t | t \geq 0\}$.*

PROOF. By dominated convergence for conditional expectations, $E(M_t^a | \mathscr{F}_s) \to E(M_t^0 | \mathscr{F}_s)$ w.p.1 as $a \to 0$ for $0 \leq s < t$. However, $E(M_t^a | \mathscr{F}_s) = M_s^a \to M_s^0$ w.p.1 as $a \to 0$ by Lemma 4.1 and the convergence previously noted. □

REMARK 4.1. When X is Brownian motion (and even more generally), Proposition 2.1 and Theorem 4.1 can be obtained from Itô's lemma, while Theorem 3.1 and Lemma 4.1 can be obtained from a generalized form of Itô's lemma; see Kella and Whitt (1990), Harrison [(1985), page 71] and Méyer [(1976), page 301].

We now give our first new proof of the generalized Pollaczek–Khinchine formula. For this purpose we use the following elementary lemma. It is also well known; for example, it is a consequence of Proposition 2 in Bingham [(1975), page 721].

LEMMA 4.2. *If X has no negative jumps, $E|X_t| < \infty$ and $EX_t < 0$, then* $ET_1^a = -a/EX_1 = -ia/\phi'(0)$.

PROOF. Since X has no negative jumps, $\{T_n^a | n \geq 0\}$ is a random walk with $0 < T_n^a < T_{n+1}^a$ w.p.1. Hence, $ET_n^a = nET_1^a$. Since $E|X_t| < \infty$, $t^{-1}X_t \to EX_1$ w.p.1 as $t \to \infty$ by the strong law of large numbers. Since $EX_t < 0$, $ET_1^a < \infty$, by Theorem 8.4.4 of Chung (1974). Finally,

$$-a = n^{-1}X_{T_n^a} = \left(n^{-1}T_n^a\right)\left(T_n^a\right)^{-1}X_{T_n^a} \to ET_1^a EX_1 \quad \text{w.p.1 as } n \to \infty,$$

so that $ET_1^a = -a/EX_1$. □

THEOREM 4.2 (Generalized Pollaczek–Khinchine formula). *If X is a Lévy process without negative jumps such that $E|X_t| < \infty$ and $EX_t < 0$, then*

$$\lim_{t\to\infty} Ee^{i\alpha R_t} = \frac{\alpha\phi'(0)}{\phi(\alpha)} \quad \text{for } \alpha \neq 0.$$

PROOF. Note that $\{e^{i\alpha R_t} | t \geq 0\}$ is a bounded aperiodic regenerative process with respect to $\{T_n^a | N \geq 0\}$, where $T_n^a = \inf\{t | X_t \leq -na\}$. (Obviously, $R_{T_n^a} = 0$.) By Lemma 4.2, $ET_1^a < \infty$. Hence,

$$\lim_{t\to\infty} Ee^{i\alpha R_t} = E\int_0^{T_1^a} e^{i\alpha R_s}\, ds / ET_1^a, \tag{4.5}$$

where $ET_1^a = -ai/\phi'(0)$ by Lemma 4.2. Finally, the main result follows from Theorem 4.1 and Doob's optional sampling theorem. First, we have $EM_{T_1^a \wedge t} = 0$ for all t. To justify the interchange of the limit as $t \to \infty$ and the expectation, note that the first term of (4.4) is dominated by t, the second and third terms are bounded and I_t in the fourth term is nondecreasing in t. Finally, note that $\phi(\alpha) = 0$ only for $\alpha = 0$ because, by (4.4),

$$\phi(\alpha) E\int_0^{T_1^a} e^{i\alpha R_s}\, ds = -i\alpha a,$$

which is not 0 for $\alpha \neq 0$. □

REMARK 4.2. We need X to have no negative jumps (in addition to the other assumptions) in order to have $ET_1^a = -ai/\phi'(0)$ in Lemma 4.2 and the proof of Theorem 4.2.

Of course, one may object to the statement that this proof is short since it essentially relies on Theorem 3.1, whose proof is not so short. Therefore we give another proof which depends only on Proposition 2.1. At the same time, we extend previous results about queueing systems with server vacations. (We will obtain a more general extension in Section 5.)

