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DISTRIBUTION-VALUED HEAVY-TRAFFIC LIMITS FOR THE $G/GI/\infty$ QUEUE

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We study the $G/GI/\infty$ queue in heavy-traffic using tempered distribution-valued processes which track the age and residual service time of each customer in the system. In both cases, we use the continuous mapping theorem together with functional central limit theorem results in order to obtain fluid and diffusion limits for these processes in the space of tempered distribution-valued processes. We find that our diffusion limits are tempered distribution-valued Ornstein–Uhlenbeck processes.

1. Introduction. Limit theorems for the infinite-server queue in heavy-traffic have a rich history starting with the seminal work of Iglehart [17] on the $M/M/\infty$ queue. This work then inspired a line of research aimed at extending the results of [17] to additional classes of service time distributions. Whitt [32] studies the $GI/PH/\infty$ queue, having phase-type service-time distributions, and Glynn and Whitt [11] consider the $GI/GI/\infty$ queue with service times taking values in a finite set. In [5], [25] and [31], the $G/GI/\infty$ queue is studied with general service time distributions. Pang et al. [29] gives a survey of these results.

In this paper, we study two Markov processes associated with the $G/GI/\infty$ queue. The first process which we study is a tempered distribution-valued process which tracks the age of each customer in the system. We refer to this process as the age process. The second process which we study is also a tempered distribution-valued process and it tracks the residual service time of each customer in the system as well as the amount of time since departure for each customer who has left the system. We refer to this process as the residual service time process. Although analyzing either of these processes might at first appear to be a difficult task, one of the key themes that runs

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throughout the present paper is that techniques originally developed for establishing heavy-traffic limits in the finite-dimensional setting may also be successfully applied in the more abstract infinite-dimensional setting.

Our main results in this paper are to obtain fluid and diffusion limits for both the age and residual service time processes. In particular, for both the age and the residual service time processes, we use the continuous mapping theorem together with functional central limit theorem results in order to establish our main results. The corresponding diffusion limits that we obtain for both the age and residual time processes may be characterized as tempered distribution-valued Ornstein–Uhlenbeck processes.

The tempered distribution-valued representation which we use for the residual service time process was also used by Decreusefond and Moyal [8] in order to analyze the $M/G/\infty$ queue. However, in the current paper we go beyond analyzing the residual service time process and also analyze the age process, which was not treated in [8]. In particular, we provide a diffusion limit result for the tempered distribution-valued age process which is fundamentally different from the diffusion limit for the age process obtained in [8]. Moreover, our general methodology for proving our main results differs from that employed in [8]. Specifically, while the approach in [8] has as its starting point the infinitesimal generator of the residual service time process, in the present paper we begin by defining the model primitives and setting up a governing system equation for both the age and residual service time processes. As alluded to above, we then rely upon the continuous mapping theorem and functional central limit theorem results when proving our main results.

A second major contribution of our work is to make a connection between the literature on infinite-dimensional heavy-traffic limits for queueing systems [7–9, 12–14, 23] and the vast literature on infinite-dimensional Ornstein–Uhlenbeck processes motivated by applications to interacting particle systems [2–4, 15, 16, 19, 20, 27, 28]. Our work especially relies upon [19] and [20] in order to prove continuity of a particular regulator map.

Another set of papers related to ours are those of Kaspi and Ramanan [23, 24]. Although these works analyze the many-server queue with general service time distributions, their infinite-dimensional representation of the system is similar to ours. Fluid limits are established for the system in [23] in the space of Radon measure-valued processes. However, when establishing corresponding diffusion limits, the limit process evidently falls out of the space of Radon measure-valued processes and distribution-valued processes are used instead in [24]. In the present paper, we follow the work of [8] in which the space of tempered distributions is used. This space may be characterized as the topological dual of Schwartz space, the space of rapidly decreasing, infinitely differentiable functions.

We also mention the work [30], where the authors build upon the work of [11] and [25] in order to prove heavy-traffic limits for the $G/GI/\infty$ queue in a two-parameter function space. They analyze both the age and the residual service time process as we do. The main difference between the present work and [30] is that in the present work tempered distribution-valued processes are used which allows one to apply the continuous mapping theorem and other standard results in order to obtain heavy-traffic limits.

The remainder of this paper is now organized as follows. In Section 2, we derive basic system equations for both the age process and the residual service time process. These equations serve as the starting point for our analysis in the remainder of the paper. In Section 3, we present a regulator map result to be used in conjunction with the continuous mapping theorem in order to prove our main results. In Section 4, we provide martingale results that are used together with the regulator map of Section 3 in order to obtain our fluid and diffusion limits. In Sections 5 and 6, we prove our fluid and diffusion limits, respectively. In the Appendix, we provide the proofs of several technical lemmas that are used throughout the paper.

1.1. *Technical background.* We now provide some technical background which is useful for the remainder of the paper. We begin with some preliminary details.

1.1.1. *Preliminaries.* All random variables and processes in this paper are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are measurable maps from $(\Omega, \mathcal{F}, \mathbb{P})$ to an arbitrary topological space with an associated Borel σ-algebra. It turns out that many of the random quantities which we study in this paper take values in a topological space which is not metrizable and so we now provide the definition of weak convergence on an arbitrary topological space. We follow the approach of [21]. Let X be an arbitrary topological space with associated Borel σ-algebra $\mathcal{B}(X)$. We say that a sequence of probability measures $(\mathbb{P}_n)_{n\geq 1}$ on $\mathcal{B}(X)$ weakly converges to a probability measure $\mathbb{P}$ on $\mathcal{B}(X)$, abbreviated as $\mathbb{P}_n \Rightarrow \mathbb{P}$, if $\int f\, d\mathbb{P}_n \to \int f\, d\mathbb{P}$ for every bounded, continuous real functional f on X (see Definition 2.2.1 of [21]). We also use the notation $\stackrel{\mathbb{P}}{\to}$ to denote convergence in probability. For any two topological spaces X and Y, we denote by $X \times Y$ the cartesian product of X and Y and we associate with $X \times Y$ the product topology. Note that using the above definition of weak convergence, it is straightforward to show that the continuous mapping theorem continues to hold (see, e.g., the proof of Theorem 3.4.1 of [33]). In particular, we have the following.

Proposition 1.1. *Let X and Y be two topological spaces and let $(x^n)_{n\geq 1}$ be a sequence of random elements of X such that $x^n \Rightarrow x$. If $g : X \mapsto Y$ is a continuous function, then $g(x^n) \Rightarrow g(x)$ in Y.*

Next, for each $0 < T < \infty$, let $\mathbb{D}([0,T],\mathbb{R})$ denote the space of functions from $[0,T]$ to $\mathbb{R}$ that are right-continuous on $[0,T)$ with left limits everywhere on $(0,T]$. We equip $\mathbb{D}([0,T],\mathbb{R})$ with the Skorokhod J_1-topology [1]. We also note that we will commonly abbreviate the notation $\mathbb{D}([0,T],\mathbb{R})$ by simply writing $\mathbb{D}$. Next, let $\mathbb{D}([0,T],\mathbb{D})$ denote the space of functions from $[0,T]$ to $\mathbb{D}$ that are right-continuous on $[0,T)$ with left limits everywhere on $(0,T]$. We equip $\mathbb{D}([0,T],\mathbb{D})$ with the Skorokhod J_1-topology [1] as well. For an element $x \in \mathbb{D}([0,T],\mathbb{R})$, we set

$$\|x\|_T = \sup_{0\le t\le T} |x(t)|.$$

We denote by $e = (t, t \in [0,T])$, the identity process on $[0,T]$.

1.1.2. *Schwartz space.* The space of rapidly decreasing functions, also known as Schwartz space, plays an important rule in this paper and so we now provide a brief review of some of the relevant facts concerning this space. Much of the material found in this subsection may also be found in [21].

Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the set of reals, nonnegative reals and nonpositive reals, respectively. Also, denote by $\mathbb{N} = \{0,1,2,\dots\}$ the set of nonnegative integers. Let $C^\infty(\mathbb{R})$ denote the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ and let $C^\infty(\mathbb{R}_+)$ denote the set of infinitely differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$. Also, let $C_b^\infty(\mathbb{R})$ denote the set of infinitely differentiable, bounded functions from $\mathbb{R}$ to $\mathbb{R}$ whose derivatives of all orders are bounded and, similarly, let $C_b^\infty(\mathbb{R}_+)$ denote the set of infinitely differentiable, bounded functions from $\mathbb{R}_+$ to $\mathbb{R}$ whose derivatives of all orders are bounded.

Now define

$$\mathcal{S} \equiv \{\varphi \in C^\infty(\mathbb{R}) : \|\varphi\|_{\alpha,\beta} < \infty \text{ for all } \alpha,\beta \in \mathbb{N}\}, \tag{1.1}$$

where

$$\|\varphi\|_{\alpha,\beta} \equiv \sup_{x\in\mathbb{R}} |x^\alpha \varphi^{(\beta)}(x)| \tag{1.2}$$

and $\varphi^{(\beta)}$ denotes the βth derivative of φ. The space $\mathcal{S}$ is commonly referred to as Schwartz space or the space of rapidly decreasing functions [21].

The topology of $\mathcal{S}$ is given by the family of seminorms $\{\|\cdot\|_{\alpha,\beta} : \alpha,\beta \in \mathbb{N}\}$ defined in (1.2). In particular, $\varphi_n \to \varphi$ in $\mathcal{S}$ if $\|\varphi_n - \varphi\|_{\alpha,\beta} \to 0$ for all $\alpha,\beta \in \mathbb{N}$. We note that by Lemma 1.3.2 and Theorem 1.3.2 of [21], the space $\mathcal{S}$ is a nuclear Frèchet space. Moreover, we also note that one may construct a sequence of seminorms $\{\|\cdot\|_p : p \in \mathbb{N}\}$ with the property that $\|\cdot\|_p \le \|\cdot\|_{p+1}$ for each $p \in \mathbb{N}$ and which also induce the same the topology on $\mathcal{S}$ as the seminorms $\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}\}$ given above. In particular, by Lemma 1.3.3 of [21], one has that for each $p \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ and $C > 0$ such that

$$\|\varphi\|_p \le C \max_{0 \le \alpha,\beta \le k+1} \|\varphi\|_{\alpha,\beta} \qquad \text{for all } \varphi \in \mathcal{S}, \tag{1.3}$$

and, by Lemma 1.3.4 of [21], one has that for each $\alpha, \beta \in \mathbb{N}$, there exists a $p \in \mathbb{N}$ and $M > 0$ such that

$$\max_{0 \le \alpha,\beta \le k+1} \|\varphi\|_{\alpha,\beta} \le M \|\varphi\|_p \qquad \text{for all } \varphi \in \mathcal{S}. \tag{1.4}$$

For a precise construction of $\|\cdot\|_p$ for each $p \in \mathbb{N}$, one may consult page 24 of [21].

The set of all linear maps from $\mathcal{S}$ to $\mathcal{S}$ is denoted by $L(\mathcal{S}, \mathcal{S})$ and the strong topology on $L(\mathcal{S}, \mathcal{S})$ is defined in the following manner. A subset B of $\mathcal{S}$ is said to be bounded if for any neighborhood U of $\varphi \equiv 0 \in \mathcal{S}$, there exists a constant $\alpha > 0$ such that $\alpha^{-1} B \subset U$ (see Definition 1.1.7 of [21]). The strong topology on $L(\mathcal{S}, \mathcal{S})$ is then given by the following definition (see Theorem 1.2.1 of [21]).

DEFINITION 1.2. For each bounded subset B of $\mathcal{S}$ and $p \in \mathbb{N}$, let

$$q_{B,p}(T) \equiv \sup_{\varphi \in B} \|T\varphi\|_p \qquad \text{for all } T \in L(\mathcal{S}, \mathcal{S}).$$

Then $\{q_{B,p}\}$ constitutes a family of seminorms on $L(\mathcal{S}, \mathcal{S})$ and the topology given by these seminorms is referred to as the strong topology on $L(\mathcal{S}, \mathcal{S})$.

1.1.3. *The space of tempered distributions.* Many of the processes studied in this paper take values in the topological dual of $\mathcal{S}$, which we denote by $\mathcal{S}'$. Recall that $\mathcal{S}'$ is the space of all continuous linear functionals on $\mathcal{S}$. Elements of $\mathcal{S}'$ are referred to as *tempered distributions* and we now review some relevant facts concerning tempered distributions as well as tempered distribution-valued processes.

For each $\mu \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we denote the *duality product* of μ and φ by $\langle \mu, \varphi \rangle \equiv \mu(\varphi)$. The *distributional derivative* of $\mu \in \mathcal{S}'$ is denoted by μ' and is defined to be the unique element of $\mathcal{S}'$ such that

$$\langle \mu', \varphi \rangle = -\langle \mu, \varphi' \rangle \qquad \text{for all } \varphi \in \mathcal{S}.$$

It is clear by the definition of $\mathcal{S}$ that μ' is well defined. For each $\mu \in \mathcal{S}'$ and $t \in \mathbb{R}$, we also define $\tau_t \mu$ as the unique element of $\mathcal{S}'$ such that

$$\langle \tau_t \mu, \varphi \rangle = \langle \mu, \tau_t \varphi \rangle \qquad \text{for all } \varphi \in \mathcal{S},$$

where $\tau_t \varphi \in \mathcal{S}$ is the function defined by $\tau_t \varphi(\cdot) \equiv \varphi(\cdot - t)$.

All statements in this paper regarding convergence in $\mathcal{S}'$ are with respect to the strong topology on $\mathcal{S}'$, which we now define. One may consult Section 1.1 of [21] for further details.

DEFINITION 1.3. For each bounded subset $B \subset \mathcal{S}$, let

$$q_B(\mu) \equiv \sup_{\varphi \in B} |\langle \mu, \varphi \rangle| \qquad \text{for all } \mu \in \mathcal{S}'. \tag{1.5}$$

Then the strong topology on $\mathcal{S}'$ is the topology induced by the family of seminorms $\{q_B\}$.

Unfortunately, the space $\mathcal{S}'$ is not metrizable with respect to the strong topology (see Section 2 of [21]). Nevertheless, as we discuss below, one may still usefully speak of weak convergence of $\mathcal{S}'$-valued random elements and processes taking values in $\mathcal{S}'$.

Let $\mathbb{D}([0,T], \mathcal{S}')$ denote the space of functions from $[0,T]$ to $\mathcal{S}'$ that are right-continuous on $[0,T)$ with left limits everywhere on $(0,T]$. If $(\mu_t)_{t \ge 0} \in \mathbb{D}([0,T], \mathcal{S}')$ and $t \in [0,T]$, we then define the tempered distribution $\int_0^t \mu_s \, ds$ to be the unique element of $\mathcal{S}'$ (see Section 2 of [19]) such that

$$\left\langle \int_0^t \mu_s \, ds, \varphi \right\rangle = \int_0^t \langle \mu_s, \varphi \rangle \, ds \qquad \text{for all } t \ge 0, \varphi \in \mathcal{S}.$$

As noted immediately following Definition 1.3 above, the space $\mathcal{S}'$ equipped with the strong topology is not metrizable and so the Skorokhod metric and ensuing Skorokhod J_1-topology may not defined on $\mathbb{D}([0,T], \mathcal{S}')$ in the usual manner. We therefore follow the approach of [21, 26] in defining an appropriate topology on $\mathbb{D}([0,T], \mathcal{S}')$. Let Λ be the set of strictly increasing continuous maps from $[0,T]$ onto itself such that for each $\lambda \in \Lambda$,

$$\gamma(\lambda) = \sup_{0 \le s < t \le T} \left| \ln\left(\frac{\lambda_t - \lambda_s}{t - s} \right) \right| < \infty.$$

We then have the following definition (see [21, 26]).

DEFINITION 1.4. For each seminorm q_B defining the strong topology on $\mathcal{S}'$, let

$$d^o_{q_B}(\mu, \nu) = \inf_{\lambda \in \Lambda} \left(\sup_{0 \le t \le T} |q_B(\mu_t - \nu_{\lambda_t}) + \gamma(\lambda)| \right) \qquad \text{for all } \mu, \nu \in \mathcal{S}'.$$

The topology on $\mathbb{D}([0,T], \mathcal{S}')$ is then defined by the family of pseudometrics $\{d^o_{q_B}\}$.

By part (c) of Theorem 2.4.1 of [21], the topology given in Definition 1.4 above is equivalent to the topology defined by the family of pseudometrics $\{d_{q_B}\}$, where

$$d_{q_B}(\mu, \nu) = \inf_{\lambda \in \Lambda} \left(\sup_{0 \le t \le T} |q_B(\mu_t - \nu_{\lambda_t})| + \sup_{0 \le t \le T} |\lambda_t - t| \right) \qquad \text{for all } \mu, \nu \in \mathcal{S}'.$$

We also note that under this topology, $\mathbb{D}([0,T],\mathcal{S}')$ is a completely regular topological space [26].

The following result is an important consequence of Proposition 5.2 of [26] regarding weak convergence of processes taking values in the dual of a nuclear Frèchet space. It provides a convenient characterization of weak convergence of processes taking values in $\mathcal{S}'$.

THEOREM 1.5 (Mitoma's theorem). *Let $(\mu^n)_{n\geq 1}$ be a sequence of random elements of $\mathbb{D}([0,T],\mathcal{S}')$. Then*

$$\mu^n \Rightarrow \mu \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}')$$

if the following two statements hold:

(1) *For each $\varphi \in \mathcal{S}$, the sequence $\{\langle \mu^n, \varphi\rangle\}_{n\geq 1}$ is tight in $\mathbb{D}([0,T],\mathbb{R})$.*
(2) *For $\varphi_1,\dots,\varphi_m \in \mathcal{S}$ and $t_1,\dots,t_m \in [0,T]$,*

$$(\langle \mu^n_{t_1},\varphi_1\rangle,\dots,\langle \mu^n_{t_m},\varphi_m\rangle) \Rightarrow (\langle \mu_{t_1},\varphi_1\rangle,\dots,\langle \mu_{t_m},\varphi_m\rangle) \qquad \text{in } \mathbb{R}^m.$$

We now conclude the technical background section with some comments regarding martingales and, in particular, $\mathcal{S}'$-valued martingales. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration on an underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$ and let M and N be two $\mathbb{R}$-valued $\mathcal{F}_t$-martingales. The quadratic covariation of M and N is denoted by $(\langle M,N\rangle_t)_{t\geq 0}$ and the quadratic variation of M is denoted by $(\langle\langle M\rangle\rangle_t)_{t\geq 0} \equiv (\langle M,M\rangle_t)_{t\geq 0}$. An $\mathcal{S}'$-valued process M is said to be an *$\mathcal{S}'$-valued $\mathcal{F}_t$-martingale* if for all $\varphi \in \mathcal{S}$, $(\langle M_t,\varphi\rangle)_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t$-martingale. For two $\mathcal{S}'$-valued martingales M and N, their *tensor quadratic covariation* $(\langle M,N\rangle_t)_{t\geq 0}$ is given for all $t\geq 0$ and all $\varphi,\psi\in\mathcal{S}$ by

$$\langle M,N\rangle_t(\varphi,\psi) \equiv \langle\langle M_\cdot,\varphi\rangle,\langle N_\cdot,\psi\rangle\rangle_t,$$

and the *tensor quadratic variation* $(\langle\langle M\rangle\rangle_t)_{t\geq 0}$ of an $\mathcal{S}'$-valued martingale is given by $(\langle\langle M\rangle\rangle_t)_{t\geq 0} \equiv (\langle M,M\rangle_t)_{t\geq 0}$. Two $\mathcal{S}'$-valued martingales, M and N, are said to be *orthogonal* if $\langle M,N\rangle = 0$ identically. Corresponding notions for the optional quadratic variation process $[M]$ are defined analogously.

2. System equations.

In this section, we obtain semi-martingale decompositions of the tempered distribution-valued age process $\mathcal{A} \equiv (\mathcal{A}_t)_{t\geq 0}$ and the tempered distribution-valued residual service time process $\mathcal{R} \equiv (\mathcal{R}_t)_{t\geq 0}$. We begin in Section 2.1 by treating the age process $\mathcal{A}$ and then move on in Section 2.2 to treating the residual service time process $\mathcal{R}$.

2.1. *Age process.* We consider a $G/GI/\infty$ queue with general arrival process $(E_t)_{t\geq 0} \in \mathbb{D}([0,\infty),\mathbb{R})$. We assume that $E_0 = 0$, $\mathbb{P}$-a.s., and, for convenience in our proofs, we also define $E_t = 0$ for $t < 0$. We also make the assumption that for each $t \geq 0$, we have that $\mathbb{E}[E_t^2] < \infty$. Next, for each $i \geq 1$, we denote by

$$\tau_i = \inf\{t \geq 0 : E_t \geq i\}$$

the time of the arrival of the ith customer to the system after time $t = 0$. We assume that $\mathbb{E}[\tau_i] < \infty$ for each $i = 1, 2, \ldots$. We denote by η_i the service time of the ith customer to arrive to the system after time $t = 0$ and we assume that $\{\eta_i, i \geq 1\}$ is an i.i.d. sequence of nonnegative, mean 1 random variables with cumulative distribution function (c.d.f.) F, complementary cumulative distribution function (c.c.d.f.) $\bar{F} = 1 - F$, and probability density function (p.d.f.) f. We also assume that the hazard rate function h of F satisfies the following assumption.

ASSUMPTION 2.1. The function $h \in C_b^\infty(\mathbb{R}_+)$.

Now let $(A_t)_{t\geq 0} \in \mathbb{D}([0,\infty),\mathbb{D})$ be such that for each $t \geq 0$ and $y \geq 0$, the quantity $A_t(y)$ represents the number of customers in the system at time $t \geq 0$ that have been in the system for less than or equal to y units of time at time t. For $y < 0$, we set $A_t(y) = 0$. At time $t = 0$, we assume that there are $A_0(y)$ customers present who have been in the system for less than or equal to $y \geq 0$ units of time and that there are a total of $A_0(\infty)$ customers present. We assume that $\mathbb{E}[A_0^2(\infty)] < \infty$. For each $i = 1, \ldots, A_0(\infty)$, we denote by

$$\tilde{\tau}_i = -\inf\{y \geq 0 : A_0(y) \geq i\}$$

the "arrival" time of the ith initial customer to the system. We denote by $\tilde{\eta}_i$ the remaining service time at time $t = 0$ of the ith initial customer in the system. The distribution of $\tilde{\eta}_i$, conditional on the arrival time $\tilde{\tau}_i$, is given by

$$\mathbb{P}(\tilde{\eta}_i > x | \tilde{\tau}_i) = \frac{1 - F(-\tilde{\tau}_i + x)}{1 - F(-\tilde{\tau}_i)}, \qquad x \geq 0. \tag{2.1}$$

We denote by $f_{\tilde{\tau}_i}$ the conditional p.d.f. associated with this distribution and we set $h_{\tilde{\tau}_i}(\cdot) = h(\cdot - \tilde{\tau}_i)$.

We now derive a convenient representation for the system equations for $(A_t)_{t\geq 0}$ and its tempered distribution-valued counterpart, $\mathcal{A}$, which we define shortly. We begin by noting that by first principles we have that for each $t \geq 0$ for $y \geq 0$,

$$A_t(y) = \sum_{i=1}^{A_0(\infty)} \mathbf{1}_{\{t-\tilde{\tau}_i \leq y\}} \mathbf{1}_{\{t<\tilde{\eta}_i\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{t-\tau_i \leq y\}} \mathbf{1}_{\{t-\tau_i<\eta_i\}}. \tag{2.2}$$

Our first result provides an alternative way to write (2.2). In the following, we set $\sum_{i=1}^{0} = 0$.