THEOREM 4.3. *Let X be a Lévy process without negative jumps for which $E|X_t| < \infty$ and $EX_t < 0$. Let $\{U_n|n \geq 0\}$ be a positive i.i.d. sequence with $EU_0 < \infty$. Let $\{T_n|n \geq 0\}$ be defined as in* (3.3). *If either X is not deterministic or if the distribution of U_0 is aperiodic, then*

$$\lim_{t\to\infty} Ee^{i\alpha Y_s} = \frac{\alpha\phi'(0)}{\phi(\alpha)} \frac{i(1 - Ee^{i\alpha U_0})}{\alpha EU_0}. \tag{4.6}$$

PROOF. As in the proof of Theorem 4.2, $\{Y_t|t \geq 0\}$ is regenerative with respect to $\{T_n|n \geq 0\}$. Since X is not deterministic or U_0 is aperiodic, the regenerative process is aperiodic. Hence,

$$\lim_{t\to\infty} Ee^{i\alpha Y_s} = \frac{1}{ET_1} E\int_0^{T_1} e^{i\alpha Y_s}\, ds. \tag{4.7}$$

Since $ET_1 = -iEU_0/\phi'(0)$ (again use Lemma 4.2, after conditioning on U_0), the result follows by multiplying the martingale of Proposition 2.1 by $e^{i\alpha U_0}$ and applying the optional sampling theorem. This with (4.7) yields (4.6) for all α such that $\phi(\alpha) \neq 0$. Since $P(X_t = 0) < 1$, we cannot have $\phi(\alpha) = 0$ for all α. Indeed, we have $\phi(\alpha) = 0$ for some $\alpha \neq 0$ if and only if X_t has a lattice distribution, see Chung [(1974), page 174], which is not possible for a Lévy process without negative jumps and $EX_t < 0$. □

REMARK 4.3. By Theorem 4.2, $\alpha\phi'(0)/\phi(\alpha)$ in (4.6) is the cf of the limiting distribution of the RLP. The other term $i(1 - Ee^{i\alpha U_0})/\alpha EU_0$ is the cf of the stationary forward-recurrence-time distribution of U_0, which has density $P(U_0 > x)/EU_0$. Hence, Y_t converges in distribution to the *convolution* of those two component distributions and thus the limiting distribution of Y_t has a *stochastic decomposition*. Theorem 4.3 extends previous results for the virtual waiting time process in the $M/G/1$ queue with multiple server vacations [(5.6) of Doshi (1990a), Cooper (1970), Fuhrmann and Cooper (1985) and Lévy and Yechiali (1975)] and Brownian motion with jumps [Theorem 2.2 of Kella and Whitt (1990)].

SECOND PROOF OF THEOREM 4.2. The special case of Theorem 4.3 in which $U_n = a > 0$ for all $n \geq 1$ results in the process $\{Y_t^a|t \geq 0\}$ Lemma 4.1. By (4.2),

$$|Ee^{i\alpha Y_t^a} - Ee^{i\alpha R_t}| \leq E|e^{i\alpha Y_t^a} - e^{i\alpha R_t}| \leq 2\alpha|Y_t^a - R_t| \leq 2\alpha a. \tag{4.8}$$

Now the result is obtained from (4.8) by taking expectations, letting $t \to \infty$ and then letting $a \to \infty$. The form of the limit is obtained by letting $U_0 \equiv a \to 0$ in (4.6). □

This is the quickest way that we know to establish Theorem 4.2.

5. The steady-state distribution of the JLP. We now characterize the limiting distribution of the JLP Y in the general framework of Section 3. Let $\to_d$ denote convergence in distribution and let $\to_p$ denote convergence in probability.

THEOREM 5.1. *Suppose that*

(i) $P(X_t = 0) \neq 1$,
(ii) $n^{-1}T_n \to_p \lambda^{-1}$ *as* $n \to \infty$ *for* $0 < \lambda^{-1} < \infty$,
(iii) $Y_t \to_d Y$ *as* $t \to \infty$,
(iv) $n^{-1}\sum_{k=0}^{n-1} e^{i\alpha Y_{T_k}} \to_p Ee^{i\alpha Y^+}$ *and* $n^{-1}\sum_{k=0}^{n-1} e^{i\alpha(Y_{T_k}-U_k)} \to_p Ee^{i\alpha Y^-}$ *as* $n \to \infty$ *for random variables* Y^+ *and* Y^- *with* $EY^+ \neq EY^-$,
(v) $\{t^{-1}N_t | t \geq 0\}$ *is uniformly integrable.*

Then necessarily $\lambda = i\phi'(0)/(EY^+ - EY^-)$ *and*

$$\lim_{t\to\infty} Ee^{i\alpha Y_t} = Ee^{i\alpha Y} = \frac{\alpha\phi'(0)}{\phi(\alpha)} \frac{i(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+})}{\alpha(EY^+ - EY^-)}. \tag{5.1}$$

REMARK 5.1. Formula (5.1) is not well defined if $\phi(\alpha) = 0$. By condition (i), we do not have $\phi(\alpha) = 0$ for all α. As noted in the proof of Theorem 4.3, $\phi(\alpha) = 0$ for $\alpha \neq 0$ if and only if X_t has a lattice distribution. Then any α such that $\phi(\alpha) = 0$ is an isolated point. For such α, we understand (5.1) to be defined by taking a limit on α, which is well defined since Y has a bona fide cf by (iii).

REMARK 5.2. If X has no negative jumps with $E|X_t| < \infty$ and $EX_t < 0$, then the term $\alpha\phi'(0)/\phi(\alpha)$ in (5.1) is the cf for the RLP in Section 4. The second term in (5.1) is considered in Theorem 5.2.

REMARK 5.3. A natural sufficient condition for condition (iv) in Theorem 5.1 is to have $\{Y_{T_k} | k \geq 0\}$ and $\{Y_{T_k} - U_k | k \geq 0\}$ be stationary and ergodic. Then the averages in (iv) converge w.p.1 to $Ee^{i\alpha Y_{T_0}}$ and $Ee^{i\alpha(Y_{T_0}-U_0)}$ as $k \to \infty$, respectively; see Karlin and Taylor [(1975), page 488]. Then Y^+ is distributed as Y_{T_0} and Y^- is distributed as $Y_{t_0} - U_0$. Another way to obtain w.p.1 convergence in (iv) is to have regenerative structure.

REMARK 5.4. Condition (v) is always satisfied if $\{N_t | t \geq 0\}$ is a renewal counting process; see Chung [(1974), page 136].

REMARK 5.5. Formula (5.1) generalizes (3.7) of Doshi (1990a). It is also similar to equation (2) of Shanthikumar (1988) and equation (4) of Fuhrmann and Cooper (1985). However, they concentrated only on the $M/G/1$ queue and mostly on the queue size, rather than the workload process. Now we see that there is one more good reason for Fuhrmann and Cooper's assumption (7). The essence of this assumption is that the waiting time and the workload process (viewing vacations as work) are one and the same.

In the proof of Theorem 5.1 we use the following lemma.

LEMMA 5.1. *Suppose that* $n^{-1}\sum_{i=0}^{n-1} W_i \to_p m$ *for random variables* W_i *with* $|W_i| < K < \infty$ *for all* i *and* $t^{-1}N_t \to_p \lambda$, $0 < \lambda < \infty$, *for a counting process* N_t.

Then

$$t^{-1}\sum_{i=0}^{N_t} X_i \to_p \lambda m.$$

PROOF. Since X_i is bounded, $N_t^{-1}\Sigma_{i=0}^{N_t} X_i$ is contained in a compact subset for every t w.p.1. Hence, every subsequence has a sub-subsequence converging w.p.1. Since $n^{-1}\Sigma_{i=0}^{n-1} X_i \to_p m$, the limit of this w.p.1 convergent sub-subsequence must be m. Hence, $N_t^{-1}\Sigma_{i=0}^{N_t} X_i \to_p m$; for example, see Chung [(1974), Problem 7, page 75]. By assumption and Theorem 4.4 of Billingsley (1968),

$$\left(t^{-1}N_t,\, N_t^{-1}\sum_{i=0}^{N_t} X_i\right) \to_p (\lambda, m).$$

The proof is completed by applying the continuous mapping theorem with multiplication; see Theorem 5.1 of Billingsley (1968). □