PROPOSITION 2.2. *For each $t \ge 0$ and $y \ge 0$,*

$$\begin{aligned}A_t(y) = A_0(y) - \sum_{i=1}^{A_0(\infty)} \mathbf{1}_{\{\tilde\eta_i \le t\wedge(y+\tilde\tau_i)\}} - \sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{\tilde\eta_i > y+\tilde\tau_i\}} \\ + E_t - \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i \le (t-\tau_i)\wedge y\}} - \sum_{i=1}^{E_{(t-y)-}} \mathbf{1}_{\{\eta_i > y\}}.\end{aligned} \tag{2.3}$$

PROOF. By (2.2), we have that

$$\begin{aligned}A_t(y) &= \sum_{i=1}^{A_0(y)} \mathbf{1}_{\{t-\tilde\tau_i \le y\}}\mathbf{1}_{\{t<\tilde\eta_i\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{t-\tau_i \le y\}}\mathbf{1}_{\{t-\tau_i<\eta_i\}} \\ &= A_0(y) + \sum_{i=1}^{A_0(y)} (\mathbf{1}_{\{t-\tilde\tau_i \le y\}}\mathbf{1}_{\{t<\tilde\eta_i\}} - 1) + E_t \\ &\quad + \sum_{i=1}^{E_t} (\mathbf{1}_{\{t-\tau_i \le y\}}\mathbf{1}_{\{t-\tau_i<\eta_i\}} - 1).\end{aligned} \tag{2.4}$$

However,

$$\begin{aligned}&1 - \mathbf{1}_{\{t-\tilde\tau_i \le y\}}\mathbf{1}_{\{t<\tilde\eta_i\}} \\ &\quad = \mathbf{1}_{\{t-\tilde\tau_i \le y\}}\mathbf{1}_{\{\tilde\eta_i \le t\}} + \mathbf{1}_{\{t-\tilde\tau_i > y\}} \\ &\quad = (\mathbf{1}_{\{t-\tilde\tau_i \le y\}}\mathbf{1}_{\{\tilde\eta_i \le t\}} + \mathbf{1}_{\{t-\tilde\tau_i > y\}}\mathbf{1}_{\{-\tilde\tau_i+\tilde\eta_i \le y\}}) + \mathbf{1}_{\{t-\tilde\tau_i > y\}}\mathbf{1}_{\{-\tilde\tau_i+\tilde\eta_i > y\}},\end{aligned} \tag{2.5}$$

and, similarly,

$$\begin{aligned}&1 - \mathbf{1}_{\{t-\tau_i \le y\}}\mathbf{1}_{\{t-\tau_i<\eta_i\}} \\ &\quad = \mathbf{1}_{\{t-\tau_i \le y\}}\mathbf{1}_{\{\eta_i \le t-\tau_i\}} + \mathbf{1}_{\{t-\tau_i > y\}} \\ &\quad = (\mathbf{1}_{\{t-\tau_i \le y\}}\mathbf{1}_{\{\eta_i \le t-\tau_i\}} + \mathbf{1}_{\{t-\tau_i > y\}}\mathbf{1}_{\{\eta_i \le y\}}) + \mathbf{1}_{\{t-\tau_i > y\}}\mathbf{1}_{\{\eta_i > y\}}.\end{aligned} \tag{2.6}$$

Substituting (2.6) and (2.5) into (2.4) and summing over $A_0(y)$ and E_t completes the proof. $\square$

We now provide an intuitive description of each of the terms appearing in (2.3). The first term represents the number of customers in the system at time $t = 0$ that have been in the system for less than or equal to y units of time, the second term represents the number of departures by time $t \ge 0$ of those initial customers that had total service less than or equal to y units of time at time $t = 0$, and the third term represents the number of initial customers whose total service time is greater than y units of time and had been in the system for less than or equal to y units of time at time $t=0$ but have been in the system for greater than y units of time at time $t\geq 0$. The fourth, fifth and sixth terms represent similar quantities but for those customers that arrived to the system after time $t=0$.

Now let $D^0=(D^0_t)_{t\geq 0}\in\mathbb{D}([0,\infty),\mathbb{D})$ be defined by setting

$$D^0_t(y)=\sum_{i=1}^{A_0(\infty)}\left(\mathbf{1}_{\{\tilde{\eta}_i\leq t\wedge(y+\tilde{\tau}_i)\}}-\int_0^{\tilde{\eta}_i\wedge t\wedge(y+\tilde{\tau}_i)} h_{\tilde{\tau}_i}(u)\,du\right), \tag{2.7}$$

for $t\geq 0$, $y\geq 0$, and set $D^0_t(y)=0$ for $y<0$. Also, let $D=(D_t)_{t\geq 0}\in\mathbb{D}([0,\infty),\mathbb{D})$ be defined by setting

$$D_t(y)=\sum_{i=1}^{E_t}\left(\mathbf{1}_{\{\eta_i\leq(t-\tau_i)\wedge y\}}-\int_0^{\eta_i\wedge(t-\tau_i)\wedge y} h(u)\,du\right),\qquad t\geq 0, y\geq 0, \tag{2.8}$$

and set $D_t(y)=0$ for $y<0$. It then follows from (2.3) that for each $t\geq 0$ and $y\geq 0$, we may write

$$\begin{aligned} A_t(y) &= A_0(y)+E_t-D^0_t(y)-D_t(y)\\ &\quad-\sum_{i=1}^{A_0(\infty)}\int_0^{\tilde{\eta}_i\wedge t\wedge(y+\tilde{\tau}_i)} h_{\tilde{\tau}_i}(u)\,du-\sum_{i=1}^{E_t}\int_0^{\eta_i\wedge(t-\tau_i)\wedge y} h(u)\,du\\ &\quad-\sum_{i=A_0(y-t)+1}^{A_0(y)}\mathbf{1}_{\{-\tilde{\tau}_i+\tilde{\eta}_i>y\}}-\sum_{i=1}^{E_{(t-y)-}}\mathbf{1}_{\{\eta_i>y\}}. \end{aligned} \tag{2.9}$$

The above expression for $A_t(y)$ will become useful in a moment. However, we next move on to expressing the age process as a tempered distribution-valued process using the Schwartz space $\mathcal{S}$ defined in (1.1). In particular, we associate with the process A defined in (2.2) the $\mathcal{S}'$-valued process $\mathcal{A}=(\mathcal{A}_t)_{t\geq 0}$ such that for each $t\geq 0$ and $\varphi\in\mathcal{S}$ we set

$$\langle\mathcal{A}_t,\varphi\rangle=\int_{\mathbb{R}}\varphi(y)\,dA_t(y). \tag{2.10}$$

In a similar manner, we associate the $\mathcal{S}'$-valued processes $\mathcal{D}^0=(\mathcal{D}^0_t)_{t\geq 0}$ and $\mathcal{D}=(\mathcal{D}_t)_{t\geq 0}$ with D^0 and D, respectively. That is, for each $t\geq 0$ and $\varphi\in\mathcal{S}$ we set

$$\langle\mathcal{D}^0_t,\varphi\rangle=\int_{\mathbb{R}}\varphi(y)\,dD^0_t(y)\quad\text{and}\quad\langle\mathcal{D}_t,\varphi\rangle=\int_{\mathbb{R}}\varphi(y)\,dD_t(y).$$

We also associate the $\mathcal{S}'$-valued random variable $\mathcal{A}_0$ with A_0 by setting

$$\langle\mathcal{A}_0,\varphi\rangle=\int_{\mathbb{R}}\varphi(y)\,dA_0(y),\qquad\varphi\in\mathcal{S}.$$

It is straightforward to see that for each $t \geq 0$, the quantities $\mathcal{A}_t$, $\mathcal{D}_t^0$ and $\mathcal{D}_t$ are well-defined elements of $\mathcal{S}'$. Moreover, since for each fixed $\varphi \in \mathcal{S}$ the sample paths of $(\langle \mathcal{A}_t, \varphi\rangle)_{t\geq 0}$, $(\langle \mathcal{D}_t^0, \varphi\rangle)_{t\geq 0}$ and $(\langle \mathcal{D}_t, \varphi\rangle)_{t\geq 0}$ all lie in $\mathbb{D}([0,\infty),\mathbb{R})$, $\mathbb{P}$-a.s., it follows that $\mathcal{A}, \mathcal{D}^0, \mathcal{D} \in \mathbb{D}([0,\infty),\mathcal{S}')$, $\mathbb{P}$-a.s.

For the remainder of the paper, we now replace the hazard rate function $h:\mathbb{R}_+ \mapsto \mathbb{R}$ with a function $\tilde{h}:\mathbb{R}\to\mathbb{R}$ such that $\tilde{h}(x) = h(x)$ for $x \geq 0$ and $\tilde{h} \in C_b^\infty(\mathbb{R})$. In a similar manner, we replace the c.d.f. F and c.c.d.f. $\bar{F}$ with corresponding functions $\tilde{F}$ and $\tilde{\bar{F}}$ such that $\tilde{F}, \tilde{\bar{F}} \in C_b^\infty(\mathbb{R})$. For ease of notation, we continue to refer to $\tilde{h}$, $\tilde{F}$ and $\tilde{\bar{F}}$ as h, F and $\bar{F}$, respectively.

Our next step is to use the expression (2.9) in order to provide a convenient expression for the tempered distribution-valued process $\mathcal{A}$. We begin by noting that integrating test functions $\varphi \in \mathcal{S}$ term-by-term in (2.9) it follows that for each $t \geq 0$ one has that

$$\begin{aligned}\langle \mathcal{A}_t, \varphi\rangle &= \langle \mathcal{A}_0, \varphi\rangle - \langle \mathcal{D}_t^0 + \mathcal{D}_t, \varphi\rangle \\ &\quad - \sum_{i=1}^{A_0(\infty)} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y)h(y)\,dy - \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y)h(y)\,dy \\ &\quad - \int_{\mathbb{R}_+} \varphi(y)\,d\Bigg(\sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde{\tau}_i + \tilde{\eta}_i > y\}} + \sum_{i=1}^{E_{(t-y)-}} \mathbf{1}_{\{\eta_i > y\}}\Bigg).\end{aligned} \tag{2.11}$$

The following two propositions now allow us to further simplify the expression in (2.11). We first have the following.

Proposition 2.3. *For each $t \geq 0$,*

$$\sum_{i=1}^{A_0(\infty)} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y)h(y)\,dy + \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y)h(y)\,dy = \int_0^t \langle \mathcal{A}_s, \varphi h\rangle\,ds.$$

Proof. For each $t \geq 0$,

$$\begin{aligned}&\sum_{i=1}^{A_0(\infty)} \int_{-\tilde{\tau}_i}^{-\tilde{\tau}_i + (\tilde{\eta}_i \wedge t)} \varphi(y)h(y)\,dy + \sum_{i=1}^{E_t} \int_0^{\eta_i \wedge (t-\tau_i)} \varphi(y)h(y)\,dy \\ &\quad = \sum_{i=1}^{A_0(\infty)} \int_0^t \mathbf{1}_{\{0 \leq s \leq \tilde{\eta}_i\}} \varphi(s-\tilde{\tau}_i)h(s-\tilde{\tau}_i)\,ds \\ &\qquad + \sum_{i=1}^{E_t} \int_0^t \mathbf{1}_{\{0 \leq s-\tau_i \leq \eta_i\}} \varphi(s-\tau_i)h(s-\tau_i)\,ds\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \Bigg(\sum_{i=1}^{A_0(\infty)} \mathbf{1}_{\{0\le s\le \tilde\eta_i\}} \varphi(s-\tilde\tau_i) h(s-\tilde\tau_i) \\
&\qquad + \sum_{i=1}^{E_t} \mathbf{1}_{\{0\le s-\tau_i\le \eta_i\}} \varphi(s-\tau_i) h(s-\tau_i) \Bigg)\, ds \\
&= \int_0^t \langle \mathcal{A}_s, \varphi h\rangle \, ds.
\end{aligned}
$$

This completes the proof. □

Next, we have the following.

PROPOSITION 2.4. *For each* $t \ge 0$,

$$
\begin{aligned}
&- \int_{\mathbb{R}_+} \varphi(y)\, d\Bigg(\sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde\tau_i+\tilde\eta_i>y\}} + \sum_{i=1}^{E_{(t-y)-}} \mathbf{1}_{\{\eta_i>y\}} \Bigg) \\
&\qquad = E_t \varphi(0) + \int_0^t \langle \mathcal{A}_s, \varphi' \rangle \, ds.
\end{aligned}
$$

PROOF. Let $t \ge 0$. Then, integrating by parts we have that

$$
\begin{aligned}
&-E_t\varphi(0) - \int_{\mathbb{R}_+} \varphi(y)\, d\Bigg(\sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde\tau_i+\tilde\eta_i>y\}} + \sum_{i=1}^{E_{(t-y)-}} \mathbf{1}_{\{\eta_i>y\}} \Bigg) \\
&\quad = \int_{\mathbb{R}_+} \Bigg(\sum_{i=A_0(y-t)+1}^{A_0(y)} \mathbf{1}_{\{-\tilde\tau_i+\tilde\eta_i>y\}} + \sum_{i=1}^{E_{(t-y)-}} \mathbf{1}_{\{\eta_i>y\}} \Bigg) \varphi'(y)\, dy \\
&\quad = \int_{\mathbb{R}_+} \Bigg(\sum_{i=1}^{A_0(\infty)} \mathbf{1}_{\{\tilde\tau_i \ge -y, -\tilde\tau_i+\tilde\eta_i>y, \tilde\tau_i+y<t\}} + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i>y, \tau_i+y<t\}} \Bigg) \varphi'(y)\, dy \\
&\quad = \sum_{i=1}^{A_0(\infty)} \int_{\mathbb{R}_+} \mathbf{1}_{\{\tilde\tau_i \ge -y, -\tilde\tau_i+\tilde\eta_i>y, \tilde\tau_i+y<t\}} \varphi'(y)\, dy \\
&\qquad + \sum_{i=1}^{E_t} \int_{\mathbb{R}_+} \mathbf{1}_{\{\eta_i>y, \tau_i+y<t\}} \varphi'(y)\, dy \\
&\quad = \sum_{i=1}^{A_0(\infty)} \int_0^t \mathbf{1}_{\{0\le s-\tilde\tau_i \le -\tilde\tau_i+\tilde\eta_i\}} \varphi'(s-\tilde\tau_i)\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{E_t}\int_0^t \mathbf{1}_{\{0\le s-\tau_i\le\eta_i\}}\varphi'(s-\tau_i)\,ds\\
&=\int_0^t\left(\sum_{i=1}^{A_0(\infty)}\mathbf{1}_{\{0\le s-\tilde\tau_i\le-\tilde\tau_i+\tilde\eta_i\}}\varphi'(s-\tilde\tau_i)+\sum_{i=1}^{E_t}\mathbf{1}_{\{0\le s-\tau_i\le\eta_i\}}\varphi'(s-\tau_i)\right)ds\\
&=\int_0^t\left(\int_{\mathbb{R}_+}\varphi'(u)\,d\mathcal{A}_s(u)\right)ds\\
&=\int_0^t\langle\mathcal{A}_s,\varphi'\rangle\,ds.
\end{aligned}
$$

This completes the proof. □

Now note that combining Propositions 2.3 and 2.4 with system equation (2.11), one finds that for each $t\ge 0$ and $\varphi\in\mathcal{S}$,

$$
\begin{aligned}
\langle\mathcal{A}_t,\varphi\rangle&=\langle\mathcal{A}_0,\varphi\rangle+\langle\mathcal{E}_t-\mathcal{D}_t^0-\mathcal{D}_t,\varphi\rangle\\
&\quad-\int_0^t\langle\mathcal{A}_s,h\varphi\rangle\,ds+\int_0^t\langle\mathcal{A}_s,\varphi'\rangle\,ds,
\end{aligned}
\tag{2.12}
$$

where we define the $\mathcal{S}'$-valued process $\mathcal{E}=(\mathcal{E}_t)_{t\ge0}$ to be such that $\langle\mathcal{E}_t,\varphi\rangle=E_t\varphi(0)$ for each $\varphi\in\mathcal{S}$ and $t\ge 0$. In general, we refer to (2.12) as the semimartingale decomposition of $\mathcal{A}$. This will become clear in Section 4 where we show that the process $(\mathcal{D}_t^0+\mathcal{D}_t)_{t\ge0}$ is a martingale.

2.2. *Residual service time process.* We next move on to analyzing the residual service time process $\mathcal{R}$. As in Section 2.1, we assume that we have a $G/GI/\infty$ queue in which customers arrive to the system according to a general arrival process $(E_t)_{t\ge0}\in\mathbb{D}([0,\infty),\mathbb{R})$, where we assume that $E_0=0$, $\mathbb{P}$-a.s. For each $i\ge1$, we denote by

$$\tau_i=\inf\{t\ge0:E_t\ge i\}$$

the time of the ith customer arrival to the system after time $t=0$ and we let η_i be the service time of the ith customer to arrive to the system after time $t=0$. We assume that $\{\eta_i,i\ge1\}$ is an i.i.d. sequence of nonnegative, mean 1 random variables with cumulative distribution function F and probability density function f. We also assume in this subsection that the hazard rate function h of F satisfies Assumption 2.1 of Section 2.1.

Now let $R=(R_t)_{t\ge0}\in\mathbb{D}([0,\infty),\mathbb{D})$ be such that for each $t\ge0$ and $y\in\mathbb{R}$, the quantity $R_t(y)$ denotes the number of customers in the system at time $t\ge0$ that have less than or equal to $y\in\mathbb{R}$ units of service remaining. For $y<0$, we interpret $R_t(y)$ as the number of customers who have departed from the system by time $t+y$. Thus, $(R_t(y))_{y\in\mathbb{R}}$ not only keeps tracks of the residual service times of those customers present in the system at time t, but it also records the departure times of all customers who have departed from the system by time t. We assume that at time $t=0$ there are $R_0(y)$ customers in the system that have less than or equal to y units of service time remaining. By first principles, it then follows that for each $t\geq 0$ and $y\in\mathbb{R}$ we may write

$$R_t(y)=R_0(t+y)+\sum_{i=1}^{E_t}\mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}. \tag{2.13}$$

The following proposition now presents an alternative expression for the right-hand side of (2.13).

PROPOSITION 2.5. *For each $t\geq 0$ and $y\in\mathbb{R}$,*

$$\begin{aligned} R_t(y)&=R_0(y)+(R_0(t+y)-R_0(y))\\ &\quad+\sum_{i=1}^{E_t}\mathbf{1}_{\{\eta_i\leq y\}}+\sum_{i=1}^{E_t}\mathbf{1}_{\{\eta_i>y,(\tau_i+\eta_i)-t\leq y\}}. \end{aligned}\tag{2.14}$$

PROOF. By (2.13),

$$\begin{aligned} R_t(y)&=R_0(t+y)+\sum_{i=1}^{E_t}\mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}\\ &=R_0(y)+(R_0(t+y)-R_0(y))+\sum_{i=1}^{E_t}\mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}. \end{aligned}\tag{2.15}$$

However, note that

$$\mathbf{1}_{\{\eta_i-(t-\tau_i)\leq y\}}=\mathbf{1}_{\{\eta_i\leq y\}}+\mathbf{1}_{\{\eta_i>y,\eta_i-(t-\tau_i)\leq y\}}. \tag{2.16}$$

Substituting (2.16) into (2.15) and summing over E_t, completes the proof. □

We now give an intuitive description for each of the terms appearing in (2.15). The first term represents the number of customers in the system at time $t=0$ with less than or equal to y units of service time remaining. The second term represents the number of customers in the system at time $t=0$ with greater than y units of total service time but at time $t\geq 0$ have less than or equal to y units of service time remaining. The third and fourth terms in (2.15) have analogous descriptions but for those customers that arrive to the system after time $t=0$.

Now let $G=(G_t)_{t\geq 0}\in\mathbb{D}([0,\infty),\mathbb{D})$ be defined by setting

$$G_t(y)=\sum_{i=1}^{E_t}(\mathbf{1}_{\{\eta_i\leq y\}}-F(y)),\qquad t\geq 0,y\in\mathbb{R}.$$

By Proposition 2.5, it then follows that for each $t \geq 0$ and $y \in \mathbb{R}$, $R_t(y)$ may be written as

$$\begin{aligned} R_t(y) &= R_0(y) + (R_0(t+y) - R_0(y)) + G_t(y) + E_t F(y) \\ &\quad + \sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}}. \end{aligned} \tag{2.17}$$

The above representation for $R_t(y)$ will be useful in a moment. However, we first proceed to define tempered distribution-valued versions of the processes defined above. In particular, we let $\mathcal{R} = (\mathcal{R}_t)_{t \geq 0}$ be the $\mathcal{S}'$-valued process associated with R such that for each $t \geq 0$ and $\varphi \in \mathcal{S}$ we have that

$$\langle \mathcal{R}_t, \varphi \rangle = \int_{\mathbb{R}} \varphi(y)\, dR_t(y)$$

and we let $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ be the $\mathcal{S}'$-valued process associated with G such that for each $t \geq 0$ and $\varphi \in \mathcal{S}$ we have that

$$\langle \mathcal{G}_t, \varphi \rangle = \int_{\mathbb{R}} \varphi(y)\, dG_t(y).$$

We also define the elements $\mathcal{F}$ and $\mathcal{F}_e$ of $\mathcal{S}'$ by setting

$$\langle \mathcal{F}, \varphi \rangle \equiv \int_0^\infty \varphi(x)\, dF(x), \qquad \varphi \in \mathcal{S},$$

and

$$\langle \mathcal{F}_e, \varphi \rangle \equiv \int_0^\infty \varphi(x)(1 - F(x))\, dx, \qquad \varphi \in \mathcal{S}.$$

Now note that integrating test functions $\varphi \in \mathcal{S}$ against each of the terms in (2.17), it follows that for each $\varphi \in \mathcal{S}$ and $t \geq 0$,

$$\begin{aligned} \langle \mathcal{R}_t, \varphi \rangle &= \langle \mathcal{R}_0, \varphi \rangle + \int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) \\ &\quad + \langle \mathcal{G}_t, \varphi \rangle + E_t \langle \mathcal{F}, \varphi \rangle + \int_{\mathbb{R}} \varphi(y)\, d\left(\sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}} \right). \end{aligned} \tag{2.18}$$

The following proposition now allows us to simplify the right-hand side of (2.18).

Proposition 2.6. *For each $t \geq 0$,*

$$\begin{aligned} &\int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) + \int_{\mathbb{R}} \varphi(y)\, d\left(\sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}} \right) \\ &\quad = -\int_0^t \langle \mathcal{R}_s, \varphi' \rangle\, ds. \end{aligned} \tag{2.19}$$

PROOF. The proof parallels the proof of Proposition 2.4. Let $t \geq 0$. Then, integrating by parts, we have that

$$\begin{aligned}
&-\int_{\mathbb{R}} \varphi(y)\, d(R_0(t+y) - R_0(y)) - \int_{\mathbb{R}} \varphi(y)\, d\left(\sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}}\right) \\
&\quad = \int_{\mathbb{R}} \left(\sum_{i=1}^{R_0(\infty)} \mathbf{1}_{\{y < \tilde{\eta}_i \leq t + y\}}\right) \varphi'(y)\, dy + \int_{\mathbb{R}} \left(\sum_{i=1}^{E_t} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}}\right) \varphi'(y)\, dy \\
&\quad = \sum_{i=1}^{R_0(\infty)} \int_{\mathbb{R}} \mathbf{1}_{\{y < \tilde{\eta}_i \leq t + y\}} \varphi'(y)\, dy + \sum_{i=1}^{E_t} \int_{\mathbb{R}} \mathbf{1}_{\{\eta_i > y, (\tau_i + \eta_i) - t \leq y\}} \varphi'(y)\, dy \\
&\quad = \sum_{i=1}^{R_0(\infty)} \int_{\mathbb{R}} \mathbf{1}_{\{\tilde{\eta}_i - t \leq y < \tilde{\eta}_i\}} \varphi'(y)\, dy + \sum_{i=1}^{E_t} \int_{\mathbb{R}} \mathbf{1}_{\{(\tau_i + \eta_i) - t \leq y < \eta_i\}} \varphi'(y)\, dy \\
&\quad = 7 \sum_{i=1}^{R_0(\infty)} \int_0^t \varphi'(\tilde{\eta}_i - s)\, ds + \sum_{i=1}^{E_t} \int_0^t \mathbf{1}_{\{\tau_i \leq s\}} \varphi'(\tau_i + \eta_i - s)\, ds \\
&\quad = \int_0^t \left(\sum_{i=1}^{R_0(\infty)} \varphi'(\tilde{\eta}_i - s) + \sum_{i=1}^{E_t} \mathbf{1}_{\{\tau_i \leq s\}} \varphi'(\tau_i + \eta_i - s)\right) ds \\
&\quad = \int_0^t \left(\int_{\mathbb{R}_+} \varphi'\, dR_s(u)\right) ds \\
&\quad = \int_0^t \langle \mathcal{R}_s, \varphi' \rangle\, ds.
\end{aligned}$$

This completes the proof. □

Substituting (2.19) into (2.18), one now obtains that for each $t \geq 0$ and $\varphi \in \mathcal{S}$,

$$\langle \mathcal{R}_t, \varphi \rangle = \langle \mathcal{R}_0, \varphi \rangle + \langle \mathcal{G}_t, \varphi \rangle + E_t \langle \mathcal{F}, \varphi \rangle - \int_0^t \langle \mathcal{R}_s, \varphi' \rangle\, ds. \tag{2.20}$$

We refer to (2.20) as the semi-martingale decomposition of $\mathcal{R}$. This is due to the fact that in Section 4 it will be shown that the process $\mathcal{G}$ in (2.20) is a martingale. We also point out the similarity of (2.20) with (4) of [8].

3. Regulator map result.

Let $B : \mathcal{S} \mapsto \mathcal{S}$ be a continuous linear operator and for each $\mu \in \mathbb{D}([0, T], \mathcal{S}')$, consider the solution $\nu \in \mathbb{D}([0, T], \mathcal{S}')$ to the integral equation

$$\langle \nu_t, \varphi \rangle = \langle \mu_t, \varphi \rangle + \int_0^t \langle \nu_s, B\varphi \rangle\, ds, \qquad t \in [0, T], \varphi \in \mathcal{S}. \tag{3.1}$$

In this section, we first show that under some mild restrictions on B, (3.1) defines a continuous function $\Psi_B : \mathbb{D}([0,T], \mathcal{S}') \to \mathbb{D}([0,T], \mathcal{S}')$ mapping μ to ν. We then proceed in Sections 3.1 and 3.2 to study particular continuous linear operators associated with the age process and the residual service time process, respectively.

We begin with the following definition from [19].

DEFINITION 3.1. A family $(S_t)_{t\geq 0}$ of linear operators on $\mathcal{S}$ is said to be a strongly-continuous $(C_0, 1)$ semigroup if the following three conditions are satisfied:

(1) $S_0 = I$, where I is the identity operator, and, for all $s, t \geq 0$, $S_s S_t = S_{s+t}$.

(2) The map $t \to S_t$ is continuous in the strong topology of $L(\mathcal{S}, \mathcal{S})$. That is, if $t_n \to t$ in $\mathbb{R}_+$, then for any bounded subset $K \subset \mathcal{S}$ and $p \geq 1$,

$$\sup_{\varphi \in K} \|S_{t_n}\varphi - S_t\varphi\|_p \to 0.$$

(3) For each $q \geq 0$, there exist numbers M_q, σ_q and $p \geq q$ such that for all $\varphi \in \mathcal{S}$ and $t \geq 0$,

$$\|S_t\varphi\|_q \leq M_q e^{\sigma_q t}\|\varphi\|_p.$$

REMARK 3.2. Note that condition (2) of Definition 3.1 is stronger than the corresponding condition for a $(C_0, 1)$ semigroup as given, for example, in [19]. The weaker definition in [19] only requires that the map $t \to S_t\varphi$ be continuous in the weak topology of $L(\mathcal{S}, \mathcal{S})$. That is, if $t_n \to t$ in $\mathbb{R}_+$, then for all $\varphi \in \mathcal{S}$ and $m \geq 1$, $\|S_{t_n}\varphi - S_t\varphi\|_m \to 0$.

Recall now that for a family $(S_t)_{t\geq 0}$ of linear operators on $\mathcal{S}$, the infinitesimal generator B of $(S_t)_{t\geq 0}$ is defined to be such that

$$B\varphi = \lim_{t \to 0} \frac{S_t\varphi - \varphi}{t} \qquad \text{in } \mathcal{S},$$

for all such $\varphi \in \mathcal{S}$ that the limit on the right-hand side above exits. We refer to such $\varphi \in \mathcal{S}$ as $\mathcal{D}(B)$, the domain of B. We now have the following result, which is our main result regarding the integral equation (3.1).

THEOREM 3.3. *Let $B \in L(\mathcal{S}, \mathcal{S})$ be the infinitesimal generator of a strongly-continuous $(C_0, 1)$ semigroup $(S_t)_{t\geq 0}$. Then, for each $\mu \in \mathbb{D}([0,T], \mathcal{S}')$, the equation (3.1) has a unique solution given by*

$$\langle \nu_t, \varphi\rangle = \langle \mu_t, \varphi\rangle + \int_0^t \langle \mu_s, S_{t-s}B\varphi\rangle\, ds, \qquad t \in [0,T], \varphi \in \mathcal{S}. \tag{3.2}$$

Furthermore, (3.2) defines a continuous function $\Psi_B : \mathbb{D}([0,T], \mathcal{S}') \to \mathbb{D}([0,T], \mathcal{S}')$ mapping μ to ν.

PROOF. That Ψ_B is a well-defined function from $\mathbb{D}([0,T],\mathcal{S}')$ to $\mathbb{D}([0,T],\mathcal{S}')$ and the form of the solution (3.2) follows from Theorem 2.1 of [19] (see also Corollary 2.2 of [19]).

We now show that Ψ_B is continuous. By (3.2), it suffices to show that the function mapping $\mathbb{D}([0,T],\mathcal{S}')$ to $\mathbb{D}([0,T],\mathcal{S}')$ defined by $\mu \mapsto \int_0^{\cdot} B^* S^*_{\cdot - s}\mu_s\,ds$, where B^* and S_t^* denote the adjoint operators of B and S_t, respectively, is continuous. Let $(\mu^n)_{n\ge 1}$ be a sequence converging to μ in $\mathbb{D}([0,T],\mathcal{S}')$. Then, by Definition 1.4 and the comment below it (see also Theorem 2.4.1 of [21]), there exists a sequence $(\lambda^n)_{n\ge1}$ of strictly increasing homeomorphisms of the interval $[0,T]$ such that for each bounded set $K\subset\mathcal{S}$,

$$\sup_{0\le t\le T}\sup_{\varphi\in K}|\langle \mu_t^n - \mu_{\lambda_t^n},\varphi\rangle| \to 0 \quad \text{and} \quad \sup_{0\le t\le T}|\lambda_t^n - t|\to 0 \qquad \text{as } n\to\infty. \tag{3.3}$$

Moreover, it suffices to consider homeomorphisms $(\lambda^n)_{n\ge1}$ that are absolutely continuous with respect to Lebesgue measure on $[0,T]$ having corresponding derivatives $(\dot\lambda^n)_{n\ge1}$ satisfying $\|\dot\lambda^n - 1\|_T\to 0$ as $n\to\infty$ (see pages 112–114 of [1]). It then follows that for each $t\in[0,T]$ and $\varphi\in\mathcal{S}$ we may write

$$\begin{aligned}
&\left|\left\langle \int_0^t B^* S^*_{t-s}\mu_s^n\,ds - \int_0^{\lambda_t^n} B^* S^*_{\lambda_t^n - s}\mu_s\,ds,\varphi\right\rangle\right| \\
&\quad = \left|\left\langle \int_0^t B^* S^*_{t-s}\mu_s^n\,ds - \int_0^t B^* S^*_{\lambda_t^n-\lambda_s^n}\mu_{\lambda_s^n}\dot\lambda_s^n\,ds,\varphi\right\rangle\right| \\
&\quad \le \|\dot\lambda^n - 1\|_T\left|\left\langle \int_0^t B^* S^*_{\lambda_t^n-\lambda_s^n}\mu_{\lambda_s^n}\,ds,\varphi\right\rangle\right| \\
&\qquad + \left|\left\langle \int_0^t B^*(S^*_{t-s} - S^*_{\lambda_t^n-\lambda_s^n})\mu_{\lambda_s^n}\,ds,\varphi\right\rangle\right| \\
&\qquad + \left|\left\langle \int_0^t B^* S^*_{t-s}(\mu_s^n - \mu_{\lambda_s^n})\,ds,\varphi\right\rangle\right| \\
&\quad = \|\dot\lambda^n - 1\|_T\left|\int_0^t \langle \mu_{\lambda_s^n}, S_{\lambda_t^n-\lambda_s^n}B\varphi\rangle\,ds\right| \\
&\qquad + \left|\int_0^t\langle \mu_{\lambda_s^n},(S_{t-s} - S_{\lambda_t^n-\lambda_s^n})B\varphi\rangle\,ds\right| + \left|\int_0^t \langle \mu_s^n - \mu_{\lambda_s^n}, S_{t-s}B\varphi\rangle\,ds\right|.
\end{aligned} \tag{3.4}$$

In order to complete the proof, it now suffices to show that for each bounded subset $K\subset\mathcal{S}$, the following three limits hold:

$$\sup_{t\in[0,T]}\sup_{\varphi\in K}\|\dot\lambda^n - 1\|_T\left|\int_0^t\langle \mu_{\lambda_s^n}, S_{\lambda_t^n-\lambda_s^n}B\varphi\rangle\,ds\right| \to 0 \qquad \text{as } n\to\infty, \tag{3.5}$$

$$\sup_{t\in[0,T]}\sup_{\varphi\in K}\left|\int_0^t \langle \mu_{\lambda_s^n}, (S_{t-s}-S_{\lambda_t^n-\lambda_s^n})B\varphi\rangle\,ds\right| \to 0 \qquad \text{as } n\to\infty, \tag{3.6}$$

$$\sup_{t\in[0,T]}\sup_{\varphi\in K}\left|\int_0^t \langle \mu_s^n-\mu_{\lambda_s^n}, S_{t-s}B\varphi\rangle\,ds\right| \to 0 \qquad \text{as } n\to\infty. \tag{3.7}$$

We begin with (3.5). First note that for each bounded subset $K\subset\mathcal{S}$,

$$\sup_{s\in[0,T]}\sup_{\varphi\in K}|\langle\mu_s,\varphi\rangle|<\infty. \tag{3.8}$$

Now let $K\subset\mathcal{S}$ be an arbitrary bounded set. We show that the set $K'\equiv\{S_uB\varphi, u\in[0,T],\varphi\in K\}$ is bounded in $\mathcal{S}$ as well. Indeed, note that by Definition 3.1, we have that for each $q\ge 0$ there exist M_q, σ_q and $p\ge q$ such that

$$\sup_{\varphi\in K'}\|\varphi\|_q=\sup_{u\in[0,T],\varphi\in K}\|S_uB\varphi\|_q\le M_q e^{\sigma_q T}\sup_{\varphi\in K}\|B\varphi\|_p<\infty, \tag{3.9}$$

where the final inequality follows from the fact that continuous linear operators from $\mathcal{S}$ to $\mathbb{R}$ map bounded sets of $\mathcal{S}$ to bounded sets of $\mathbb{R}$ (see Theorem 1.2.1 of [21]). Thus, $K'=\{S_uB\varphi,u\in[0,T],\varphi\in K\}$ is bounded and so by (3.8),

$$\sup_{s\in[0,T]}\sup_{u\in[0,T],\varphi\in K}|\langle\mu_s,S_uB\varphi\rangle|<\infty.$$

It therefore follows since $\|\dot\lambda^n-1\|_T\to 0$ as $n\to\infty$, that

$$\begin{aligned}&\sup_{t\in[0,T]}\sup_{\varphi\in K}\|\dot\lambda^n-1\|_T\left|\int_0^t\langle\mu_{\lambda_s^n},S_{\lambda_t^n-\lambda_s^n}B\varphi\rangle\,ds\right|\\&\quad\le\|\dot\lambda^n-1\|_T T\sup_{s\in[0,T]}\sup_{u\in[0,T],\varphi\in K}|\langle\mu_s,S_uB\varphi\rangle|\,ds\to 0,\end{aligned}\tag{3.10}$$

as $n\to\infty$, which implies (3.5).

We next prove the limit (3.6). First note that by Lemma 2.2 of [18], there exist $\theta\ge 0$ and $q\ge 1$ such that

$$\begin{aligned}&\sup_{t\in[0,T]}\sup_{\varphi\in K}\left|\int_0^t\langle\mu_{\lambda_s^n},(S_{t-s}-S_{\lambda_t^n-\lambda_s^n})B\varphi\rangle\,ds\right|\\&\quad\le T\sup_{s\in[0,T]}\sup_{0\le w\le v\le T}\sup_{\varphi\in K}|\langle\mu_s,(S_{v-w}-S_{\lambda_v^n-\lambda_w^n})B\varphi\rangle|\\&\quad\le T\theta\sup_{0\le w\le v\le T}\sup_{\varphi\in K}\|(S_{v-w}-S_{\lambda_v^n-\lambda_w^n})B\varphi\|_q.\end{aligned}\tag{3.11}$$

Next, note that for $w\le v$ we may write

$$\begin{aligned}S_{v-w}-S_{\lambda_v^n-\lambda_w^n}&=\mathbf{1}_{\{(v-w)-(\lambda_v^n-\lambda_w^n)\ge 0\}}(S_{(v-w)-(\lambda_v^n-\lambda_w^n)}-I)S_{\lambda_v^n-\lambda_w^n}\\&\quad+\mathbf{1}_{\{(\lambda_v^n-\lambda_w^n)-(v-w)\ge 0\}}(I-S_{(\lambda_v^n-\lambda_w^n)-(v-w)})S_{v-w}.\end{aligned}$$

Thus, recalling the definition of the set $K' = \{S_u B\varphi, u \in [0,T], \varphi \in K\}$, it follows that

$$
\begin{aligned}
&\sup_{0\le w\le v\le T}\sup_{\varphi\in K}\|(S_{v-w} - S_{\lambda_v^n-\lambda_w^n})B\varphi\|_q \\
&\quad\le \sup_{0\le w\le v\le T}\sup_{\varphi\in K}\|\mathbf{1}_{\{(v-w)-(\lambda_v^n-\lambda_w^n)\ge 0\}}(S_{(v-w)-(\lambda_v^n-\lambda_w^n)} - I)S_{\lambda_v^n-\lambda_w^n}B\varphi\|_q \\
&\qquad + \sup_{0\le w\le v\le T}\sup_{\varphi\in K}\|\mathbf{1}_{\{(\lambda_v^n-\lambda_w^n)-(v-w)\ge 0\}}(I - S_{(\lambda_v^n-\lambda_w^n)-(v-w)})S_{v-w}B\varphi\|_q \qquad (3.12)\\
&\quad\le \sup_{0\le w\le v\le T}\sup_{\varphi\in K'}\|\mathbf{1}_{\{(v-w)-(\lambda_v^n-\lambda_w^n)\ge 0\}}(S_{(v-w)-(\lambda_v^n-\lambda_w^n)} - I)\varphi\|_q \\
&\qquad + \sup_{0\le w\le v\le T}\sup_{\varphi\in K'}\|\mathbf{1}_{\{(\lambda_v^n-\lambda_w^n)-(v-w)\ge 0\}}(I - S_{(\lambda_v^n-\lambda_w^n)-(v-w)})\varphi\|_q.
\end{aligned}
$$

However, note that since $\|\dot{\lambda}^n - 1\|_T \to 0$ as $n \to \infty$, it follows that

$$\sup_{0\le w\le v\le T}|(v-w) - (\lambda_v^n - \lambda_w^n)| \to 0 \qquad \text{as } n\to\infty.$$

Thus, since by (3.9) the set $K' \subset \mathcal{S}$ is bounded, it follows by part (2) of Definition 3.1 that

$$
\begin{aligned}
&\sup_{0\le w\le v\le T}\sup_{\varphi\in K'}\|\mathbf{1}_{\{(v-w)-(\lambda_v^n-\lambda_w^n)\ge 0\}}(S_{(v-w)-(\lambda_v^n-\lambda_w^n)} - I)\varphi\|_q \\
&\quad + \sup_{0\le w\le v\le T}\sup_{\varphi\in K'}\|\mathbf{1}_{\{(\lambda_v^n-\lambda_w^n)-(v-w)\ge 0\}}(I - S_{(\lambda_v^n-\lambda_w^n)-(v-w)})\varphi\|_q \\
&\quad \to 0 \qquad \text{as } n\to\infty,
\end{aligned}
$$

which, by (3.11) and (3.12), implies (3.6).

Finally, (3.7) follows from the fact that

$$
\begin{aligned}
&\sup_{t\in[0,T]}\sup_{\varphi\in K}\left|\int_0^t \langle \mu_s^n - \mu_{\lambda_s^n}, S_{t-s}B\varphi\rangle\, ds\right| \\
&\qquad \le T\sup_{s\in[0,T]}\sup_{u\in[0,T],\varphi\in K}|\langle \mu_s^n - \mu_{\lambda_s^n}, S_u B\varphi\rangle| \to 0,
\end{aligned} \qquad (3.13)
$$

as $n\to\infty$, where the final convergence follows from (3.3) and (3.9). This completes the proof. □

3.1. *Age process.* Now let $B^{\mathcal{A}}$ be the linear operator defined on $\mathcal{S}$ such that

$$B^{\mathcal{A}}\varphi = \varphi' - h\varphi \qquad \text{for all } \varphi\in\mathcal{S}. \qquad (3.14)$$

Our main result in this subsection is to verify that $B^{\mathcal{A}}$ generates a strongly-continuous $(C_0,1)$ semigroup. This will then be useful in Sections 5.1 and 6.1, where we prove our fluid and diffusion limits, respectively, for the age process. In particular, by Theorem 3.3 of the preceding subsection and (2.12) of Section 2.1, this will then allow us to write

$$\mathcal{A} = \Psi_{B^{\mathcal{A}}}(\mathcal{A}_0 + \mathcal{E} - (\mathcal{D}^0 + \mathcal{D})), \tag{3.15}$$

where the map $\Psi_{B^{\mathcal{A}}} : \mathbb{D}([0,T], \mathcal{S}') \mapsto \mathbb{D}([0,T], \mathcal{S}')$ is a continuous map. We begin with the following lemma. Its proof may be found in the Appendix.

LEMMA 3.4. *(1) For each $n \geq 1$ and $t \geq 0$,*

$$\sup_{x \geq 0} \left| \left(\frac{\bar{F}(x+t)}{\bar{F}(x)} \right)^{(n)} \right| < \infty. \tag{3.16}$$

(2) For each $T > 0$, there exists a sequence $(M_n)_{n \geq 0}$ such that for each $s, t \in [0, T]$,

$$\sup_{x \geq 0} \left| \left(\frac{\bar{F}(x+t)}{\bar{F}(x)} - \frac{\bar{F}(x+s)}{\bar{F}(x)} \right)^{(n)} \right| \leq M_n |t - s|. \tag{3.17}$$

PROOF. See the Appendix. □

Next, we have the following.

LEMMA 3.5. *For each $\varphi \in \mathcal{S}$, $t \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}$, we have*

$$\|\tau_t \varphi\|_{\alpha,\beta} \leq (1 \vee |t|^\alpha) 2^\alpha \max_{0 \leq i \leq \alpha} \|\varphi\|_{i,\beta}. \tag{3.18}$$

PROOF. For each $\varphi \in \mathcal{S}$, $t \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}$, we have

$$\begin{aligned}
\|\tau_t \varphi\|_{\alpha,\beta} = \|\varphi(\cdot - t)\|_{\alpha,\beta} &= \sup_{x \in \mathbb{R}} |x^\alpha \varphi^{(\beta)}(x-t)| \\
&= \sup_{x \in \mathbb{R}} |[t + (x-t)]^\alpha \varphi^{(\beta)}(x-t)| \\
&= \sup_{x \in \mathbb{R}} \left| \sum_{i=0}^{\alpha} \binom{\alpha}{i} t^{\alpha-i} (x-t)^i \varphi^{(\beta)}(x-t) \right| \\
&\leq \sum_{i=0}^{\alpha} \binom{\alpha}{i} |t|^{\alpha-i} \|\varphi\|_{i,\beta} \leq (1 \vee |t|^\alpha) 2^\alpha \max_{0 \leq i \leq \alpha} \|\varphi\|_{i,\beta}.
\end{aligned}$$

This completes the proof. □

The following is now our main result of this subsection.