PROOF OF THEOREM 5.1. (a) We apply Theorem 3.1(a). By condition (ii), $T_n \to \infty$ w.p.1 as $n \to \infty$, as required there. Also (ii) implies that $t^{-1}N_t \to_p \lambda$ as $t \to \infty$; for example, see Theorem 3 of Glynn and Whitt (1988). Together with (iv) and Lemma 5.1, this implies that

$$(5.2)\quad t^{-1}\sum_{k=0}^{N_t}\left(e^{i\alpha(Y_{T_k}-U_k)} - e^{i\alpha Y_{T_k}}\right) \to_p \lambda\left(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+}\right) \quad \text{as } t \to \infty.$$

Together with (v), (5.2) implies that

$$t^{-1}E\sum_{k=0}^{N_t}\left(e^{i\alpha Y_{T_k}-U_k} - e^{i\alpha Y_{T_k}}\right) \to \lambda\left(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+}\right) \quad \text{as } t \to \infty.$$

By (iii), $Ee^{i\alpha Y_t} \to Ee^{i\alpha Y}$ as $t \to \infty$. From Theorem 3.1(a), after dividing by t and letting $t \to \infty$, we conclude that

$$(5.3)\qquad \phi(\alpha)Ee^{i\alpha Y} = \lambda\left(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+}\right),$$

because $EM_t = 0$ for all t for M_t in (3.4). We divide by $\phi(\alpha)$ in (5.3) for α such that $\phi(\alpha) \neq 0$. For α such that $\phi(\alpha) = 0$, we take a limit, as indicated in Remark 5.1. Finally differentiating with respect to α in (5.3) and setting $\alpha = 0$ gives the expression for λ. By condition (iv), $EY^+ \neq EY^-$, so we can divide by $(EY^+ - EY^-)$. □

We now consider the second term in (5.1). Following Shanthikumar (1988) and Doshi (1990a, Section 4), we provide necessary and sufficient conditions for the second term to be a bona fide cf and sufficient conditions for it to be the product of two cf's (so that we have a further stochastic decomposition).

THEOREM 5.2. (a) *The term* $i(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+})/\alpha(EY^+ - EY^-)$ *in* (5.1) *is the* cf *of a bona fide probability distribution if and only if* $Y^- \leq_{st} Y^+$.

(b) *Suppose that the assumptions of Theorem* 5.1 *are satisfied. If* $\{U_n | n \geq 0\}$ *is a nonnegative i.i.d. sequence with* U_n *independent of* X_{T_n} *and* $1_{\{N_t \geq n\}}$, *then*

$$\frac{i(Ee^{i\alpha Y^-} - Ee^{i\alpha Y^+})}{\alpha(EY^+ - EY^-)} = \frac{i(1 - Ee^{i\alpha U_0})}{\alpha EU_0} Ee^{i\alpha Y^-}. \tag{5.4}$$

PROOF. (a) Note that this term is the Fourier transform $\int_{-\infty}^{\infty} e^{i\alpha y} f(y)\, dy$ of the function

$$f(y) = \frac{P(Y^+ > y) - P(Y^- > y)}{EY^+ - EY^-}, \tag{5.5}$$

which is a bona fide probability density if and only if $P(Y^+ > y) \geq P(Y^- > y)$ for all y, that is, if and only if $Y^- \leq_{st} Y^+$. For this last step, recall that

$$E \max\{0, Y\} = \int_0^{\infty} P(Y \geq y)\, dy$$

and

$$\begin{aligned} E \min\{0, Y\} &= -E \max\{0, -Y\} \\ &= -\int_0^{\infty} P(-Y \geq y)\, dy = -\int_{-\infty}^{0} P(Y \leq y)\, dy. \end{aligned}$$

(b) Since U_n is independent of $Y_{T_n} - U_n = X_{T_n} + \sum_{i=0}^{n-1} U_i$ and the event $\{N_t \geq n\}$, the monotone convergence theorem yields

$$\begin{aligned} E \sum_{k=0}^{N_t} \left(e^{i\alpha(Y_{T_k} - U_k)} - e^{i\alpha Y_{T_k}}\right) &= E \sum_{k=0}^{\infty} e^{i\alpha(Y_{T_k} - U_k)}(1 - e^{i\alpha U_k}) 1_{\{N_t \geq k\}} \\ &= \sum_{k=0}^{\infty} Ee^{i\alpha(Y_{T_k} - U_k)} 1_{\{N_t \geq k\}}(1 - Ee^{i\alpha U_0}) \\ &= E \sum_{k=0}^{N_t} e^{i\alpha(Y_{T_k} - U_k)}(1 - Ee^{i\alpha U_0}). \end{aligned} \tag{5.6}$$

Hence, instead of (5.3), we now have

$$\phi(\alpha) Ee^{i\alpha Y} = \lambda E(e^{i\alpha Y^-})(1 - Ee^{i\alpha U_0}). \tag{5.7}$$

Differentiating with respect to α in (5.7), we obtain $\lambda = i\phi'(0)/EU_0$. Substituting this in (5.7) gives (5.4). □

Combining Theorems 5.1 and 5.2, we obtain the following corollary.