PROPOSITION 3.6. *The linear operator $B^{\mathcal{A}}$ defined by (3.14) generates a strongly-continuous $(C_0, 1)$ semigroup $(S_t^{\mathcal{A}})_{t\geq 0}$ given by*

$$S_t^{\mathcal{A}}\varphi = (1/\bar{F})\tau_{-t}(\bar{F}\varphi) \qquad \text{for all } \varphi \in \mathcal{S}. \tag{3.19}$$

PROOF. We first check that $B^{\mathcal{A}}$ is indeed the infinitesimal generator of the semi-group given by (3.19). In order to do so, it suffices to check that for each $\alpha, \beta \in \mathbb{N}$,

$$\lim_{t\to 0}\left\|\frac{S_t^{\mathcal{A}}\varphi - \varphi}{t} - (\varphi' - h\varphi)\right\|_{\alpha,\beta} = 0. \tag{3.20}$$

We begin by noting that for each $t \geq 0, x \in \mathbb{R}$ and $\varphi \in \mathcal{S}$, we have that

$$|S_t^{\mathcal{A}}\varphi(x)| = \left|\frac{1 - F(x-t)}{1 - F(x)} \cdot \tau_t\varphi(x)\right| \leq e^{\|h\|_\infty t} \cdot |\tau_t\varphi(x)|,$$

and so it follows that $S_t^{\mathcal{A}}\varphi \in \mathcal{S}$. Now let $x \in \mathbb{R}$ be fixed and note that for each $t \geq 0$, we may write

$$S_t^{\mathcal{A}}\varphi(x) = \exp\left(-\int_x^{x+t} h(v)\,dv\right)\varphi(x+t).$$

Hence, by Taylor's theorem, expanding in terms of t we obtain that we may write

$$S_t^{\mathcal{A}}\varphi(x) = \varphi(x) + (\varphi'(x) - h(x)\varphi(x))t + R(x,t), \tag{3.21}$$

where the remainder term $R(x,t)$ has the from

$$R(x,t) = \frac{1}{2}\int_0^t \frac{d^2}{du^2}\left(\exp\left(-\int_x^{x+u} h(v)\,dv\right)\varphi(x+u)\right)u\,du.$$

Now differentiating with respect to x in (3.21), we obtain that for each $\alpha, \beta \in \mathbb{N}$,

$$\lim_{t\to 0}\left\|\frac{S_t^{\mathcal{A}}\varphi - \varphi}{t} - (\varphi' - h\varphi)\right\|_{\alpha,\beta} = \lim_{t\to 0}\sup_{x\in\mathbb{R}}\left|x^\alpha\frac{R^{(\beta)}(x,t)}{t}\right|,$$

where $R^{(\beta)}(x,t)$ denotes the βth derivative of $R(x,t)$ with respect to x. Hence, in order to verify (3.20) it now suffices to show that for each $\alpha, \beta \in \mathbb{N}$,

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}\left|x^\alpha\frac{R^{(\beta)}(x,t)}{t}\right| = 0.$$

First note that for each $x \in \mathbb{R}$ fixed, we may write

$$\begin{aligned} &x^\alpha R^{(\beta)}(x,t) \\ &\quad = \frac{1}{2}\int_0^t x^\alpha\left(\frac{d^2}{du^2}\left(\exp\left(-\int_x^{x+u} h(v)\,dv\right)\varphi(x+u)\right)\right)^{(\beta)} u\,du. \end{aligned} \tag{3.22}$$

However, since $h \in C_b(\mathbb{R}_+)$ and $\varphi \in \mathcal{S}$, it is straightforward to show that for each $\alpha, \beta \in \mathbb{N}$, there exists a constant $C_{\alpha,\beta} < \infty$ such that for u sufficiently small,

$$\sup_{x\in\mathbb{R}} \left| x^\alpha \left(\frac{d^2}{du^2} \left(\exp\left(- \int_x^{x+u} h(v)\, dv \right) \varphi(x+u) \right) \right)^{(\beta)} \right| < C_{\alpha,\beta}.$$

Hence, by (3.22) we obtain that

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}} \left| x^\alpha \frac{R^{(\beta)}(x,t)}{t} \right| < \lim_{t\to 0} \left| \frac{C_{\alpha,\beta}}{2t} \int_0^t u\, du \right| = \lim_{t\to 0} \frac{C_{\alpha,\beta}}{4} t = 0.$$

This now completes the verification of the fact that $B^{\mathcal{A}}$ is the infinitesimal generator of the semigroup $(S_t^{\mathcal{A}})_{t\geq 0}$ given by (3.19).

We next verify that the semigroup $(S_t^{\mathcal{A}})_{t\geq 0}$ is a strongly-continuous $(C_0, 1)$ semi-group. It is straightforward to see that part (1) of Definition 3.1 is satisfied. We next check that part (2) of Definition 3.1 is satisfied. Consider $0 \leq s < t$, a bounded set $K \subset \mathcal{S}$, and $\alpha, \beta \in \mathbb{N}$. We then have that we may write

$$\begin{aligned}
&\sup_{\varphi\in K} \| S_s^{\mathcal{A}}\varphi - S_t^{\mathcal{A}}\varphi \|_{\alpha,\beta} \\
&\quad = \sup_{\varphi\in K} \| (1/\bar F)\tau_{-s}(\bar F\varphi) - (1/\bar F)\tau_{-t}(\bar F\varphi) \|_{\alpha,\beta} \\
&\quad = \sup_{\varphi\in K} \sup_{x\in\mathbb{R}} \left| x^\alpha \left(\frac{\bar F(x+s)}{\bar F(x)} \varphi(x+s) - \frac{\bar F(x+t)}{\bar F(x)} \varphi(x+t) \right)^{(\beta)} \right| \\
&\quad \leq \sup_{\varphi\in K} \sup_{x\in\mathbb{R}} \left| x^\alpha \left(\left(\left(\frac{\bar F(x+s)}{\bar F(x)} - \frac{\bar F(x+t)}{\bar F(x)} \right) \varphi(x+s) \right)^{(\beta)} \right) \right| \\
&\qquad + \sup_{\varphi\in K} \sup_{x\in\mathbb{R}} \left| x^\alpha \left(\frac{\bar F(x+t)}{\bar F(x)} (\varphi(x+s) - \varphi(x+t)) \right)^{(\beta)} \right|.
\end{aligned} \tag{3.23}$$

We now handle each of the terms in (3.23) separately.

For the first term in (3.23), note that by Lemmas 3.4 and 3.5 we have that

$$\begin{aligned}
&\sup_{\varphi\in K} \sup_{x\in\mathbb{R}} \left| x^\alpha \left[\left(\left(\frac{\bar F(x+s)}{\bar F(x)} - \frac{\bar F(x+t)}{\bar F(x)} \right) \varphi(x+s) \right)^{(\beta)} \right] \right| \\
&\quad = \sup_{\varphi\in K} \sup_{x\in\mathbb{R}} \sum_{i=0}^{\beta} \binom{\beta}{i} \left| x^\alpha \left(\frac{\bar F(x+s)}{\bar F(x)} - \frac{\bar F(x+t)}{\bar F(x)} \right)^{(\beta-i)} \varphi^{(i)}(x+s) \right| \\
&\quad = |t-s| \sum_{i=0}^{\beta} M_{\beta-i} \binom{\beta}{i} \sup_{\varphi\in K} \sup_{x\in\mathbb{R}} |x^\alpha \varphi^{(i)}(x+s)|
\end{aligned} \tag{3.24}$$

$$\leq |t-s|(1 \vee |t|^{\alpha})2^{\alpha} \sum_{i=0}^{\beta} M_{\beta-i} \binom{\beta}{i} \sup_{\varphi \in K} \max_{0 \leq j \leq \alpha} \|\varphi\|_{j,i}.$$

We next focus on the second term in (3.23). For each $n \geq 1$, denote the left-hand side of (3.16) by L_n. By the mean value theorem, for each $x \in \mathbb{R}$ there exists an $r_x \in [s,t]$ such that

$$\begin{aligned}
&\sup_{\varphi \in K} \sup_{x \in \mathbb{R}} \left| x^{\alpha} \left(\frac{\bar{F}(x+t)}{\bar{F}(x)} (\varphi(x+s) - \varphi(x+t)) \right)^{(\beta)} \right| \\
&\quad = \sup_{\varphi \in K} \sup_{x \in \mathbb{R}} \sum_{i=0}^{\beta} \binom{\beta}{i} \left| x^{\alpha} \frac{\bar{F}(x+t)}{\bar{F}(x)}^{(\beta-i)} (\varphi^{(i)}(x+s) - \varphi^{(i)}(x+t)) \right| \\
&\quad = \sup_{\varphi \in K} \sup_{x \in \mathbb{R}} \sum_{i=0}^{\beta} \binom{\beta}{i} L_{\beta-i} |x^{\alpha}(\varphi^{(i)}(x+s) - \varphi^{(i)}(x+t))| \\
&\quad \leq |t-s| \sup_{\varphi \in K} \sup_{x \in \mathbb{R}} \sum_{i=0}^{\beta} \binom{\beta}{i} L_{\beta-i} |x^{\alpha} \varphi^{(i+1)}(x+r_x)| \\
&\quad \leq |t-s|(1 \vee |t|^{\alpha})2^{\alpha} \sum_{i=0}^{\beta} \binom{\beta}{i} L_{\beta-i} \sup_{\varphi \in K} \max_{0 \leq j \leq \alpha} \|\varphi\|_{j,i+1},
\end{aligned} \tag{3.25}$$

where the final inequality follows as a consequence of Lemma 3.5. Combining (3.24) and (3.25) with (3.23) and taking the limit as $s \to t$ now yields part (2) of Definition 3.1.

We now complete the proof by verifying that part (3) of Definition 3.1 is satisfied. First note that for each $\varphi \in \mathcal{S}$, $t \geq 0$ and $\alpha, \beta \in \mathbb{N}$, we may write

$$\begin{aligned}
\|S_t^{\mathcal{A}} \varphi\|_{\alpha,\beta} &= \sup_{x \in \mathbb{R}} \left| x^{\alpha} \left(\frac{\bar{F}(x+t)}{\bar{F}(x)} \varphi(x+t) \right)^{(\beta)} \right| \\
&= \sup_{x \in \mathbb{R}} \left| x^{\alpha} \sum_{i=0}^{\beta} \binom{\beta}{i} \left(\frac{\bar{F}(x+t)}{\bar{F}(x)} \right)^{(\beta-i)} \varphi^{(i)}(x+t) \right| \\
&\leq \sum_{i=0}^{\beta} \binom{\beta}{i} L_{\beta-i} \sup_{x \in \mathbb{R}} |x^{\alpha} \varphi^{(i)}(x+t)| \\
&\leq (1 \vee |t|^{\alpha})2^{\alpha} \sum_{i=0}^{\beta} \binom{\beta}{i} L_{\beta-i} \max_{0 \leq j \leq \alpha} \|\varphi\|_{j,i} \\
&\leq (1 \vee |t|^{\alpha})2^{\alpha+\beta} \max_{0 \leq i \leq \beta} L_i \max_{0 \leq j \leq \alpha} \max_{0 \leq i \leq \beta} \|\varphi\|_{j,i},
\end{aligned}$$

where the final inequality above follows from Lemma 3.5. Part (3) of Definition 3.1 now follows from (1.3) and (1.4) of Section 1.1 and the fact that $\|\cdot\|_p \leq \|\cdot\|_{p+1}$ for each $p \in \mathbb{N}$. This completes the proof. $\square$

3.2. *Residual service time process.* Now let $B^{\mathcal{R}}$ be the linear operator defined on $\mathcal{S}$ such that

$$B^{\mathcal{R}}\varphi = -\varphi' \qquad \text{for all } \varphi \in \mathcal{S}. \tag{3.26}$$

In this subsection, we verify that $B^{\mathcal{R}}$ generates a strongly-continuous $(C_0, 1)$ semi-group. This will be useful in Sections 5.2 and 6.2, where we prove our fluid and diffusion limits, respectively, for the residual service time process. In particular, by (2.20) of Section 2.2 and Theorem 3.3, this then implies that we may write

$$\mathcal{R} = \Psi_{B^{\mathcal{R}}}(\mathcal{R}_0 + E\mathcal{F} + \mathcal{G}), \tag{3.27}$$

where the map $\Psi_{B^{\mathcal{R}}} : \mathbb{D}([0,T], \mathcal{S}') \mapsto \mathbb{D}([0,T], \mathcal{S}')$ is a continuous map.

Our main result of this subsection is the following.

Proposition 3.7. *The linear operator $B^{\mathcal{R}}$ defined by (3.26) generates the strongly-continuous $(C_0, 1)$ semigroup $(\tau_t)_{t\geq 0}$.*

Proof. First note that it is clear by Lemma 3.5 that for each $t \geq 0$ and $\varphi \in \mathcal{S}$, we have that $\tau_t\varphi \in \mathcal{S}$. We next check that for each $\alpha, \beta \in \mathbb{N}$, we have the convergence

$$\lim_{t\to 0}\left\|\frac{\tau_t\varphi - \varphi}{t} - (-\varphi')\right\|_{\alpha,\beta} = 0. \tag{3.28}$$

This will then be sufficient to verify that $B^{\mathcal{R}}$ as defined by (3.26) generates the semigroup $(\tau_t)_{t\geq 0}$. Let $x \in \mathbb{R}$ and $\varphi \in \mathcal{S}$ be fixed. It then follows by Taylor's theorem, expanding in terms of t, that we may write

$$\tau_t\varphi(x) = \varphi(x) - \varphi'(x)t + R(x,t), \tag{3.29}$$

where the remainder term $R(x,t)$ is given by

$$R(x,t) = \frac{1}{2}\int_0^{-t} \varphi''(x+u)(-t-u)\,du. \tag{3.30}$$

Now differentiating in (3.29) with respect to x, we obtain that

$$\lim_{t\to 0}\left\|\frac{\tau_t\varphi - \varphi}{t} - (-\varphi')\right\|_{\alpha,\beta} = \lim_{t\to 0}\sup_{x\in\mathbb{R}}\left|x^\alpha \frac{R^{(\beta)}(t,x)}{t}\right|,$$

where $R^{(\beta)}(x,t)$ denotes the βth derivative of R with respect to x. Thus, in order to prove (3.28), it now suffices to check that

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}\left|x^\alpha \frac{R^{(\beta)}(t,x)}{t}\right| = 0. \tag{3.31}$$

First note that by (3.30) we may write

$$\begin{aligned} x^\alpha R^{(\beta)}(x,t) &= \frac{x^\alpha}{2}\int_0^{-t}\varphi^{(\beta+2)}(x+u)(-t-u)\,du \\ &= \frac{1}{2}\int_0^{-t} x^\alpha\varphi^{(\beta+2)}(x+u)(-t-u)\,du. \end{aligned}$$

However, by Lemma 3.5 there exists a constant $C_{\alpha,\beta} < \infty$ such that for sufficiently small u,

$$\sup_{x\in\mathbb{R}}|x^\alpha\varphi^{(\beta+2)}(x+u)| < C_{\alpha,\beta}.$$

We therefore obtain that

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}}\left|x^\alpha\frac{R^{(\beta)}(t,x)}{t}\right| < \lim_{t\to 0}\left|\frac{C_{\alpha,\beta}}{2t}\int_0^{-t}(-t-u)\,du\right| = \lim_{t\to 0}\frac{C_{\alpha,\beta}}{4}t = 0,$$

thus proving (3.31). This completes our verification of the fact that $B^{\mathcal{R}}$ as defined by (3.26) generates the semigroup $(\tau_t)_{t\geq 0}$.

We next proceed to verify that $(\tau_t)_{t\geq 0}$ is a strongly-continuous $(C_0,1)$ semigroup. In order to check this fact, we will verify that $(\tau_t)_{t\geq 0}$ satisfies parts (1) through (3) of Definition 3.1. It is clear that $(\tau_t)_{t\geq 0}$ satisfies part (1) of Definition 3.1. We next check that $(\tau_t)_{t\geq 0}$ satisfies part (2) of Definition 3.1. Let $s<t$, $K\subset\mathcal{S}$ be a bounded set, and let $\alpha,\beta\in\mathbb{N}$. Then, by the mean value theorem, for each $x\in\mathbb{R}$ there exists an $r_x\in[x-t,x-s]$ such that

$$\begin{aligned} &\sup_{\varphi\in K}\|\tau_s\varphi-\tau_t\varphi\|_{\alpha,\beta} \\ &\qquad = \sup_{\varphi\in K}\sup_{x\in\mathbb{R}}|x^\alpha(\varphi^{(\beta)}(x-s)-\varphi^{(\beta)}(x-t))| \\ &\qquad = \sup_{\varphi\in K}\sup_{x\in\mathbb{R}}|(r_x+(x-r_x))^\alpha(\varphi^{(\beta)}(x-s)-\varphi^{(\beta)}(x-t))| \\ &\qquad = \sup_{\varphi\in K}\sup_{x\in\mathbb{R}}\left|\sum_{i=0}^{\alpha}\binom{\alpha}{i}r_x^{\alpha-i}(x-r_x)^i(\varphi^{(\beta)}(x-s)-\varphi^{(\beta)}(x-t))\right| \\ &\qquad = |t-s|\sup_{\varphi\in K}\sup_{x\in\mathbb{R}}\left|\sum_{i=0}^{\alpha}\binom{\alpha}{i}r_x^{\alpha-i}(x-r_x)^i\varphi^{(\beta+1)}(x-r_x)\right| \\ &\qquad \leq |t-s|\sum_{i=0}^{\alpha}\binom{\alpha}{i}(t+1)^{\alpha-i}\sup_{\varphi\in K}\|\varphi\|_{i,\beta+1}. \end{aligned}$$

Part (2) of Definition 3.1 now follows from the bound (1.3) and the fact that K is a bounded set.

We now complete the proof by verifying that $(\tau_t)_{t\geq 0}$ satisfies part (3) of Definition 3.1. Note that for each $\varphi \in \mathcal{S}$, $t \geq 0$ and $\alpha, \beta \in \mathbb{N}$, we have that

$$\begin{aligned}
\|\tau_t\varphi\|_{\alpha,\beta} &= \|\varphi(\cdot - t)\|_{\alpha,\beta} \\
&= \sup_{x\in\mathbb{R}} |x^\alpha \varphi^{(\beta)}(x-t)| \\
&= \sup_{x\in\mathbb{R}} |[t + (x-t)]^\alpha \varphi^{(\beta)}(x-t)| \\
&= \sup_{x\in\mathbb{R}} \left| \sum_{i=0}^{\alpha} \binom{\alpha}{i} t^{\alpha-i}(x-t)^i \varphi^{(\beta)}(x-t) \right| \\
&\leq \sum_{i=0}^{\alpha} \binom{\alpha}{i} t^{\alpha-i} \|\varphi\|_{i,\beta} \\
&\leq (1 \wedge t^\alpha) 2^\alpha \max_{0\leq i\leq\alpha} \|\varphi\|_{i,\beta}.
\end{aligned}$$

Part (3) of Definition 3.1 now follows as a result of the bounds (1.3) and (1.4). $\square$

4. Martingale results. In this section, we show that the process $\mathcal{D}^0 + \mathcal{D}$ defined in Section 2.1 and the process $\mathcal{G}$ defined in Section 2.2 are both $\mathcal{S}'$-valued martingales. The fact that $\mathcal{D}^0 + \mathcal{D}$ is an $\mathcal{S}'$-valued martingale will ultimately be used together with the martingale functional central limit theorem [10] and the continuous mapping theorem [1] in Sections 5.1 and 6.1 in order to prove our fluid and diffusion limits, respectively, for the age process. The fact that $\mathcal{G}$ is an $\mathcal{S}'$-valued martingale is not necessarily needed in order to prove limit theorems for the residual service time process but may be used to show that the residual service time process is in fact a Markov process. We begin by studying $\mathcal{D}^0 + \mathcal{D}$.

4.1. *Age process.* In this subsection, we show that the process $\mathcal{D}^0 + \mathcal{D}$ defined in Section 2.1 is an $\mathcal{S}'$-valued martingale with respect to the filtration $(\mathcal{F}_t^{\mathcal{A}})_{t\geq 0}$ defined by

$$\begin{aligned}
\mathcal{F}_t^{\mathcal{A}} = \sigma\{&\mathbf{1}_{\{\tilde{\eta}_i \leq s - \tilde{\tau}_i\}}, s \leq t, i = 1, 2, \ldots, A_0(\infty)\} \\
&\vee \sigma\{\tilde{\tau}_i, i = 1, \ldots, A_0(\infty)\} \\
&\vee \sigma\{\mathbf{1}_{\{\eta_i \leq s - \tau_i\}}, s \leq t, i = 1, 2, \ldots, E_t\} \\
&\vee \sigma\{E_s, s \leq t\} \vee \mathcal{N}.
\end{aligned}$$

Moreover, we explicitly identify the tensor quadratic variation of $\mathcal{D}^0 + \mathcal{D}$. The following is our main result of this subsection.

PROPOSITION 4.1. *The process $\mathcal{D}^0+\mathcal{D}$ is an $\mathcal{S}'$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale with tensor quadratic variation process given for all $\varphi,\psi\in\mathcal{S}$ by*

$$\langle\langle \mathcal{D}^0+\mathcal{D}\rangle\rangle_t(\varphi,\psi) = \sum_{i=1}^{A_0(\infty)} \int_0^{\tilde{\eta}_i\wedge t} \varphi(x-\tilde{\tau}_i)\psi(x-\tilde{\tau}_i)h_{\tilde{\tau}_i}(x)\,dx + \sum_{i=1}^{E_t} \int_0^{\eta_i\wedge(t-\tau_i)^+} \varphi(x)\psi(x)h(x)\,dx. \tag{4.1}$$

PROOF. Let $\varphi\in\mathcal{S}$. We claim that $(\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle)_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale, which is sufficient to show that $\mathcal{D}^0+\mathcal{D}$ is an $\mathcal{S}'$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale. Let $t\geq 0$. We first show that $E[|\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle|]<\infty$. Note that by (2.7) and (2.8) we may write

$$\begin{aligned}
&\mathbb{E}[|\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle|]\\
&\quad\leq \mathbb{E}\left[\left|\sum_{i=1}^{A_0(\infty)}\int_0^t \varphi(x-\tilde{\tau}_i)\,d\left(\mathbf{1}_{\{\tilde{\eta}_i\leq x\}} - \int_0^{\tilde{\eta}_i\wedge x} h_{\tilde{\tau}_i}(u)\,du\right)\right|\right]\\
&\qquad + \mathbb{E}\left[\left|\sum_{i=1}^{E_t}\int_0^{(t-\tau_i)^+} \varphi(x)\,d\left(\mathbf{1}_{\{\eta_i\leq x\}} - \int_0^{\eta_i\wedge x} h(u)\,du\right)\right|\right]\\
&\quad\leq \mathbb{E}\left[\sup_{0\leq s<\infty}|\varphi(s)|\sum_{i=1}^{A_0(\infty)}\left(\mathbf{1}_{\{\tilde{\eta}_i\leq t\}} + \int_0^{\tilde{\eta}_i\wedge t} h_{\tilde{\tau}_i}(u)\,du\right)\right]\\
&\qquad + \mathbb{E}\left[\sup_{0\leq s<\infty}|\varphi(s)|\sum_{i=1}^{E_t}\left(\mathbf{1}_{\{\eta_i\leq (t-\tau_i)^+\}} + \int_0^{\eta_i\wedge(t-\tau_i)^+} h(u)\,du\right)\right]\\
&\quad\leq \sup_{0\leq s<\infty}|\varphi(s)|(1+t\|h\|_\infty)(\mathbb{E}[A_0(\infty)]+\mathbb{E}[E_t])\\
&\quad<\infty,
\end{aligned}\tag{4.2}$$

where the final inequality follows from the assumptions made on $\mathbb{E}[A_0(\infty)]$ and $\mathbb{E}[E_t]$ in Section 2.1. Thus, $E[|\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle|]<\infty$ as desired.

Next, we show that $(\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle)_{t\geq 0}$ possesses the martingale property with respect to the filtration $(\mathcal{F}_t^{\mathcal{A}})_{t\geq 0}$. That is, we show that for each $0\leq s\leq t$,

$$\mathbb{E}[\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}] = \langle(\mathcal{D}^0+\mathcal{D})_s,\varphi\rangle. \tag{4.3}$$

First note that by (2.7) and (2.8), we may write

$$\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle = \sum_{i=1}^{\infty}\mathbf{1}_{\{i\leq A_0(\infty)\}}\langle\mathcal{D}_t^{0,i},\varphi\rangle + \sum_{i=1}^{\infty}\langle\mathcal{D}_t^i,\varphi\rangle, \tag{4.4}$$

where, for each $i \geq 1$, we set

$$\mathbf{1}_{\{i\leq A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle = \mathbf{1}_{\{i\leq A_0(\infty)\}}\int_0^t \varphi(x-\tilde{\tau}_i)\,d\left(\mathbf{1}_{\{\tilde{\eta}_i\leq x\}} - \int_0^{\tilde{\eta}_i\wedge x} h_{\tilde{\tau}_i}(u)\,du\right)$$

and

$$\langle \mathcal{D}_t^i,\varphi\rangle = \int_0^{(t-\tau_i)^+} \varphi(x)\,d\left(\mathbf{1}_{\{\eta_i\leq x\}} - \int_0^{\eta_i\wedge x} h(u)\,du\right). \tag{4.5}$$

We now show that for each $i \geq 1$,

$$\mathbb{E}[\langle \mathcal{D}_t^i,\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}] = \langle \mathcal{D}_s^i,\varphi\rangle. \tag{4.6}$$

The proof that

$$\mathbb{E}[\mathbf{1}_{\{i\leq A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}] \tag{4.7}$$

$$= \mathbf{1}_{\{i\leq A_0(\infty)\}}\mathbb{E}[\langle \mathcal{D}_t^{0,i},\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}] = \mathbf{1}_{\{i\leq A_0(\infty)\}}\langle \mathcal{D}_s^{0,i},\varphi\rangle, \tag{4.8}$$

for each $i \geq 1$, is similar and will not be included. For each $i \geq 1$ and $y \geq 0$, set

$$D_t^i(y) = \mathbf{1}_{\{\eta_i\leq (t-\tau_i)^+\wedge y\}} - \int_0^{\eta_i\wedge(t-\tau_i)^+\wedge y} h(u)\,du. \tag{4.9}$$

We now claim that in order to show (4.6), it suffices to show that $\mathbb{E}[D_t^i(y)|\mathcal{F}_s^{\mathcal{A}}] = D_s^i(y)$ for each $y \geq 0$. This is true since it will then follow that

$$\begin{aligned}\mathbb{E}[\langle \mathcal{D}_t^i,\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}] &= -\mathbb{E}\left[\int_{\mathbb{R}_+} D_t^i(y)\varphi'(y)\,dy\middle|\mathcal{F}_s^{\mathcal{A}}\right] \\ &= -\int_{\mathbb{R}_+} \mathbb{E}[D_t^i(y)|\mathcal{F}_s^{\mathcal{A}}]\varphi'(y)\,dy \\ &= -\int_{\mathbb{R}_+} D_s^i(y)\varphi'(y)\,dy, \\ &= \langle \mathcal{D}_s^i,\varphi\rangle.\end{aligned}$$

First note that since $y\wedge(t-\tau_i)^+ = (t\wedge(\tau_i+y)-\tau_i)^+$, we may write

$$D_t^i(y) = D_{t\wedge(\tau_i+y)}^i(\infty).$$

Next note that since by the assumptions in Section 2.1, we have that $\mathbb{E}[\tau_i] < \infty$, it is straightforward to verify that $\tau_i + y$ is an $\mathcal{F}_t^{\mathcal{A}}$-stopping time for each $y \geq 0$. Thus, by Problem 3.2.4 of [22], it suffices to show that $D^i(\infty) = (D_t^i(\infty))_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale. First note that since $\|h\|_\infty < \infty$, it is straightforward to show that $\mathbb{E}[|D_t^i(\infty)|] < \infty$ for each $t \geq 0$. Next note that using the independence of τ_i and η_i, one may also verify that

$$\mathbb{E}[D_t^i(\infty)|\mathbf{1}_{\{\tau_i \leq s\}}, \mathbf{1}_{\{\eta_i \leq s-\tau_i\}}] = D_s^i(\infty).$$

Hence, since η_i is assumed to be independent of $A_0, \{\tilde{\eta}_k, k = 1, \ldots, A_0(\infty)\}$, $E = (E_t)_{t \geq 0}$ and $\eta_k, k \neq i$, we obtain that

$$\begin{aligned} E[D_t^i(\infty)|\mathcal{F}_s^{\mathcal{A}}] &= E[D_t^i(\infty)|\mathbf{1}_{\{\tau_i \leq s\}}, \mathbf{1}_{\{\eta_i \leq s-\tau_i\}}] \\ &= E[D_s^i(\infty)|\mathcal{F}_s^{\mathcal{A}}] \\ &= D_s^i(\infty), \end{aligned}$$

and so $D^i(\infty)$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale as desired. This completes the proof of (4.6). The proof of (4.8) is similar.

We now show that (4.6) and (4.8) imply (4.3). We claim that for each $k \geq 1$, the sum

$$\sum_{i=1}^{k} \mathbf{1}_{\{i \leq A_0(\infty)\}} \langle \mathcal{D}_t^{0,i}, \varphi \rangle + \sum_{i=1}^{k} \langle \mathcal{D}_t^i, \varphi \rangle$$

is dominated uniformly over $k \geq 1$ by an integrable random variable. By Lebesgue's dominated convergence theorem for conditional expectations [6], this then implies that

$$\begin{aligned} \mathbb{E}[\langle \mathcal{D}_t, \varphi \rangle|\mathcal{F}_s^{\mathcal{A}}] &= \mathbb{E}\left[\sum_{i=1}^{\infty} \langle \mathcal{D}_t^i, \varphi \rangle \middle| \mathcal{F}_s^{\mathcal{A}}\right] \\ &= \sum_{i=1}^{\infty} \mathbb{E}[\langle \mathcal{D}_t^i, \varphi \rangle|\mathcal{F}_s^{\mathcal{A}}] \qquad (4.10) \\ &= \sum_{i=1}^{\infty} \langle \mathcal{D}_s^i, \varphi \rangle = \langle \mathcal{D}_s, \varphi \rangle, \end{aligned}$$

as desired. However, to obtain the bound is straightforward since

$$\begin{aligned} &\left| \sum_{i=1}^{k} \mathbf{1}_{\{i \leq A_0(\infty)\}} \langle \mathcal{D}_t^{0,i}, \varphi \rangle + \sum_{i=1}^{k} \langle \mathcal{D}_t^i, \varphi \rangle \right| \\ &\quad \leq \left| \sum_{i=1}^{k} \mathbf{1}_{\{i \leq A_0(\infty)\}} \int_0^t \varphi(x - \tilde{\tau}_i) \, d\left(\mathbf{1}_{\{\tilde{\eta}_i \leq x\}} - \int_0^{\tilde{\eta}_i \wedge x} h_{\tilde{\tau}_i}(u) \, du \right) \right| \\ &\qquad + \left| \sum_{i=1}^{k} \int_0^{(t-\tau_i)^+} \varphi(x) \, d\left(\mathbf{1}_{\{\eta_i \leq x\}} - \int_0^{\eta_i \wedge x} h(u) \, du \right) \right| \\ &\quad \leq \sup_{0 \leq s < \infty} |\varphi(s)| (1 + t\|h\|_\infty)(A_0(\infty) + E_t), \end{aligned}$$

and as in (4.2) we have that

$$\sup_{0\le s<\infty} |\varphi(s)|(1+t\|h\|_\infty)(\mathbb{E}[A_0(\infty)]+\mathbb{E}[E_t])<\infty.$$

We have therefore shown that $(\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle)_{t\ge0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale for each $\varphi\in\mathcal{S}$, which implies that $\mathcal{D}^0+\mathcal{D}$ is an $\mathcal{S}'$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale.

We next proceed to calculate the tensor quadratic variation of $\mathcal{D}^0+\mathcal{D}$. Let $i\ge1$ and recall that by (4.5) we have that for each $\varphi\in\mathcal{S}$ and $t\ge0$,

$$\langle \mathcal{D}_t^i,\varphi\rangle=\int_0^{(t-\tau_i)^+}\varphi(x)\,d\biggl(\mathbf{1}_{\{\eta_i\le x\}}-\int_0^{\eta_i\wedge x}h(u)\,du\biggr). \tag{4.11}$$

Therefore, as on page 259 of [25], it follows that

$$\begin{aligned}\langle\langle\,\langle\mathcal{D}^i,\varphi\rangle\,\rangle\rangle_t&=\int_0^{(t-\tau_i)^+}\varphi(x)^2\,d\biggl\langle\mathbf{1}_{\{\eta_i\le x\}}-\int_0^{\eta_i\wedge x}h(u)\,du\biggr\rangle\\&=\int_0^{\eta_i\wedge(t-\tau_i)^+}\varphi(x)^2h(x)\,dx,\end{aligned}\tag{4.12}$$

which implies that

$$\begin{aligned}\langle\langle\mathcal{D}^i\rangle\rangle_t(\varphi,\psi)&=\langle\langle\mathcal{D}^i,\varphi\rangle,\langle\mathcal{D}^i,\psi\rangle\rangle_t\\&=\frac14(\langle\langle\,\langle\mathcal{D}^i,\varphi+\psi\rangle\,\rangle\rangle_t-\langle\langle\,\langle\mathcal{D}^i,\varphi-\psi\rangle\,\rangle\rangle_t)\\&=\frac14\biggl(\int_0^{\eta_i\wedge(t-\tau_i)^+}(\varphi(x)+\psi(x))^2h(x)\,dx\\&\qquad-\int_0^{\eta_i\wedge(t-\tau_i)^+}(\varphi(x)-\psi(x))^2h(x)\,dx\biggr)\\&=\int_0^{\eta_i\wedge(t-\tau_i)^+}\varphi(x)\psi(x)h(x)\,dx,\end{aligned}$$

where the second equality in the above follows from the polarization identity and the third equality follows from (4.12). In a similar manner, one may also show that for all $i\ge1$,

$$\langle\langle\mathbf{1}_{\{i\le A_0(\infty)\}}\mathcal{D}^{0,i}\rangle\rangle_t(\varphi,\psi)=\mathbf{1}_{\{i\le A_0(\infty)\}}\int_0^{\tilde\eta_i\wedge t}\varphi(x-\tilde\tau_i)\psi(x-\tilde\tau_i)h_{\tilde\tau_i}(x)\,dx,$$

for all $\varphi,\psi\in\mathcal{S}$.

We now claim that in order to show that the tensor quadratic variation of $\mathcal{D}^0+\mathcal{D}$ is given by (4.1), it suffices to show the following three facts:

(1) $\mathcal{D}^i$ is orthogonal to $\mathcal{D}^j$ for $i\ne j$,

(2) $\mathbf{1}_{\{i\le A_0(\infty)\}}\mathcal{D}^{0,i}$ is orthogonal to $\mathbf{1}_{\{j\le A_0(\infty)\}}\mathcal{D}^{0,j}$ for $i\neq j$,
(3) $\mathbf{1}_{\{i\le A_0(\infty)\}}\mathcal{D}^{0,i}$ is orthogonal to $\mathcal{D}^j$ for all $i,j\ge 1$.

The fact that the tensor quadratic variation of $\mathcal{D}^0+\mathcal{D}$ is given by (4.1) can then be shown in the following manner. For each $k\ge 1,\varphi\in\mathcal{S}$ and $t\ge 0$, let

$$\langle(\mathcal{D}^0+\mathcal{D})_t^k,\varphi\rangle=\sum_{i=1}^k\mathbf{1}_{\{i\le A_0(\infty)\}}\langle\mathcal{D}_t^{0,i},\varphi\rangle+\sum_{i=1}^k\langle\mathcal{D}_t^i,\varphi\rangle,$$

and set $(\mathcal{D}^0+\mathcal{D})^k=((\mathcal{D}^0+\mathcal{D})_t^k,t\ge 0)$. It is then clear that claims (1) through (3) above imply that

$$\begin{aligned}&\langle\langle(\mathcal{D}^0+\mathcal{D})^k\rangle\rangle_t(\varphi,\psi)\\&\quad=\sum_{i=1}^k\mathbf{1}_{\{i\le A_0(\infty)\}}\int_0^{\tilde\eta_i\wedge t}\varphi(x-\tilde\tau_i)\psi(x-\tilde\tau_i)h_{\tilde\tau_i}(x)\,dx\\&\qquad+\sum_{i=1}^k\int_0^{\eta_i\wedge(t-\tau_i)^+}\varphi(x)\psi(x)h(x)\,dx.\end{aligned}$$

Moreover, using similar arguments as above and the simple inequality $(x_1+x_2)^2\le 4(x_1^2+x_2^2)$, it is straightforward to show that for each $k\ge 1$, one has $\mathbb{P}$-a.s. the bound

$$\begin{aligned}&\Bigg|\langle(\mathcal{D}^0+\mathcal{D})_t^k,\varphi\rangle\langle(\mathcal{D}^0+\mathcal{D})_t^k,\psi\rangle\\&\quad-\left(\sum_{i=1}^k\mathbf{1}_{\{i\le A_0(\infty)\}}\int_0^{\tilde\eta_i\wedge t}\varphi(x-\tilde\tau_i)\psi(x-\tilde\tau_i)h_{\tilde\tau_i}(x)\,dx\right.\\&\qquad\left.+\sum_{i=1}^k\int_0^{\eta_i\wedge(t-\tau_i)^+}\varphi(x)\psi(x)h(x)\,dx\right)\Bigg|\\&\le 4\Big(\sup_{0\le s<\infty}(|\varphi(s)|+|\psi(s)|)\Big)^2(1+t\|h\|_\infty)^2(E_t^2+A_0^2(\infty))\\&\quad+\sup_{0\le s<\infty}(|\varphi(s)||\psi(s)|)(1+t\|h\|_\infty)(E_t+A_0(\infty)).\end{aligned}$$

However, by the assumptions in Section 2.1, one has that $\mathbb{E}[E_t^2+A_0^2(\infty)]<\infty$ and $\mathbb{E}[E_t+A_0(\infty)]<\infty$. Hence, using the dominated convergence theorem for conditional expectations [6], it follows that for $0\le s\le t$,

$$\begin{aligned}&\mathbb{E}[\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle\langle(\mathcal{D}^0+\mathcal{D})_t,\varphi\rangle-\langle\langle(\mathcal{D}^0+\mathcal{D})\rangle\rangle_t(\varphi,\psi)|\mathcal{F}_s^A]\\&\quad=\mathbb{E}\Big[\lim_{k\to\infty}(\langle(\mathcal{D}^0+\mathcal{D})_t^k,\varphi\rangle\langle(\mathcal{D}^0+\mathcal{D})_t^k,\psi\rangle-\langle\langle(\mathcal{D}^0+\mathcal{D})^k\rangle\rangle_t(\varphi,\psi))|\mathcal{F}_s^A\Big]\end{aligned}$$

$$= \lim_{k\to\infty} \mathbb{E}[\langle (\mathcal{D}^0+\mathcal{D})_t^k, \varphi\rangle \langle (\mathcal{D}^0+\mathcal{D})_t^k, \psi\rangle - \langle\langle (\mathcal{D}^0+\mathcal{D})^k\rangle\rangle_t(\varphi,\psi)|\mathcal{F}_s^{\mathcal{A}}]$$
$$= \lim_{k\to\infty} (\langle (\mathcal{D}^0+\mathcal{D})_s^k, \varphi\rangle \langle (\mathcal{D}^0+\mathcal{D})_s^k, \psi\rangle - \langle\langle (\mathcal{D}^0+\mathcal{D})^k\rangle\rangle_s(\varphi,\psi))$$
$$= \langle (\mathcal{D}^0+\mathcal{D})_s, \varphi\rangle \langle (\mathcal{D}^0+\mathcal{D})_s, \psi\rangle - \langle\langle (\mathcal{D}^0+\mathcal{D})\rangle\rangle_s(\varphi,\psi).$$

This then implies that the tensor quadratic variation of $\mathcal{D}^0+\mathcal{D}$ is given by (4.1). We now proceed to prove claims (1) through (3), which is sufficient to complete the proof.

We begin with claim (1). Let $\varphi,\psi\in\mathcal{S}$ and $i\neq j$. We show that $(\langle\mathcal{D}_t^i, \varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle)_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale, which is sufficient to show that $\mathcal{D}^i$ is orthogonal to $\mathcal{D}^j$. First note that it is clear as in (4.2) that for each $t\geq 0$, we have that $\mathbb{E}[|\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle|]<\infty$. Next, let $0\leq s\leq t$. By the independence of η_i from $A_0, \{\tilde\eta_k, k=1,\dots,A_0(\infty)\}, E=(E_t)_{t\geq 0}$ and $\eta_k, k\neq i$, and, similarly, the independence of η_j from $A_0, \{\tilde\eta_k, k=1,\dots,A_0(\infty)\}, E=(E_t)_{t\geq 0}$ and $\eta_k, k\neq j$, it follows that

$$\mathbb{E}[\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle|\mathcal{F}_s^{\mathcal{A}}]$$
$$= \mathbb{E}[\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle|\mathbf{1}_{\{\tau_i\leq s\}}, \mathbf{1}_{\{\tau_j\leq s\}}, \mathbf{1}_{\{\eta_i\leq s-\tau_i\}}, \mathbf{1}_{\{\eta_j\leq s-\tau_j\}}].$$

However, by the independence of η_i from η_j and τ_j, and, similarly, the independence of η_j from η_i and τ_i, we have that

$$\mathbb{E}[\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle|\mathbf{1}_{\{\tau_i\leq s\}}, \mathbf{1}_{\{\tau_j\leq s\}}, \mathbf{1}_{\{\eta_i\leq s-\tau_i\}}, \mathbf{1}_{\{\eta_j\leq s-\tau_j\}}]$$
$$= \mathbb{E}[\langle\mathcal{D}_t^i,\varphi\rangle|\mathbf{1}_{\{\tau_i\leq s\}}, \mathbf{1}_{\{\eta_i\leq s-\tau_i\}}]\mathbb{E}[\langle\mathcal{D}_t^j,\psi\rangle|\mathbf{1}_{\{\tau_j\leq s\}}, \mathbf{1}_{\{\eta_j\leq s-\tau_j\}}]$$
$$= \langle\mathcal{D}_s^i,\varphi\rangle\langle\mathcal{D}_s^j,\psi\rangle,$$

where the final equality follows from (4.6) and the fact that for $k\geq 1$,

$$\mathbb{E}[\langle\mathcal{D}_t^k,\varphi\rangle|\mathbf{1}_{\{\tau_k\leq s\}}, \mathbf{1}_{\{\eta_k\leq s-\tau_k\}}] = \mathbb{E}[\langle\mathcal{D}_t^k,\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}]. \tag{4.13}$$

Thus, it is clear that $(\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle)_{t\geq 0}$ possesses the martingale property and so $(\langle\mathcal{D}_t^i,\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle)_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale, and hence $\mathcal{D}^i$ is orthogonal to $\mathcal{D}^j$. The proof of claim (2) above follows similarly. The proof of claim (3) above follows in a similar manner as well. In particular, let $i,j\geq 1$. We show that $(\mathbf{1}_{\{i\leq A_0(\infty)\}}\langle\mathcal{D}_t^{0,i},\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle)_{t\geq 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale for each $\varphi,\psi\in\mathcal{S}$, which is sufficient to show that $\mathbf{1}_{\{i\leq A_0(\infty)\}}\mathcal{D}^{0,i}$ is orthogonal to $\mathcal{D}^j$. For each $t\geq 0$, it is clear as in (4.2) that $\mathbb{E}[|\mathbf{1}_{\{i\leq A_0(\infty)\}}\langle\mathcal{D}_t^{0,i},\varphi\rangle\langle\mathcal{D}_t^j,\psi\rangle|]<\infty$. Next, note that since $\tilde\eta_i$ is independent of $A_0, \{\tilde\eta_k, k=1,\dots,A_0(\infty); k\neq i\}, E=(E_t)_{t\geq 0}$ and $\eta_k, k\geq 1$, and, similarly, η_j is independent of $A_0, \{\tilde\eta_k, k=1,\dots,A_0(\infty)\}, E=(E_t)_{t\geq 0}$ and

$\eta_k, k \neq j$, we have that

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle\langle \mathcal{D}_t^{j},\psi\rangle|\mathcal{F}_s^{\mathcal{A}}]\\ &\qquad = \mathbb{E}[\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle\langle \mathcal{D}_t^{j},\psi\rangle|\tilde{\tau}_i, \mathbf{1}_{\{\tau_j\le s\}}, \mathbf{1}_{\{\tilde{\eta}_i\le s-\tilde{\tau}_i\}}, \mathbf{1}_{\{\eta_j\le s-\tau_j\}}].\end{aligned}$$

However, by the independence of $\tilde{\eta}_i$ from η_j and τ_j and, similarly, the independence of η_j from $\tilde{\eta}_i$ and $\tilde{\tau}_i$, we have that

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle\langle \mathcal{D}_t^{j},\psi\rangle|\tilde{\tau}_i, \mathbf{1}_{\{\tau_j\le s\}}, \mathbf{1}_{\{\tilde{\eta}_i\le s-\tilde{\tau}_i\}}, \mathbf{1}_{\{\eta_j\le s-\tau_j\}}]\\ &\qquad = \mathbb{E}[\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle|\tilde{\tau}_i, \mathbf{1}_{\{\tilde{\eta}_i\le s-\tilde{\tau}_i\}}]\mathbb{E}[\langle \mathcal{D}_t^{j},\psi\rangle|\mathbf{1}_{\{\tau_j\le s\}}, \mathbf{1}_{\{\eta_j\le s-\tau_j\}}]\\ &\qquad = \mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_s^{0,i},\varphi\rangle\langle \mathcal{D}_s^{j},\psi\rangle,\end{aligned}$$

where the final equality follows by (4.6), (4.8), (4.13) and the fact that for $k \ge 1$,

$$\mathbb{E}[\mathbf{1}_{\{k\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,k},\varphi\rangle|\tilde{\tau}_k, \mathbf{1}_{\{\tilde{\eta}_k\le s-\tilde{\tau}_k\}}] = \mathbb{E}[\mathbf{1}_{\{k\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,k},\varphi\rangle|\mathcal{F}_s^{\mathcal{A}}].$$

Thus, it is clear that $(\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle\langle \mathcal{D}_t^{j},\psi\rangle)_{t\ge 0}$ possesses the martingale property and so $(\mathbf{1}_{\{i\le A_0(\infty)\}}\langle \mathcal{D}_t^{0,i},\varphi\rangle\langle \mathcal{D}_t^{j},\psi\rangle)_{t\ge 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{A}}$-martingale, and hence $\mathcal{D}^{0,i}$ is orthogonal to $\mathcal{D}^j$. This proves claim (3), which completes the proof. □

4.2. *Residuals.* In this subsection, we show that the process $\mathcal{G}$ defined in Section 2.2 is a martingale. This fact may be useful in future work where one wishes to show that the residual service time process is a Markov process. Let $(\mathcal{F}_t^{\mathcal{G}})_{t\ge 0}$ be the natural filtration generated by $\mathcal{G}$. We then have the following result.

PROPOSITION 4.2. *The process $\mathcal{G}$ is an $\mathcal{S}'$-valued $\mathcal{F}_t^{\mathcal{G}}$-martingale with tensor optional quadratic variation process given for all $\varphi, \psi \in \mathcal{S}$ by*

$$[\mathcal{G}]_t(\varphi,\psi) = \sum_{i=1}^{E_t}(\varphi(\eta_i) - \langle \mathcal{F},\varphi\rangle)(\psi(\eta_i) - \langle \mathcal{F},\psi\rangle). \tag{4.14}$$

PROOF. Let $\varphi \in \mathcal{S}$. We first show that $(\langle \mathcal{G}_t,\varphi\rangle)_{t\ge 0}$ is an $\mathbb{R}$-valued $\mathcal{F}_t^{\mathcal{G}}$-martingale. Define the filtration $(\mathcal{H}_k)_{k\ge 1}$ by setting $\mathcal{H}_k = \sigma\{E_t, t \ge 0\} \vee \sigma\{\eta_1, \eta_2, \ldots, \eta_k\} \vee \mathcal{N}$ for each $k \ge 1$. Next, define the discrete-time $\mathbb{D}$-valued process $(G^k)_{k\ge 1}$ by

$$G^k(y) = \sum_{i=1}^{k}(\mathbf{1}_{\{\eta_i\le y\}} - F(y)), \qquad y \ge 0, \tag{4.15}$$

and, for convenience, let $G^k(y)=0$ for $y<0$. Then let $(\mathcal{G}^k)_{k\geq 1}$ be the $\mathcal{S}'$-valued process associated with G^k. Since $\varphi\in\mathcal{S}$ is bounded, it is clear that $E[|\langle\mathcal{G}^k,\varphi\rangle|]<\infty$ for each $k\geq 1$. Moreover, by the independence of the service times from the arrival process, one has that $(\langle\mathcal{G}^k,\varphi\rangle)_{k\geq 1}$ possesses the martingale property with respect to $(\mathcal{H}_k)_{k\geq 1}$. Hence, $(\langle\mathcal{G}^k,\varphi\rangle)_{k\geq 1}$ is an $\mathbb{R}$-valued $\mathcal{H}_k$-martingale. However, since for each $t\geq 0$ we have by the assumptions in Section 2.1 that $\mathbb{E}[E_t]<\infty$, it is straightforward to see that E_t is a stopping time with respect to the filtration $(\mathcal{H}_k)_{k\geq 1}$. Thus, the filtration $(\mathcal{H}_{E_t})_{t\geq 0}$ is well defined and, furthermore, it follows by the optional sampling theorem [22] that $(\langle\mathcal{G}_t,\varphi\rangle)_{t\geq 0}=(\langle\mathcal{G}^{E_t},\varphi\rangle)_{t\geq 0}$ is an $\mathcal{H}_{E_t}$-martingale. The result now follows since any martingale is a martingale relative to its natural filtration.