COROLLARY 5.1. *If, in addition to the conditions of Theorems* 5.1 *and* 5.2(a), X *has no negative jumps with* $E|X_t| < \infty$ *and* $EX_t < 0$, *then the distribution of* Y *is the convolution of two distributions one of which is the distribution of* R. *If, in addition, the assumptions of Theorem* 5.2(b) *hold, then* Y *is the convolution of three distributions: the distributions of* R *and* Y^- *and the stationary forward-recurrence-time distribution associated with* U_0.

REMARK 5.6. Under the conditions of Theorems 5.1 and 5.2, but without the condition on X in Corollary 5.1, we do *not* necessarily obtain a valid stochastic decomposition. To see this, suppose that $-X$ is a Poisson process. Then X_t has a lattice distribution, so that $\phi(2\pi) = 0$. Hence, $\alpha\phi'(0)/\phi(\alpha)$ is not a bona fide cf. Then (5.1) can be defined for $\alpha = 2\pi$ by taking a limit as $\alpha \to 2\pi$.

REMARK 5.7. Theorem 5.2 and Corollary 5.1 are closely related to Sections 4 and 5 of Doshi (1990a). Equation (5.4) is also similar to equation (3) of Fuhrmann and Cooper (1985).

REMARK 5.8. Note that Theorem 4.3 is a simple consequence of Theorems 5.1 and 5.2, but we prefer the direct proof given before.

6. The reflected JLP. We conclude by considering a reflected JLP, which we refer to as a RJLP. Let Y be a JLP as defined in Section 3 and let $L_t = -\inf_{0\le s\le t} Y_s$, $t \ge 0$. Then the RJLP is $R_t^0 = Y_t + L_t$, $t \ge 0$. Here we require that the underlying Lévy process X have no negative jumps.

As in Section 4, we consider R^0 as the limit of associated JLPs with small positive jumps to keep it positive. By the argument of Lemma 4.1 and Theorem 4.1, we obtain the following result.

LEMMA 6.1. *Suppose that X has no negative jumps and $U_n \ge 0$ for all n w.p.1. Then $EL_t < \infty$ w.p.1.* (a) *If $N_t < \infty$ w.p.1, then $\{M_t | t \ge 0\}$ is a local martingale with respect to $\{\mathscr{F}_t | t \ge 0\}$ with localizing sequence $\{T_n\}$, where*

$$(6.1)\qquad \begin{aligned} M_t \equiv{}& \phi(\alpha)\int_0^t e^{i\alpha R_s^0}\,ds + 1 - e^{i\alpha R_t^0} \\ &- \sum_{k=0}^{N_t} e^{i\alpha(R^0_{T_k}-U_k)} - e^{i\alpha R^0_{T_k}} + i\alpha L_t, \qquad t\ge 0. \end{aligned}$$

(b) *If $EN_t < \infty$ for all t, then $\{M_t | t \ge 0\}$ in* (6.1) *is a zero-mean complex-valued martingale with respect to $\{\mathscr{F}_t | t \ge 0\}$.*

PROOF. To see that $EL_t < \infty$, note that $L_t \le I_t$ for all t w.p.1; since $U_n \ge 0$ for all n w.p.1, $Y_t \ge X_t$ for all t w.p.1. ($EI_t < \infty$ by Lemma 4.1.) Paralleling the definition of Y^a in Section 4, let R^a be the JLP associated with Y that approximates the RJLP R^0; that is, let $T_n^a = \inf\{t \ge 0 | Y_t = -na\}$, $n \ge 1$, and $R_t^a = y_t + L_t^a$, where $L_t^a = a(N_t^a + 1)$ with N_t^a being the counting process associated with $\{T_n^a\}$. Then, as in (4.2), $L_t^a - L_t = R_t^a - R_t^0 \le a$ for all t w.p.1. Moreover, R_t^a is itself a bona fide JLP, so that we can apply Theorem 3.1 to it to obtain the analog of Lemma 4.1. The assumptions imply that $N_t + N_t^a < \infty$ w.p.1 in (a) as needed for Theorem 3.1(a) and that $EN_t + EN_t^a < \infty$ in (b) as needed for Theorem 3.1(b). Finally, let $a \to 0$ as in the proof of Theorem 4.1 to obtain the desired conclusion. □