The form of the tensor optional quadratic variation (4.14) is immediate by Theorem 3.3 of [29]. $\square$

5. Fluid limits.

In this section, we provide our main fluid limit results. We begin in Section 5.1 by studying the age process and in Section 5.2 we study the residual service time process. Our setup in both subsections is the same. In particular, we consider a sequence of $G/GI/\infty$ queues indexed by $n\geq 1$, where the arrival rate to the system grows large with n while the service time distribution does not change with n.

5.1. *Ages.*

We begin by studying the age process $\mathcal{A}$ defined in Section 2.1. For each $n\geq 1$, define the fluid scaled quantities

$$\begin{aligned}
&\bar{\mathcal{A}}_0^n\equiv\frac{\mathcal{A}_0^n}{n},\qquad \bar{E}^n\equiv\frac{E^n}{n},\qquad \bar{\mathcal{D}}^{0,n}\equiv\frac{\mathcal{D}^{0,n}}{n},\\
&\bar{\mathcal{D}}^n\equiv\frac{\mathcal{D}^n}{n},\qquad \bar{\mathcal{A}}^n\equiv\frac{\mathcal{A}^n}{n},
\end{aligned}\tag{5.1}$$

and set $\bar{\mathcal{E}}^n\equiv\bar{E}^n\delta_0$. Using (2.12), Theorem 3.3 and Proposition 3.6, it is straightforward to show that one may write

$$\bar{\mathcal{A}}^n=\Psi_{B^{\mathcal{A}}}(\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n-(\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^n)),\tag{5.2}$$

where the map $\Psi_{B^{\mathcal{A}}}:\mathbb{D}([0,T],\mathcal{S}')\mapsto\mathbb{D}([0,T],\mathcal{S}')$ is continuous. We now prove that if $(\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n)_{n\geq 1}$ weakly converges, then so too does $(\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n-(\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^n))_{n\geq 1}$.

PROPOSITION 5.1. *If $\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n\Rightarrow\bar{\mathcal{A}}_0+\bar{\mathcal{E}}$ in $\mathbb{D}([0,T],\mathcal{S}')$ as $n\to\infty$, then*

$$\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n-(\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^n)\Rightarrow\bar{\mathcal{A}}_0+\bar{\mathcal{E}}\qquad\text{in }\mathbb{D}([0,T],\mathcal{S}')\text{ as }n\to\infty.$$

PROOF. We first note that by Theorem 1.5, it is sufficient to show that if $\bar{\mathcal{A}}_0^n+\bar{\mathcal{E}}^n\Rightarrow\bar{\mathcal{A}}_0+\bar{\mathcal{E}}$ as $n\to\infty$, then

$$\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^n\Rightarrow 0\qquad\text{in }\mathbb{D}([0,T],\mathcal{S}')\text{ as }n\to\infty.\tag{5.3}$$

Let $T>0$ and $0\le t\le T$. Then, for each $\varphi\in\mathcal{S}$, we have by Proposition 4.1 that

$$
\begin{aligned}
&|\langle\langle \bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n}\rangle\rangle_t(\varphi,\varphi)| \\
&\quad = \left| \frac{1}{n^2}\left(\sum_{i=1}^{A_0^n(\infty)} \int_0^{\tilde{\eta}_i\wedge t} \varphi^2(x-\tilde{\tau}_i^n) h_{\tilde{\tau}_i^n}(x)\,dx \right.\right. \\
&\qquad\qquad \left.\left. + \sum_{i=1}^{E_t^n} \int_0^{\eta_i\wedge(t-\tau_i^n)^+} \varphi^2(x)h(x)\,dx \right)\right| \qquad (5.4)\\
&\quad \le \frac{\|h\|_\infty}{n^2} \sum_{i=1}^{A_0^n(\infty)} \int_0^t \varphi^2(x-\tilde{\tau}_i^n)\,dx + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n. \qquad (5.5)
\end{aligned}
$$

Thus, from (5.5) we obtain that for each $0\le t\le T$,

$$
\begin{aligned}
&|\langle\langle \bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n}\rangle\rangle_t(\varphi,\varphi)| \\
&\quad \le \frac{\|h\|_\infty}{n^2} \sum_{i=1}^{A_0^n(\infty)} \int_0^t \varphi^2(x-\tilde{\tau}_i^n)\,dx + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n \\
&\quad = \frac{\|h\|_\infty}{n^2} \int_0^t \sum_{i=1}^{A_0^n(\infty)} \varphi^2(x-\tilde{\tau}_i^n)\,dx + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n \qquad (5.6)\\
&\quad = \frac{\|h\|_\infty}{n} \int_0^t \langle \bar{\mathcal{A}}_0^n, \tau_{-x}\varphi^2\rangle\,dx + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n \\
&\quad \le \frac{\|h\|_\infty t}{n} q_K(\bar{\mathcal{A}}_0^n) + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n,
\end{aligned}
$$

where the set K is given by $K=\{\tau_{-x}\varphi^2, 0\le x\le t\}$. By Lemma 3.5, the set K is bounded in $\mathcal{S}$, and hence q_K is a seminorm on $\mathcal{S}'$ by Definition 1.3. This then implies that q_K is a continuous function on $\mathcal{S}'$. Hence, since by assumption $\bar{\mathcal{A}}_0^n\Rightarrow\bar{\mathcal{A}}_0$ and $\bar{E}_T^n\Rightarrow\bar{E}_T$, it follows by Slutsky's theorem that

$$
\frac{\|h\|_\infty t}{n} q_K(\bar{\mathcal{A}}_0^n) + \frac{\|\varphi^2 h\|_\infty}{n}\bar{E}_T^n \Rightarrow 0 \qquad \text{as } n\to\infty.
$$

By (5.6), this then implies that

$$
\langle\langle \bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n}\rangle\rangle(\varphi,\varphi)\Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n\to\infty. \qquad (5.7)
$$

We now verify that parts (1) and (2) of Theorem 1.5 are satisfied for the sequence $(\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n})_{n\ge1}$, with the limit point being the function which is identically 0. We begin with part (1). Using the fact that the maximum jump of both $\langle\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n},\varphi\rangle$ and $\langle\langle\bar{\mathcal{D}}^{0,n}+\bar{\mathcal{D}}^{n}\rangle\rangle(\varphi,\varphi)$ is bounded over the interval $[0,T]$ uniformly in n, we obtain by (5.7) and the martingale FCLT (see Theorem 7.1.4 of [10] or [34]) that

$$\langle \bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n, \varphi \rangle \Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n \to \infty. \tag{5.8}$$

Thus, part (1) of Theorem 1.5 holds. We next check that condition (2) holds. Let $m \geq 1$ and let $t_1,\ldots,t_m \in [0,T]$ and $\varphi_1,\ldots,\varphi_m \in \mathcal{S}$. By (5.8), we have that for each $1 \leq i \leq m$,

$$\langle (\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n)_{t_i}, \varphi_i \rangle \Rightarrow 0 \qquad \text{in } \mathbb{R} \text{ as } n \to \infty.$$

However, by Theorem 3.9 of [1] this now implies that

$$(\langle (\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n)_{t_1}, \varphi_1 \rangle, \ldots, \langle (\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n)_{t_m}, \varphi_m \rangle) \Rightarrow (0,\ldots,0) \qquad \text{in } \mathbb{R}^m,$$

as $n \to \infty$. Thus, we have shown that part (2) of Theorem 1.5 holds and so (5.3) is proven. This completes the proof. $\square$

We are now in a position to prove the main result of this subsection. We have the following.

THEOREM 5.2. *If $\bar{\mathcal{A}}_0^n + \bar{\mathcal{E}}^n \Rightarrow \bar{\mathcal{A}}_0 + \bar{\mathcal{E}}$ in $\mathbb{D}([0,T],\mathcal{S}')$ as $n \to \infty$, then*

$$\bar{\mathcal{A}}^n \Rightarrow \bar{\mathcal{A}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty,$$

where $\bar{\mathcal{A}}$ is the unique solution to the integral equation

$$\langle \bar{\mathcal{A}}_t, \varphi \rangle = \langle \bar{\mathcal{A}}_0, \varphi \rangle + \langle \bar{\mathcal{E}}_t, \varphi \rangle - \int_0^t \langle \bar{\mathcal{A}}_s, h\varphi \rangle \, ds + \int_0^t \langle \bar{\mathcal{A}}_s, \varphi' \rangle \, ds, \qquad t \in [0,T], \tag{5.9}$$

for all $\varphi \in \mathcal{S}$.

PROOF. By the assumption that $\bar{\mathcal{A}}_0^n + \bar{\mathcal{E}}^n \Rightarrow \bar{\mathcal{A}}_0 + \bar{\mathcal{E}}$, it follows immediately by Proposition 5.1 that

$$\bar{\mathcal{A}}_0^n + \bar{\mathcal{E}}^n - (\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n) \Rightarrow \bar{\mathcal{A}}_0 + \bar{\mathcal{E}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty. \tag{5.10}$$

Next, recall that by (5.2) we have that $\bar{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\bar{\mathcal{A}}_0^n + \bar{\mathcal{E}}^n - (\bar{\mathcal{D}}^{0,n} + \bar{\mathcal{D}}^n))$, where the map $\Psi_{B^{\mathcal{A}}} : \mathbb{D}([0,T],\mathcal{S}') \mapsto \mathbb{D}([0,T],\mathcal{S}')$ is continuous. The result now follows by (5.10) and Proposition 1.1 applied to $\Psi_{B^{\mathcal{A}}}$. $\square$

REMARK 5.3. Note that one may now use Theorem 5.2 along with Theorem 3.3 in order to obtain an explicit expression for $\bar{\mathcal{A}}$. Similarly, one may obtain explicit expressions for $\bar{\mathcal{R}}$, $\hat{\mathcal{A}}$ and $\hat{\mathcal{R}}$ in Theorems 5.6, 6.5 and 6.9, respectively, below.

We also note that at a heuristic level, one may attempt to substitute the function $\mathbf{1}_{\{x\geq 0\}}$ into the explicit formula provided by Theorem 3.3 for $\bar{\mathcal{A}}$ in order to obtain an expression for the limiting, fluid scaled total number of customers in the system. For instance, suppose that $\bar{\mathcal{A}}_0 = 0$ so that the system is initially empty and that $\langle \bar{\mathcal{E}}, \varphi\rangle = \lambda\varphi(0)e$ for each $\varphi \in \mathcal{S}$. Then, using the form of the generator $B^{\mathcal{A}}$ from (3.14) and the semigroup $(S_t^{\mathcal{A}})_{t\geq 0}$ from Proposition 3.6, one obtains after substituting into Theorem 3.3 that heuristically the total number of customers in the system at time $t \geq 0$ is given by

$$\lambda t - \lambda \int_0^t s f(t-s)\,ds = \lambda \int_0^t \bar{F}(t-s)\,ds.$$

We now conclude this subsection by providing an additional condition on the arrival process under which a stationary solution to the fluid limit equation (5.9) may be explicitly found. Note also that our condition in Proposition 5.4 below holds, for example, if the arrival process to the nth system is a renewal process which has been sped up by a factor of n (as will be the case for the $GI/GI/\infty$ queue).

PROPOSITION 5.4. *If $\langle \bar{\mathcal{E}}, \varphi\rangle = \lambda\varphi(0)e$ for each $\varphi \in \mathcal{S}$, then $\bar{\mathcal{A}} = \lambda \mathcal{F}_e$ is a stationary solution to the fluid limit equation (5.9).*

PROOF. Substituting $\bar{\mathcal{A}} = \lambda\mathcal{F}_e$ and $\langle \bar{\mathcal{E}}, \varphi\rangle = \lambda\varphi(0)e$ into (5.9), we see that it suffices to verify that

$$\begin{aligned}\lambda &\int_{\mathbb{R}_+} \varphi(y)\,dF_e(y)\\ &= \lambda \int_{\mathbb{R}_+} \varphi(y)\,dF_e(y) + \lambda t\varphi(0) - \lambda t \int_{\mathbb{R}_+} (h(y)\varphi(y) - \varphi'(y))\,dF_e(y).\end{aligned}$$

However, this follows since

$$\begin{aligned}\lambda t &\int_{\mathbb{R}_+} (h(y)\varphi(y) - \varphi'(y))\,dF_e(y)\\ &= \lambda t \int_{\mathbb{R}_+} (h(y)\varphi(y) - \varphi'(y))\bar{F}(y)\,dy\\ &= \lambda t \int_{\mathbb{R}_+} (f(y)\varphi(y) - \bar{F}(y)\varphi'(y))\,dy\\ &= -\lambda t \int_{\mathbb{R}_+} (\bar{F}(y)\varphi(y))'\,dy\\ &= \lambda t\varphi(0).\end{aligned}$$

This completes the proof. □

5.2. *Residuals.* We next proceed to analyze the residual service time process $\mathcal{R}$ of Section 2.2. Our setup in this subsection is the same as that in the previous subsection. However, in addition to the fluid scaled quantities already defined in Section 5.1, we also now define for each $n \geq 1$ the new fluid scaled quantities

$$\bar{\mathcal{R}}^n \equiv \frac{\mathcal{R}^n}{n}, \qquad \bar{\mathcal{R}}_0^n \equiv \frac{\mathcal{R}_0^n}{n} \quad \text{and} \quad \bar{\mathcal{G}} \equiv \frac{\mathcal{G}^n}{n}.$$

Using (2.20), Theorem 3.3 and Proposition 3.7, it is now straightforward to show that

$$\bar{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} + \bar{\mathcal{G}}^n), \tag{5.11}$$

where the map $\Psi_{B^{\mathcal{R}}} : \mathbb{D}([0,T], \mathcal{S}') \mapsto \mathbb{D}([0,T], \mathcal{S}')$ is continuous. In our first result of this subsection, we prove that if $(\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F})_{n \geq 1}$ weakly converges, then so too does $(\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} + \bar{\mathcal{G}}^n)_{n \geq 1}$.

PROPOSITION 5.5. *If $\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$ in $\mathbb{D}([0,T], \mathcal{S}')$ as $n \to \infty$, then*

$$\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} + \bar{\mathcal{G}}^n \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F} \qquad \text{in } \mathbb{D}([0,T], \mathcal{S}') \text{ as } n \to \infty. \tag{5.12}$$

PROOF. We first note that by Theorem 1.5, it is sufficient to show that if $\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$ as $n \to \infty$, then

$$\bar{\mathcal{G}}^n \Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T], \mathcal{S}') \text{ as } n \to \infty. \tag{5.13}$$

Let $T > 0$ and $0 \leq t \leq T$. We then have by Proposition 4.2 and the assumption that $\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$, that for each $\varphi, \psi \in \mathcal{S}$,

$$\begin{aligned} |[\bar{\mathcal{G}}^n]_t(\varphi, \psi)| &= \left| \frac{1}{n^2} \sum_{i=1}^{E_t^n} \varphi(\eta_i)\psi(\eta_i) \right| \\ &\leq \frac{1}{n^2} E_T^n \sup_{0 \leq s < \infty} |\varphi(s)\psi(s)| \Rightarrow 0 \qquad \text{in } \mathbb{R} \text{ as } n \to \infty. \end{aligned} \tag{5.14}$$

The remainder of the proof now proceeds in a similar manner to the proof of Proposition 5.1. We omit the details. $\square$

The following is now our main result of this subsection.

THEOREM 5.6. *If $\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$ in $\mathbb{D}([0,T], \mathcal{S}')$ as $n \to \infty$, then*

$$\bar{\mathcal{R}}^n \Rightarrow \bar{\mathcal{R}} \qquad \text{in } \mathbb{D}([0,T], \mathcal{S}') \text{ as } n \to \infty,$$

where $\bar{\mathcal{R}}$ is the unique solution to the integral equation

$$\langle \bar{\mathcal{R}}_t, \varphi \rangle = \langle \bar{\mathcal{R}}_0, \varphi \rangle + \bar{E}_t \langle \mathcal{F}, \varphi \rangle - \int_0^t \langle \bar{\mathcal{R}}_s, \varphi' \rangle \, ds, \qquad t \in [0,T], \tag{5.15}$$

for all $\varphi \in \mathcal{S}$.

Proof. By the assumption that $\bar{\mathcal{R}}_0^n + \bar{E}^n\mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$, it follows immediately by Proposition 5.5 that

$$\bar{\mathcal{R}}_0^n + \bar{E}^n\mathcal{F} + \bar{\mathcal{G}}^n \Rightarrow \bar{\mathcal{R}}_0 + \bar{E}\mathcal{F} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty. \tag{5.16}$$

Next, recall that by (5.11) we have that $\bar{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\bar{\mathcal{R}}_0^n + \bar{E}^n\mathcal{F} + \bar{\mathcal{G}}^n)$, where the map $\Psi_{B^{\mathcal{R}}} : \mathbb{D}([0,T],\mathcal{S}') \mapsto \mathbb{D}([0,T],\mathcal{S}')$ is continuous. The result now follows by (5.16) and Proposition 1.1 applied to $\Psi_{B^{\mathcal{R}}}$. □

6. Diffusion limits.

In this section, we prove our main diffusion limit results. In Section 6.1, we study the age process and in Section 6.2 we study the residual service time process. Before we provide our main results, however, we first must provide the definition of an $\mathcal{S}'$-valued Wiener process and a generalized $\mathcal{S}'$-valued Ornstein–Uhlenbeck process. Our definitions are the same as those in [2].

Definition 6.1. A continuous $\mathcal{S}'$-valued Gaussian process $W = (W_t)_{t\ge 0}$ is called a *generalized $\mathcal{S}'$-valued Wiener process* with *covariance functional*

$$K(s,\varphi;t,\psi) = \mathbb{E}[\langle W_s,\varphi\rangle\langle W_t,\psi\rangle], \qquad s,t \ge 0 \text{ and } \varphi,\psi \in \mathcal{S},$$

if it has continuous trajectories and, for each $s,t \ge 0$ and $\varphi,\psi \in \mathcal{S}$, $K(s,\varphi;t,\psi)$ is of the form

$$K(s,\varphi;t,\psi) = \int_0^{s\wedge t} \langle Q_u\varphi,\psi\rangle\, du,$$

where the operators $Q_u : \mathcal{S} \to \mathcal{S}'$, $u \ge 0$, possess the following two properties:

(1) Q_u is linear, continuous, symmetric and positive for each $u \ge 0$,
(2) the function $u \mapsto \langle Q_u\varphi,\psi\rangle$ is in $\mathbb{D}([0,\infty),\mathbb{R})$ for each $\varphi,\psi \in \mathcal{S}$.

If Q_u does not depend on $u \ge 0$, then the process W is called an *$\mathcal{S}'$-valued Wiener process.*

Now, using the above definition of a generalized $\mathcal{S}'$-valued Wiener process, we may provide the following definition of a generalized $\mathcal{S}'$-valued Ornstein–Uhlenbeck process.

Definition 6.2. An $\mathcal{S}'$-valued process $X = (X_t)_{t\ge 0}$ is called a (*generalized*) *$\mathcal{S}'$-valued Ornstein–Uhlenbeck process* if for each $\varphi \in \mathcal{S}$ and $t \ge 0$,

$$\langle X_t,\varphi\rangle = \langle X_0,\varphi\rangle + \int_0^t \langle X_u, A\varphi\rangle\, du + \langle W_t,\varphi\rangle,$$

where $W \equiv (W_t)_{t\ge 0}$ is a (generalized) $\mathcal{S}'$-valued Wiener process and $A : \mathcal{S} \to \mathcal{S}$ is a continuous operator.

6.1. *Ages.* In this subsection, we prove our main diffusion limit result for the age process $\mathcal{A}$ defined in Section 2.1. Our setup is the same as that in Section 5. That is, we consider a sequence of $G/GI/\infty$ queues indexed by $n \geq 1$, where the arrival rate to the system grows large with n while the service time distribution does not change with n. For the remainder of this subsection, we assume that $\bar{\mathcal{A}}_0^n + \bar{\mathcal{E}}^n \Rightarrow \bar{\mathcal{A}}_0 + \bar{\mathcal{E}}$ as $n \to \infty$, where $\bar{\mathcal{A}}_0 + \bar{\mathcal{E}}$ is a nonrandom quantity. By Theorem 5.2 of Section 5.1, this then implies that $\bar{\mathcal{A}}$ is a nonrandom quantity as well. Setting $\bar{A}_0^n(\infty) = n^{-1} A_0^n(\infty)$ for each $n \geq 1$ and letting $T \geq 0$, we also assume that the sequences $\{\bar{A}_0^n(\infty), n \geq 1\}$ and $\{\bar{E}_T^n, n \geq 1\}$ are uniformly integrable.

Now, for each $n \geq 1$, define the diffusion scaled quantities

$$\hat{\mathcal{A}}^n \equiv \sqrt{n}(\bar{\mathcal{A}}^n - \bar{\mathcal{A}}), \qquad \hat{\mathcal{A}}_0^n \equiv \sqrt{n}(\bar{\mathcal{A}}_0^n - \bar{\mathcal{A}}_0), \qquad \hat{E}^n \equiv \sqrt{n}(\bar{E}^n - \bar{E}),$$

$$\hat{\mathcal{D}}^{0,n} \equiv \sqrt{n}\bar{\mathcal{D}}^{0,n}, \qquad \hat{\mathcal{D}}^n \equiv \sqrt{n}\bar{\mathcal{D}}^n,$$

and set $\hat{\mathcal{E}}^n \equiv \hat{E}^n \delta_0$. Then, recalling the form of the fluid limit $\bar{\mathcal{A}}$ from Theorem 5.2, note that using system equation (2.12) in conjunction with Theorem 3.3 and Proposition 3.6, one has that for each $n \geq 1$,

$$\hat{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n)), \tag{6.1}$$

where the map $\Psi_{B^{\mathcal{A}}} : \mathbb{D}([0,T], \mathcal{S}') \mapsto \mathbb{D}([0,T], \mathcal{S}')$ is continuous. Our strategy now is to first prove a weak convergence result for the sequence $(\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n))_{n \geq 1}$ and then to apply Theorem 1.1 together with (6.1) in order to prove our diffusion limit result for the sequence $(\hat{\mathcal{A}}^n)_{n \geq 1}$.

We begin with the following result. Its proof may be found in the Appendix.

LEMMA 6.3. *If $\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n \Rightarrow \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}$ in $\mathbb{D}([0,T], \mathcal{S}')$ as $n \to \infty$, then*

$$\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n \Rightarrow \hat{\mathcal{D}}^0 + \hat{\mathcal{D}} \qquad \text{in } \mathbb{D}([0,T], \mathcal{S}') \text{ as } n \to \infty, \tag{6.2}$$

where $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ is a generalized $\mathcal{S}'$-valued Wiener process with covariance functional given for each $\varphi, \psi \in \mathcal{S}$ and $s, t \geq 0$ by

$$K_{\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}}(s, \varphi; t, \psi) = \int_0^{s \wedge t} \langle \bar{\mathcal{A}}_u, \varphi\psi h \rangle \, du. \tag{6.3}$$

PROOF. See the Appendix. $\square$

We next have the following result, which provides a weak limit for the sequence $(\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n))_{n \geq 1}$.

PROPOSITION 6.4. *If $\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n \Rightarrow \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}$ in $\mathbb{D}([0,T], \mathcal{S}')$ as $n \to \infty$, then*

$$\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n) \Rightarrow \hat{\mathcal{A}}_0 + \hat{\mathcal{E}} - (\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}) \qquad \text{in } \mathbb{D}([0,T], \mathcal{S}') \text{ as } n \to \infty, \tag{6.4}$$

where $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ is as given in Lemma 6.3 and is independent of $\hat{\mathcal{A}}_0 + \hat{\mathcal{E}}$.

Proof. We will check that the sequence $(\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n))_{n\geq 1}$ satisfies parts (1) and (2) of Theorem 1.5. We begin with part (1). Let $\varphi \in \mathcal{S}$. By assumption, the sequence $(\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n, \varphi\rangle)_{n\geq 1}$ weakly converges and hence is tight in $\mathbb{D}([0,T],\mathbb{R})$, and, by Lemma 6.3, the sequence $(\langle (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n), \varphi\rangle)_{n\geq 1}$ weakly converges, and hence is tight in $\mathbb{D}([0,T],\mathbb{R})$. Therefore, there must exist a subsequence $(n)_{n\geq 1}$ along which we have the joint convergence

$$(\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n, \varphi\rangle, \langle (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n), \varphi\rangle) \Rightarrow (\langle \check{\mathcal{A}}_0 + \check{\mathcal{E}}, \varphi\rangle, \langle (\check{\mathcal{D}}^0 + \check{\mathcal{D}}), \varphi\rangle)$$

in $\mathbb{D}^2([0,T],\mathbb{R})$ as $n\to\infty$. Now note that clearly $\langle \check{\mathcal{A}}_0 + \check{\mathcal{E}}, \varphi\rangle$ has the same distribution as $\langle \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}, \varphi\rangle$ and, similarly, $\langle (\check{\mathcal{D}}^0 + \check{\mathcal{D}}), \varphi\rangle$ has the same distribution as $\langle (\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}), \varphi\rangle$. We now verify that $\langle \check{\mathcal{A}}_0 + \check{\mathcal{E}}, \varphi\rangle$ and $\langle (\check{\mathcal{D}}^0 + \check{\mathcal{D}}), \varphi\rangle$ are independent of one another. This will then imply the convergence

$$\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n, \varphi\rangle - \langle (\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n), \varphi\rangle \Rightarrow \langle \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}, \varphi\rangle - \langle (\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}), \varphi\rangle$$

in $\mathbb{D}([0,T],\mathbb{R})$ as $n\to\infty$, along the given subsequence. However, since the subsequence was arbitrary, this will then imply convergence along the entire sequence, thus verifying part (1) of Theorem 1.5.

Let $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ and let $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and let $x, y \in \mathbb{R}$. We will show that

$$\begin{aligned}
&\mathbb{P}(a_1\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_1}^n, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_2}^n, \varphi\rangle \leq x,\\
&\quad b_1\langle (\hat{\mathcal{D}}_{t_1}^{0,n} + \hat{\mathcal{D}}_{t_1}^n), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^{0,n} + \hat{\mathcal{D}}_{t_2}^n), \varphi\rangle \leq y)\\
&\qquad \to \mathbb{P}(a_1\langle \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}_{t_1}, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}_{t_2}, \varphi\rangle \leq x)\\
&\qquad\quad \times \mathbb{P}(b_1\langle (\hat{\mathcal{D}}_{t_1}^0 + \hat{\mathcal{D}}_{t_1}), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^0 + \hat{\mathcal{D}}_{t_2}), \varphi\rangle \leq y)
\end{aligned} \tag{6.5}$$

as $n\to\infty$. The analogous proof for $t_1,\ldots,t_m \in [0,T]$ with $m>2$ follows similarly. This will then be sufficient to show that $\langle \check{\mathcal{A}}_0 + \check{\mathcal{E}}, \varphi\rangle$ and $\langle (\check{\mathcal{D}}^0 + \check{\mathcal{D}}), \varphi\rangle$ are independent of one another. First note that we may write

$$\begin{aligned}
&\mathbb{P}(a_1\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_1}^n, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_2}^n, \varphi\rangle \leq x,\\
&\quad b_1\langle (\hat{\mathcal{D}}_{t_1}^{0,n} + \hat{\mathcal{D}}_{t_1}^n), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^{0,n} + \hat{\mathcal{D}}_{t_2}^n), \varphi\rangle \leq y)\\
&\qquad = \mathbb{E}[\mathbf{1}_{\{a_1\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_1}^n, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_2}^n, \varphi\rangle \leq x\}}\mathbf{1}_{\{b_1\langle (\hat{\mathcal{D}}_{t_1}^{0,n} + \hat{\mathcal{D}}_{t_1}^n), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^{0,n} + \hat{\mathcal{D}}_{t_2}^n), \varphi\rangle \leq y\}}].
\end{aligned}$$

However, by the tower property of conditional expectations [6], we have that

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{a_1\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_1}^n, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_2}^n, \varphi\rangle \leq x\}}\mathbf{1}_{\{b_1\langle (\hat{\mathcal{D}}_{t_1}^{0,n} + \hat{\mathcal{D}}_{t_1}^n), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^{0,n} + \hat{\mathcal{D}}_{t_2}^n), \varphi\rangle \leq y\}}]\\
&\quad = \mathbb{E}[\mathbf{1}_{\{a_1\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_1}^n, \varphi\rangle + a_2\langle \hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}_{t_2}^n, \varphi\rangle \leq x\}}\\
&\qquad \times \mathbb{E}[\mathbf{1}_{\{b_1\langle (\hat{\mathcal{D}}_{t_1}^{0,n} + \hat{\mathcal{D}}_{t_1}^n), \varphi\rangle + b_2\langle (\hat{\mathcal{D}}_{t_2}^{0,n} + \hat{\mathcal{D}}_{t_2}^n), \varphi\rangle \leq y\}}|\mathcal{A}^n, \mathcal{E}^n]].
\end{aligned}$$

We now claim that

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{b_1\langle(\hat{\mathcal{D}}^{0,n}_{t_1}+\hat{\mathcal{D}}^n_{t_1}),\varphi\rangle+b_2\langle(\hat{\mathcal{D}}^{0,n}_{t_2}+\hat{\mathcal{D}}^n_{t_2}),\varphi\rangle\le y\}}|\mathcal{A}^n,\mathcal{E}^n]\\&\quad\xrightarrow{\mathbb{P}}\mathbb{P}(b_1\langle(\hat{\mathcal{D}}^0_{t_1}+\hat{\mathcal{D}}_{t_1}),\varphi\rangle+b_2\langle(\hat{\mathcal{D}}^0_{t_2}+\hat{\mathcal{D}}_{t_2}),\varphi\rangle\le y)\end{aligned}\tag{6.6}$$

as $n\to\infty$. Then, since by assumption

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{a_1\langle\hat{\mathcal{A}}^n_0+\hat{\mathcal{E}}^n_{t_1},\varphi\rangle+a_2\langle\hat{\mathcal{A}}^n_0+\hat{\mathcal{E}}^n_{t_2},\varphi\rangle\le x\}}]\\&\quad\to\mathbb{P}(a_1\langle\hat{\mathcal{A}}_0+\hat{\mathcal{E}}_{t_1},\varphi\rangle+a_2\langle\hat{\mathcal{A}}_0+\hat{\mathcal{E}}_{t_2},\varphi\rangle\le x)\end{aligned}$$

as $n\to\infty$, this will then imply (6.5), thus verifying part (1) of Theorem 1.5.

In order to see that (6.6) holds, first note that

$$\begin{aligned}&\mathbb{E}[\mathbf{1}_{\{b_1\langle(\hat{\mathcal{D}}^{0,n}_{t_1}+\hat{\mathcal{D}}^n_{t_1}),\varphi\rangle+b_2\langle(\hat{\mathcal{D}}^{0,n}_{t_2}+\hat{\mathcal{D}}^n_{t_2}),\varphi\rangle\le y\}}|\mathcal{A}^n,\mathcal{E}^n]\\&\quad=\mathbb{P}(b_1\langle(\hat{\mathcal{D}}^{0,n}_{t_1}+\hat{\mathcal{D}}^n_{t_1}),\varphi\rangle+b_2\langle(\hat{\mathcal{D}}^{0,n}_{t_2}+\hat{\mathcal{D}}^n_{t_2}),\varphi\rangle\le y|\mathcal{A}^n,\mathcal{E}^n).\end{aligned}$$

Now recall from (4.4) that we may write

$$\begin{aligned}&b_1\langle(\hat{\mathcal{D}}^{0,n}_{t_1}+\hat{\mathcal{D}}^n_{t_1}),\varphi\rangle+b_2\langle(\hat{\mathcal{D}}^{0,n}_{t_2}+\hat{\mathcal{D}}^n_{t_2}),\varphi\rangle\\&\quad=\sum_{i=1}^{A^n_0(\infty)}(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)+\sum_{i=1}^{E^n_{t_2}}(\langle\hat{\mathcal{D}}^{n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{n,i}_{t_2},b_2\varphi\rangle).\end{aligned}$$

Moreover, given $\mathcal{A}^n$ and $\mathcal{E}^n$, we have that the random variables $(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)$, $i=1,\dots,A^n_0(\infty)$, and $(\langle\hat{\mathcal{D}}^{n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{n,i}_{t_2},b_2\varphi\rangle)$, $i=1,\dots,E^n_{t_2}$, are mutually independent, with mean zero. In addition, it is straightforward to calculate that

$$\begin{aligned}&E\left[\sum_{i=1}^{A^n_0(\infty)}(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)^2\Big|\mathcal{A}^n_0,\mathcal{E}^n\right]\\&\quad+E\left[\sum_{i=1}^{E^n_{t_2}}(\langle\hat{\mathcal{D}}^{n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{n,i}_{t_2},b_2\varphi\rangle)^2\Big|\mathcal{A}^n_0,\mathcal{E}^n\right]\\&\quad=E\left[\int_0^{t_1}\langle\bar{\mathcal{A}}^n_u,(b_1\varphi+b_2\varphi)^2h\rangle\,du\Big|\mathcal{A}^n_0,\mathcal{E}^n\right].\end{aligned}\tag{6.7}$$

Also, note that for each $i=1,\dots,A^n_0(\infty)$,

$$\begin{aligned}&E[(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)^3|\mathcal{A}^n_0,\mathcal{E}^n]\\&\quad=E[(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)^2(\langle\hat{\mathcal{D}}^{0,n,i}_{t_1},b_1\varphi\rangle+\langle\hat{\mathcal{D}}^{0,n,i}_{t_2},b_2\varphi\rangle)|\mathcal{A}^n_0,\mathcal{E}^n]\end{aligned}\tag{6.8}$$

$$\leq \frac{(|b_1|+|b_2|)\|\varphi\|_\infty(1+t_2\|h\|_\infty)}{\sqrt{n}} \times E[(\langle \hat{\mathcal{D}}^{0,n,i}_{t_1}, b_1\varphi\rangle + \langle \hat{\mathcal{D}}^{0,n,i}_{t_2}, b_2\varphi\rangle)^2 | \mathcal{A}^n_0, \mathcal{E}^n],$$

and, similarly, for each $i = 1, \ldots, E^n_{t_2}$,

$$\begin{aligned} &E[(\langle \hat{\mathcal{D}}^{n,i}_{t_1}, b_1\varphi\rangle + \langle \hat{\mathcal{D}}^{0,n,i}_{t_2}, b_2\varphi\rangle)^3 | \mathcal{A}^n_0, \mathcal{E}^n] \\ &\quad \leq \frac{(|b_1|+|b_2|)\|\varphi\|_\infty(1+t_2\|h\|_\infty)}{\sqrt{n}} \\ &\qquad \times E[(\langle \hat{\mathcal{D}}^{n,i}_{t_1}, b_1\varphi\rangle + \langle \hat{\mathcal{D}}^{n,i}_{t_2}, b_2\varphi\rangle)^2 | \mathcal{A}^n_0, \mathcal{E}^n]. \end{aligned} \tag{6.9}$$

Now let Φ denote the c.d.f. of a standard, normal random variable. It then follows by (6.7), (6.8), (6.9) and an application of the Berry–Esseen theorem [6] for independent (but not necessarily identically distributed) random variables that

$$\begin{aligned} &\left| \mathbb{P}\left(\frac{b_1\langle(\hat{\mathcal{D}}^{0,n}_{t_1} + \hat{\mathcal{D}}^n_{t_1}), \varphi\rangle + b_2\langle(\hat{\mathcal{D}}^{0,n}_{t_2} + \hat{\mathcal{D}}^n_{t_2}), \varphi\rangle}{(\mathbb{E}[\int_0^{t_1} \langle \bar{\mathcal{A}}^n_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du | \mathcal{A}^n_0, \mathcal{E}^n])^{1/2}} \leq y | \mathcal{A}^n, \mathcal{E}^n \right) - \Phi(y) \right| \\ &\quad \leq \frac{1}{\sqrt{n}} \frac{(|b_1|+|b_2|)\|\varphi\|_\infty(1+t_2\|h\|_\infty)}{(\mathbb{E}[\int_0^{t_1} \langle \bar{\mathcal{A}}^n_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du | \mathcal{A}^n_0, \mathcal{E}^n])^{1/2}}. \end{aligned}$$

Hence, in order to complete the proof of (6.6) and hence verify part (1) of Theorem 1.5, it suffices to show that

$$E\left[\int_0^{t_1} \langle \bar{\mathcal{A}}^n_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du \middle| \mathcal{A}^n_0, \mathcal{E}^n\right] \xrightarrow{\mathbb{P}} \int_0^{t_1} \langle \bar{\mathcal{A}}_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du \quad \text{as } n \to \infty. \tag{6.10}$$

However, note that by Theorem 5.2 and the continuity of the integral map [1], we have that

$$\int_0^{t_1} \langle \bar{\mathcal{A}}^n_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du \xrightarrow{\mathbb{P}} \int_0^{t_1} \langle \bar{\mathcal{A}}_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du, \tag{6.11}$$

as $n \to \infty$. Next, note that the uniform integrability of $\{\bar{A}^n_0(\infty), n \geq 1\}$ and $\{\bar{E}^n_T, n \geq 1\}$ implies the uniform integrability of

$$\left\{ \int_0^{t_1} \langle \bar{\mathcal{A}}^n_u, (b_1\varphi + b_2\varphi)^2 h\rangle\, du, n \geq 1 \right\}. \tag{6.12}$$

It is then straightforward to show that (6.11) implies (6.10), thus completing the verification of part (1) of Theorem 1.5. The proof of the verification of part (2) of Theorem 1.5 follows in a similar manner to the above and has been omitted for the sake of brevity. This completes the proof. $\square$

The following is now our main result of this section. Its proof is a straightforward consequence of Theorem 1.1, (6.1) and Proposition 6.4.

THEOREM 6.5. *If $\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n \Rightarrow \hat{\mathcal{A}}_0 + \hat{\mathcal{E}}$ in $\mathbb{D}([0,T],\mathcal{S}')$ as $n\to\infty$, then*

$$\hat{\mathcal{A}}^n \Rightarrow \hat{\mathcal{A}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty, \tag{6.13}$$

where $\hat{\mathcal{A}}$ is the solution to the stochastic integral equation

$$\begin{aligned} \langle \hat{\mathcal{A}}_t,\varphi\rangle &= \langle \hat{\mathcal{A}}_0,\varphi\rangle + \langle \hat{\mathcal{E}}_t,\varphi\rangle - \langle \hat{\mathcal{D}}^0+\hat{\mathcal{D}},\varphi\rangle \\ &\qquad - \int_0^t \langle \hat{\mathcal{A}}_s, h\varphi\rangle\,ds + \int_0^t \langle \hat{\mathcal{A}}_s,\varphi'\rangle\,ds, \end{aligned} \tag{6.14}$$

for each $t\in[0,T]$ and $\varphi\in\mathcal{S}$. In addition, if $\hat{\mathcal{E}}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi)=\sigma^2(s\wedge t)\varphi(0)\psi(0)$, then $\hat{\mathcal{A}}$ is a generalized $\mathcal{S}'$-valued Ornstein–Uhlenbeck process driven by a generalized $\mathcal{S}'$-valued Wiener process with covariance functional

$$K_{\hat{\mathcal{E}}-(\hat{\mathcal{D}}^0+\hat{\mathcal{D}})}(s,\varphi;t,\psi) = \sigma^2(s\wedge t)\varphi(0)\psi(0) + \int_0^{s\wedge t} \langle \bar{\mathcal{A}}_u h, \varphi\psi\rangle\,du. \tag{6.15}$$

PROOF. First note that by (6.1) we have that $\hat{\mathcal{A}}^n = \Psi_{B^{\mathcal{A}}}(\hat{\mathcal{A}}_0^n + \hat{\mathcal{E}}^n - (\hat{\mathcal{D}}^{0,n}+\hat{\mathcal{D}}^n))$, where the map $\Psi_{B^{\mathcal{A}}}:\mathbb{D}([0,T],\mathcal{S}')\mapsto\mathbb{D}([0,T],\mathcal{S}')$ is a continuous map. The convergence (6.13) now follows by Theorem 1.1 and Proposition 6.4.

Next, suppose that $\hat{\mathcal{E}}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi)=\sigma^2(s\wedge t)\varphi(0)\psi(0)$. Then, combining this with (6.3) and the fact that $\hat{\mathcal{D}}^0+\hat{\mathcal{D}}$ and $\hat{\mathcal{A}}_0+\hat{\mathcal{E}}$ are independent from Proposition 6.4, yields (6.15). Thus, by Definition 6.2, $\hat{\mathcal{A}}$ is an $\mathcal{S}'$-valued Ornstein–Uhlenbeck process. $\square$

Recall now from Proposition 5.4 of Section 5.1 that if $\langle\bar{\mathcal{E}},\varphi\rangle=\lambda\varphi(0)e$ for each $\varphi\in\mathcal{S}$ and some $\lambda\geq 0$, then a stationary solution to the fluid limit equation (5.9) is given by $\bar{\mathcal{A}}=\lambda\mathcal{F}_e$. We now show that under the additional condition that $\hat{\mathcal{E}}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi)=\sigma^2(s\wedge t)\varphi(0)\psi(0)$, then the resulting limiting diffusion scaled age process $\hat{\mathcal{A}}$ of Theorem 6.5 is a time-homogeneous Markov process. Our result is the following. Note also that a similar approach may be used to analyze the diffusion limit of the residual service time process in Theorem 6.9 in the following subsection.

PROPOSITION 6.6. *If $\langle\bar{\mathcal{E}},\varphi\rangle=\lambda\varphi(0)e$ for each $\varphi\in\mathcal{S}$, $\bar{\mathcal{A}}_0=\lambda\mathcal{F}_e$ and $\hat{\mathcal{E}}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi)=$*

$\sigma^2(s\wedge t)\varphi(0)\psi(0)$, then $\hat{\mathcal{A}}$ is an $\mathcal{S}'$-valued Ornstein–Uhlenbeck process driven by an $\mathcal{S}'$-valued Wiener process with covariance functional given for each $\varphi,\psi\in\mathcal{S}$ and $s,t\geq 0$ by

$$K_{\hat{\mathcal{E}}-(\hat{\mathcal{D}}^0+\hat{\mathcal{D}})}(s,\varphi;t,\psi)=(s\wedge t)\langle\sigma^2\delta_0+\lambda\mathcal{F},\varphi\psi\rangle. \tag{6.16}$$

PROOF. It is clear that the covariance functional of $\hat{\mathcal{E}}$ is given by

$$K_{\hat{\mathcal{E}}}(s,\varphi;t,\psi)=(s\wedge t)\langle\sigma^2\delta_0,\varphi\psi\rangle. \tag{6.17}$$

We now show that the covariance functional of $\hat{\mathcal{D}}^0+\hat{\mathcal{D}}$ is given by

$$K_{\hat{\mathcal{D}}^0+\hat{\mathcal{D}}}(s,\varphi;t,\psi)=\lambda(s\wedge t)\langle\mathcal{F},\varphi\psi\rangle. \tag{6.18}$$

Then, since $\hat{\mathcal{E}}$ and $\hat{\mathcal{D}}^0+\hat{\mathcal{D}}$ are independent, summing (6.17) and (6.18) will prove (6.16), which will complete the proof.