We now apply Lemma 6.1 to characterize the limiting distribution of the RJLP.

THEOREM 6.1. *Suppose that X has no negative jumps,* $E|X_t| < \infty$, $EX_t < 0$, $U_n \geq 0$ *for all n w.p.1 and the conditions of Theorem* 5.1 *hold with* (iii) *replaced by*

$$\text{(iii)}' \qquad R_t^0 \to_d R^0 \quad \text{and} \quad t^{-1}ER_t^0 \to 0 \quad \text{as } t \to \infty$$

and (iv) *replaced by*

$$\text{(iv)}' \quad n^{-1}\sum_{k=0}^{n-1} e^{i\alpha R_{T_k}^0} \to_P Ee^{i\alpha R^+} \quad \text{and} \quad n^{-1}\sum_{k=0}^{n-1} e^{i\alpha(R_{T_k}^0 - U_k)} \to_P Ee^{i\alpha R^-}$$

as $n \to \infty$ *for random variables* R^+ *and* R^- *with* $ER^+ \neq ER^-$. *Then*

$$\lim_{t\to\infty} t^{-1}L_t = \pi|EX_1| \quad \text{and} \quad \lim_{t\to\infty} t^{-1}\sum_{i=0}^{N_t} U_i = (1-\pi)|EX_1|, \tag{6.2}$$

where $0 \leq \pi \leq 1$ *and*

$$Ee^{i\alpha R^0} = \frac{\alpha\phi'(0)}{\phi(\alpha)}\left[\frac{i(Ee^{i\alpha R^-} - Ee^{i\alpha R^+})(1-\pi)}{\alpha(ER^+ - ER^-)} + \pi\right]. \tag{6.3}$$

PROOF. By Lemma 6.1, $L_t < \infty$ for all t w.p.1. By condition (v) of Theorem 5.1, $N_t + L_t < \infty$ w.p.1, so that we can apply Lemma 6.1(a). Dividing (6.1) by t and taking expected values we know from the proof of Theorem 5.1 and (iii)′ that all terms converge except possibly for $t^{-1}i\alpha EL_t$. Hence, $t^{-1}EL_t$ converges too. Since

$$t^{-1}ER_t^0 = EX_1 + t^{-1}E\sum_{i=0}^{N_t} U_i + t^{-1}EL_t,$$

where $t^{-1}ER_t^0 \to 0$ by (iii)′,

$$\lim_{t\to\infty} Et^{-1}\sum_{j=0}^{N_t} U_j = -EX_1 - \lim_{t\to\infty} t^{-1}EL_t.$$

Hence, we have established (6.2). Formula (6.3) follows by the proof of Theorem 5.1, again differentiating with respect to α to determine λ. □

As in Section 5, Theorem 6.1 provides a stochastic decomposition.

COROLLARY 6.1. *Under the assumptions of Theorem* 6.1, (6.3) *holds with* $0 \leq \pi \leq 1$ *and there is a random variable V such that*

$$\frac{i(Ee^{i\alpha R^-} - Ee^{i\alpha R^+})}{\alpha(ER^+ - ER^-)} = Ee^{i\alpha V}. \tag{6.4}$$

Hence, R^0 *is distributed as the convolution of the distribution of R in Section* 4 *and another distribution. The second distribution is the mixture of a point* *mass at* 0 *with probability* π *and the distribution of* V *with probability* $1 - \pi$. *If, in addition, the assumptions of Theorem* 5.2(b) *hold, then the distribution of* V *is the convolution of the distribution of* R^- *and the stationary forward recurrence-time distribution of* U_0.

REMARK 6.1. Corollary 6.1 generalizes Theorem 3.3 of Kella and Whitt (1990).

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