Note that by Proposition 5.4 we have that since by assumption $\bar{\mathcal{A}}_0=\lambda\mathcal{F}_e$, it follows that $\bar{\mathcal{A}}=\lambda\mathcal{F}_e$ is the unique solution to the fluid limit equation (5.9). Therefore, by Lemma 6.3 we have that for each $\varphi,\psi\in\mathcal{S}$ and $s,t\geq 0$,

$$\begin{aligned}K_{\hat{\mathcal{D}}^0+\hat{\mathcal{D}}}(s,\varphi;t,\psi)&=\int_0^{s\wedge t}\langle\bar{\mathcal{A}}_u,\varphi\psi h\rangle\,du\\&=\int_0^{s\wedge t}\int_{\mathbb{R}_+}\varphi(x)\psi(x)h(x)\,d\bar{\mathcal{A}}_u(x)\,du\\&=\lambda(s\wedge t)\int_{\mathbb{R}_+}\varphi(x)\psi(x)h(x)\bar{F}(x)\,dx\\&=\lambda(s\wedge t)\int_{\mathbb{R}_+}\varphi(x)\psi(x)f(x)\,dx.\end{aligned}$$

This proves (6.18), which completes the proof. □

We now note that using (3.2) of Theorem 3.3 and (3.19) of Proposition 3.6, one may obtain an explicit representation of $\hat{\mathcal{A}}$ in terms of the $\mathcal{S}'$-valued Wiener process given in Proposition 6.6 above. Direct calculations may then be used in order to obtain the transient and limiting distribution of $\hat{\mathcal{A}}$. In particular, assuming that $\hat{\mathcal{A}}_0$ is a Gaussian random variable, one may then show that for each $t\in[0,T]$, $\hat{\mathcal{A}}_t$ is a Gaussian random variable with mean

$$\mathbb{E}[\langle\hat{\mathcal{A}}_t,\varphi\rangle]=\langle\hat{\mathcal{A}}_0,\bar{F}^{-1}\tau_{-t}(\varphi\bar{F})\rangle,\qquad\varphi\in\mathcal{S}, \tag{6.19}$$

and covariance functional given for each $\varphi,\psi\in\mathcal{S}$ and $t\in[0,T]$ by

$$\begin{aligned}\mathbb{E}[\langle\hat{\mathcal{A}}_t,\varphi\rangle\langle\hat{\mathcal{A}}_t,\psi\rangle]&=\lambda\langle\mathcal{F}_e,\bar{F}^{-1}\tau_{-t}(\varphi\psi\bar{F})(1-\bar{F}^{-1}\tau_{-t}\bar{F})\rangle\\&\quad+\int_0^t\varphi(u)\psi(u)(\lambda F(u)+\sigma^2\bar{F}(u))\bar{F}(u)\,du.\end{aligned} \tag{6.20}$$

In addition, taking limits as $t \to \infty$, one also finds that $\hat{\mathcal{A}}_t$ weakly converges as $t \to \infty$ to a Gaussian random variable $\mathcal{A}_\infty$ with mean zero and covariance functional given for each $\varphi, \psi \in \mathcal{S}$ by

$$\mathbb{E}[\langle \hat{\mathcal{A}}_\infty, \varphi \rangle \langle \hat{\mathcal{A}}_\infty, \psi \rangle] = \langle \mathcal{F}_e, (\lambda F + \sigma^2 \bar{F}) \varphi \psi \rangle.$$

We now conclude this subsection by noting that one may heuristically attempt to substitute the test function $\mathbf{1}_{\{x \geq 0\}}$ into the formula for $\hat{\mathcal{A}}$ provided by Theorem 3.3 in order to obtain an expression for the limiting diffusion scaled total number of customers in the system. For instance, suppose that $\hat{\mathcal{A}}_0 = 0$ and that, as in the statement of Proposition 6.6, we have that $\langle \bar{\mathcal{E}}, \varphi \rangle = \lambda \varphi(0) e$ for each $\varphi \in \mathcal{S}$ and $\bar{\mathcal{A}}_0 = \lambda \mathcal{F}_e$ and $\hat{\mathcal{E}}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional $K_{\hat{\mathcal{E}}}(s, \varphi; t, \psi) = \sigma^2 (s \wedge t) \varphi(0) \psi(0)$. Then, using the form of the generator $B^{\mathcal{A}}$ from (3.14), the semi-group $(S_t^{\mathcal{A}})_{t \geq 0}$ from Proposition 3.6 and (6.16) of Proposition 6.6, one obtains after a substitution into Theorem 3.3 that the limiting diffusion scaled number of customers in the system at time $t \geq 0$ is heuristically given by

$$\hat{B}_t - \int_0^t \hat{B}_s f(t-s)\, ds = \int_0^t \bar{F}(t-s)\, d\hat{B}_s,$$

where $\hat{B} = (\hat{B}_t)_{t \geq 0}$ is a Brownian motion with infinitesimal variance $\sigma^2 + \lambda$.

6.2. *Residuals.* We next proceed to study the residual service time process $\mathcal{R}$ defined in Section 2.2. Our setup is the same as in Section 5.2. That is, we consider a sequence of $G/GI/\infty$ queues indexed by n, where the arrival rate to the nth system is of order n and the service time distribution does not change with n. For the remainder of this subsection, we also assume that $\bar{\mathcal{R}}_0^n + \bar{E}^n \mathcal{F} \Rightarrow \bar{\mathcal{R}}_0 + \bar{E} \mathcal{F}$ as $n \to \infty$, where $\bar{\mathcal{R}}_0 + \bar{E}\mathcal{F}$ is a nonrandom quantity. By Theorem 5.6 of Section 5.2, this implies that $\bar{\mathcal{R}}$ is nonrandom as well.

Now, for each $n \geq 1$, in addition to the diffusion scaled quantities defined in Section 6.1, let us also now define the diffusion scaled quantities

$$\hat{\mathcal{R}}^n \equiv \sqrt{n}(\bar{\mathcal{R}}^n - \bar{\mathcal{R}}), \qquad \hat{\mathcal{R}}_0^n \equiv \sqrt{n}(\bar{\mathcal{R}}_0^n - \bar{\mathcal{R}}_0) \quad \text{and} \quad \hat{\mathcal{G}}^n \equiv \sqrt{n} \bar{\mathcal{G}}^n.$$

Then, after recalling the form of the fluid limit $\bar{\mathcal{R}}$ from Theorem 5.6, note that using system equation (2.20) in conjunction with Theorem 3.3 and Proposition 3.7, one has that

$$\hat{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\hat{\mathcal{R}}_0^n + \hat{E}^n \mathcal{F} + \hat{\mathcal{G}}^n), \tag{6.21}$$

where the map $\Psi_{B^{\mathcal{R}}} : \mathbb{D}([0,T], \mathcal{S}') \mapsto \mathbb{D}([0,T], \mathcal{S}')$ is a continuous map. Our strategy now is to proceed similar to as in Section 6.1. That is, we first prove a weak convergence result for the sequence $(\hat{\mathcal{R}}_0^n + \hat{E}^n \mathcal{F} + \hat{\mathcal{G}}^n)_{n \geq 1}$ and then we apply Theorem 1.1 together with (6.21) in order to obtain a diffusion limit result for the sequence $(\hat{\mathcal{R}}^n)_{n \geq 1}$.

We first show that for each $n \geq 1$, the process $\hat{\mathcal{G}}^n$ may be well approximated by a process which is independent of $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F}$. For each $n \geq 1$, let $\check{\mathcal{G}}^n$ be the $\mathcal{S}'$-valued process defined for $\varphi \in \mathcal{S}$ by

$$\langle \check{\mathcal{G}}_t^n, \varphi \rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n\bar{E}_t \rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi \rangle), \qquad t \geq 0.$$

Note that it is clear that $\check{\mathcal{G}}^n$ is independent of $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F}$. We now have the following result. Its proof may be found in the Appendix.

LEMMA 6.7. *If* $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} \Rightarrow \hat{\mathcal{R}}_0 + \hat{E}\mathcal{F}$ *in* $\mathbb{D}([0,T],\mathcal{S}')$ *as* $n \to \infty$, *then*

$$\hat{\mathcal{G}}^n - \check{\mathcal{G}}^n \Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty, \tag{6.22}$$

and

$$\check{\mathcal{G}}^n \Rightarrow \hat{\mathcal{G}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty, \tag{6.23}$$

where $\hat{\mathcal{G}}$ *is an* $\mathcal{S}'$*-valued Wiener process with covariance functional given for* $\varphi, \psi \in \mathcal{S}$ *and* $s, t \geq 0$ *by*

$$K_{\hat{G}}(s,\varphi;t,\psi) = (\bar{E}_s \wedge \bar{E}_t)\operatorname{Cov}(\varphi(\eta),\psi(\eta)), \tag{6.24}$$

where η *is a random variable with c.d.f.* F.

PROOF. See Appendix. □

Using Lemma 6.7, we may now prove the following result on the weak convergence of $(\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \hat{\mathcal{G}}^n)_{n\geq 1}$. We have the following.

PROPOSITION 6.8. *If* $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} \Rightarrow \hat{\mathcal{R}}_0 + \hat{E}\mathcal{F}$ *in* $\mathbb{D}([0,T],\mathcal{S}')$ *as* $n \to \infty$, *then*

$$\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \hat{\mathcal{G}}^n \Rightarrow \hat{\mathcal{R}}_0 + \hat{E}\mathcal{F} + \hat{\mathcal{G}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty, \tag{6.25}$$

where $\hat{\mathcal{G}}$ *is as given in Lemma 6.7 and is independent of* $\hat{\mathcal{R}}_0 + \hat{E}\mathcal{F}$.

PROOF. Since $\check{G}^n$ is independent of $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F}$ for each $n \geq 1$, it follows Theorem 1.5 and (6.23) of Lemma 6.7 that we have the convergence

$$\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \check{G}^n \Rightarrow \hat{\mathcal{R}}_0 + \hat{E}\mathcal{F} + \hat{\mathcal{G}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n \to \infty. \tag{6.26}$$

The result now follows by (6.26), Theorem 1.5, (6.22) of Lemma 6.7 and the fact that we may write

$$\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \check{G}^n = \hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \check{G}^n + (\hat{G}^n - \check{G}^n). \qquad \square$$

The following is now the main result of this subsection. It provides a weak limit for the sequence $(\hat{\mathcal{R}}^n)_{n\geq 1}$.

THEOREM 6.9. *If $\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} \Rightarrow \hat{\mathcal{R}}_0 + \hat{E}\mathcal{F}$ in $\mathbb{D}([0,T],\mathcal{S}')$ as $n\to\infty$, then*

$$\hat{\mathcal{R}}^n \Rightarrow \hat{\mathcal{R}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty, \tag{6.27}$$

where $\hat{\mathcal{R}}$ is the solution to the stochastic integral equation

$$\langle \hat{\mathcal{R}}_t,\varphi\rangle = \langle \hat{\mathcal{R}}_0,\varphi\rangle + \langle \hat{\mathcal{G}}_t,\varphi\rangle + \hat{E}_t\langle \mathcal{F},\varphi\rangle - \int_0^t \langle \hat{\mathcal{R}}_s,\varphi'\rangle\,ds, \qquad t\in[0,T],\varphi\in\mathcal{S}. \tag{6.28}$$

In addition, if $\hat{E}$ is a Brownian motion with diffusion coefficient σ, then $\hat{\mathcal{R}}$ is a generalized $\mathcal{S}'$-valued Ornstein–Uhlenbeck process driven by a generalized $\mathcal{S}'$-valued Wiener process with covariance functional

$$\begin{aligned} &K_{\hat{E}\mathcal{F}+\hat{G}}(s,\varphi;t,\psi) \\ &\quad = \sigma^2(s\wedge t)\mathbb{E}[\varphi(\eta)]\mathbb{E}[\psi(\eta)] + (\bar{E}_s\wedge\bar{E}_t)\operatorname{Cov}(\varphi(\eta),\psi(\eta)), \end{aligned} \tag{6.29}$$

where η is a random variable with c.d.f. F.

PROOF. First note that by (6.21) we have that $\hat{\mathcal{R}}^n = \Psi_{B^{\mathcal{R}}}(\hat{\mathcal{R}}_0^n + \hat{E}^n\mathcal{F} + \hat{\mathcal{G}}^n)$, where the map $\Psi_{B^{\mathcal{R}}} : \mathbb{D}([0,T],\mathcal{S}') \mapsto \mathbb{D}([0,T],\mathcal{S}')$ is a continuous map. The convergence (6.27) now follows by Theorem 1.1 and Proposition 6.8.

Next, note that if $\hat{E}$ is a Brownian motion with diffusion coefficient σ, then it is easily checked that $\hat{E}\mathcal{F}$ is an $\mathcal{S}'$-valued Wiener process with covariance functional

$$K_{\hat{E}\mathcal{F}}(s,\varphi;t,\psi) = \sigma^2(s\wedge t)\mathbb{E}[\varphi(\eta)]\mathbb{E}[\psi(\eta)]. \tag{6.30}$$

Combining (6.24) with (6.30) and the fact that $\hat{E}\mathcal{F}$ and $\hat{\mathcal{G}}$ are independent, yields (6.29). $\square$

REMARK 6.10. Note that in the special case when the arrival process to the nth system is a Poisson process with rate λn, we then have that $\bar{E} = \lambda e$ and so $\hat{E}$ turns out to be a Brownian motion with diffusion coefficient λ. It then follows that $K_{\hat{E}\mathcal{F}}(s,\varphi;t,\psi) = \lambda(s\wedge t)\mathbb{E}[\varphi(\eta)\psi(\eta)]$ and so Theorem 6.9 gives us a version of Theorem 3 of *[8]*.

APPENDIX

In the Appendix, we provide the proofs of several supporting lemmas from the main body of the paper. We begin with the proof of Lemma 3.4.

PROOF OF LEMMA 3.4. We prove part (1) by induction. For each $n\geq 0$ and $t\geq 0$ fixed, denote the quantity on the lefthand side of (3.16) by L_n. For the base case of $n=0$, it is straightforward to see that $L_0\leq 1$. Next, for the inductive step, suppose that (3.16) holds for $n = 0, \ldots, k-1$, and $t \geq 0$. Then, we have that

$$
\begin{aligned}
\left|\left(\frac{\bar{F}(x+t)}{\bar{F}(x)}\right)^{(k)}\right| &= \left|\left(\left(\frac{\bar{F}(x+t)}{\bar{F}(x)}\right)^{(1)}\right)^{(k-1)}\right| \\
&= \left|\left((h(x) - h(x+t))\frac{\bar{F}(x+t)}{\bar{F}(x)}\right)^{(k-1)}\right| \\
&\leq \sum_{i=0}^{k-1} \binom{k-1}{i} |(h(x) - h(x+t))^{(k-1-i)}| \left|\left(\frac{\bar{F}(x+t)}{\bar{F}(x)}\right)^{(i)}\right| \\
&\leq 2 \sum_{i=0}^{k-1} \binom{k-1}{i} \|h^{(k-1-i)}\|_\infty L_i \\
&< \infty.
\end{aligned}
$$

This completes the proof of part (1).

We next prove part (2) by induction as well. First recall that for $s, t \geq 0$, we may write

$$
\frac{\bar{F}(x+t)}{\bar{F}(x+s)} = \exp\left(-\int_{x+s}^{x+t} h(u)\, du\right). \tag{A.1}
$$

Hence, for the base case of $n = 0$, we have that

$$
\begin{aligned}
\sup_{x\geq 0}\left|\frac{\bar{F}(x+t)}{\bar{F}(x)} - \frac{\bar{F}(x+s)}{\bar{F}(x)}\right| &= \sup_{x\geq 0}\left|\frac{\bar{F}(x+s)}{\bar{F}(x)}\left(1 - \frac{\bar{F}(x+t)}{\bar{F}(x+s)}\right)\right| \\
&\leq \sup_{x\geq 0}\left|\frac{\bar{F}(x+s)}{\bar{F}(x)}\right| \sup_{x\geq 0}\left|1 - \frac{\bar{F}(x+t)}{\bar{F}(x+s)}\right| \\
&\leq \sup_{x\geq 0}\left|1 - \frac{\bar{F}(x+t)}{\bar{F}(x+s)}\right| \\
&\leq 1 - e^{-\|h\|_\infty |t-s|} \\
&\leq \|h\|_\infty |t-s|,
\end{aligned}
$$

where the third inequality above follows from (A.1) and the final inequality follows from the mean value theorem. Next, for the inductive step, suppose that (3.17) holds for $n = 0, 1, \ldots, k-1$. We then have that

$$
\begin{aligned}
&\sup_{x\geq 0}\left|\left(\frac{\bar{F}(x+t)}{\bar{F}(x)} - \frac{\bar{F}(x+s)}{\bar{F}(x)}\right)^{(k)}\right| \\
&\quad = \sup_{x\geq 0}\left|\left((h(x) - h(x+s))\frac{\bar{F}(x+s)}{\bar{F}(x)} - (h(x) - h(x+t))\frac{\bar{F}(x+t)}{\bar{F}(x)}\right)^{(k-1)}\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{x\geq 0}\left|\left((h(x+t)-h(x+s))\frac{\bar F(x+s)}{\bar F(x)}\right.\right.\\
\text{(A.2)}\qquad & \qquad \left.\left. -(h(x)-h(x+t))\left(\frac{\bar F(x+t)}{\bar F(x)}-\frac{\bar F(x+s)}{\bar F(x)}\right)\right)^{(k-1)}\right|\\
&= \sup_{x\geq 0}\left|\sum_{i=0}^{k-1}\binom{k-1}{i}[(h(x+t)-h(x+s))]^{(k-1-i)}\left(\frac{\bar F(x+s)}{\bar F(x)}\right)^{(i)}\right.\\
&\qquad \left. -\sum_{i=0}^{k-1}\binom{k-1}{i}(h(x)-h(x+t))^{(k-1-i)}\left(\frac{\bar F(x+t)}{\bar F(x)}-\frac{\bar F(x+s)}{\bar F(x)}\right)^{(i)}\right|.
\end{aligned}
$$

Now note that since by Assumption 2.1 we have that $h\in C_b^\infty(\mathbb{R}_+)$, it follows that all of the derivatives of h are bounded and hence uniformly continuous as well. Using this fact, part (1) and the inductive hypothesis it now follows that (A.2) is less than or equal to

$$
2|t-s|\sum_{i=0}^{k-1}\binom{k-1}{i}(\|h^{((k-1-i)-1)}\|_\infty L_i+\|h^{(k-1-i)}\|_\infty M_i)\equiv M_k|t-s|.
$$

This proves part (2) and completes the proof. $\square$

We next provide the proof of Lemma 6.3

Proof of Lemma 6.3. We must show that

$$
\hat{\mathcal{D}}^{0,n}+\hat{\mathcal{D}}^n\Rightarrow\hat{\mathcal{D}}^0+\hat{\mathcal{D}}\qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty. \tag{A.3}
$$

Let $\varphi,\psi\in\mathcal{S}$ and let $t\geq 0$. It then follows by Proposition 4.1 that we may write

$$
\begin{aligned}
&\langle\langle\hat{\mathcal{D}}^{0,n}+\hat{\mathcal{D}}^n\rangle\rangle_t(\varphi,\psi)\\
&\quad=\frac{1}{n}\left(\sum_{i=1}^{A_0^n(\infty)}\int_0^{\tilde\eta_i\wedge t}\varphi(u-\tilde\tau_i^n)\psi(u-\tilde\tau_i^n)h_{\tilde\tau_i^n}(u)\,du\right.\\
&\qquad\qquad\left.+\sum_{i=1}^{E_t^n}\int_0^{\eta_i\wedge(t-\tau_i^n)}\varphi(u)\psi(u)h(u)\,du\right)\\
&\quad=\int_0^t\langle\bar A_s^n,\varphi\psi h\rangle\,ds,
\end{aligned}
$$

where the second equality above follows as a result of Proposition 2.3. However, by the continuity of the integral mapping on $\mathbb{D}([0,T],\mathbb{R})$, it now follows by Theorem 5.2 that

$$\int_0^e \langle \bar{\mathcal{A}}_s^n, \varphi\psi h\rangle\,ds \Rightarrow \int_0^e \langle \bar{\mathcal{A}}_s, \varphi\psi h\rangle\,ds \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \tag{A.4}$$

as $n \to \infty$.

We now verify that $(\hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n)_{n\geq 1}$ satisfies parts (1) and (2) of Theorem 1.5. We begin with part (1). Let $\varphi = \psi$ in (A.4) and note that using the fact that the maximum jumps of both $\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi\rangle$ and $\langle\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n\rangle\rangle(\varphi,\varphi)$ over the interval $[0,T]$ are uniformly bounded in n, along with the martingale FCLT [10] and (A.4), yields the limit

$$\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi\rangle \Rightarrow \langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi\rangle \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n\to\infty.$$

Thus, part (1) of Theorem 1.5 is satisfied. We next proceed to verify part (2) of Theorem 1.5. Let $m \geq 1$ and let $\varphi_1,\ldots,\varphi_m \in \mathcal{S}$ and $1 \leq i,j \leq m$. Then, using the fact that the maximum jumps of both $(\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi_1\rangle, \ldots, \langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi_m\rangle)$ and $\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n\rangle(\varphi_i,\varphi_j)$ are bounded over the interval $[0,T]$, uniformly in n, it follows by the martingale FCLT [10] and (A.4) that

$$\begin{aligned}(\langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi_1\rangle, \ldots, \langle \hat{\mathcal{D}}^{0,n} + \hat{\mathcal{D}}^n, \varphi_m\rangle)\\ \Rightarrow (\langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi_1\rangle, \ldots, \langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi_m\rangle)\end{aligned}$$

in $\mathbb{D}^m([0,T],\mathbb{R})$ as $n\to\infty$. This limit then provides convergence of the finite-dimensional distributions of the random vector on the lefthand side above, which is sufficient to verify part (2) of Theorem 1.5. Thus, (A.3) holds, where $\hat{\mathcal{D}}^0 + \hat{\mathcal{D}}$ is an $\mathcal{S}'$-valued Gaussian martingale with tensor quadratic covariation given by (A.4). Equation (6.3) now holds since $\langle \hat{\mathcal{D}}^0 + \hat{\mathcal{D}}, \varphi\rangle$ has independent increments for each $\varphi \in \mathcal{S}$. □

Next, we provide the proof of Lemma 6.7.

PROOF OF LEMMA 6.7. We first prove that

$$\hat{\mathcal{G}}^n \Rightarrow \hat{\mathcal{G}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty. \tag{A.5}$$

In order to do so, we will verify that parts (1) and (2) of Theorem 1.5 are satisfied. We begin with part (1). Let $\varphi,\psi \in \mathcal{S}$ and note that by Proposition 4.2, the functional strong law of large numbers [33] and the random time change theorem [1], we have that

$$\begin{aligned}[\hat{\mathcal{G}}^n](\varphi,\psi) &= \frac{1}{n}\sum_{i=1}^{E^n}(\varphi(\eta_i) - \langle \mathcal{F},\varphi\rangle)(\psi(\eta_i) - \langle \mathcal{F},\psi\rangle)\\ &\Rightarrow \bar{E}\,\mathrm{Cov}(\varphi(\eta),\psi(\eta)) \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n\to\infty,\end{aligned} \tag{A.6}$$

where η is a random variable with c.d.f. F. Now letting $\varphi = \psi$ in (A.6) and using the fact that the maximum jump of $\langle \hat{\mathcal{G}}^n, \varphi\rangle$ over the interval $[0,T]$ is bounded uniformly in n, along with the martingale FCLT [10], yields the limit

$$\langle \hat{\mathcal{G}}^n, \varphi\rangle \Rightarrow \langle \hat{\mathcal{G}}, \varphi\rangle \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n\to\infty.$$

Thus, part (1) of Theorem 1.5 holds. We next prove that part (2) of Theorem 1.5 holds. Let $m \geq 1$ and let $\varphi_1,\ldots,\varphi_m \in \mathcal{S}$. Then, using the limit (A.6) and the fact that the maximum jump of $(\langle \hat{\mathcal{G}}^n, \varphi_1\rangle,\ldots,\langle \hat{\mathcal{G}}^n, \varphi_m\rangle)$ over the interval $[0,T]$ is bounded uniformly in n, along with the martingale FCLT [10], yields the limit

$$(\langle \hat{\mathcal{G}}^n, \varphi_1\rangle,\ldots,\langle \hat{\mathcal{G}}^n, \varphi_m\rangle) \Rightarrow (\langle \hat{\mathcal{G}}, \varphi_1\rangle,\ldots,\langle \hat{\mathcal{G}}, \varphi_m\rangle) \qquad \text{in } \mathbb{D}^m([0,T],\mathbb{R})$$

as $n\to\infty$. This limit then provides convergence of the finite-dimensional distributions of the random vector on the left-hand side above, which shows that part (2) of Theorem 1.5 holds. Thus, (A.5) it proven.

In order to complete the proof, it now suffices to show that

$$\hat{\mathcal{G}}^n - \check{\mathcal{G}}^n \Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty. \tag{A.7}$$

However, in order to show (A.7), it suffices by Theorem 1.5 to show that for each $\varphi\in\mathcal{S}$,

$$\langle \hat{\mathcal{G}}^n, \varphi\rangle - \langle \check{\mathcal{G}}^n, \varphi\rangle \Rightarrow 0 \qquad \text{in } \mathbb{D}([0,T],\mathbb{R}) \text{ as } n\to\infty. \tag{A.8}$$

We proceed as follows. In a similar manner to the above, one may show using the martingale FCLT that

$$\check{\mathcal{G}}^n \Rightarrow \hat{\mathcal{G}} \qquad \text{in } \mathbb{D}([0,T],\mathcal{S}') \text{ as } n\to\infty. \tag{A.9}$$

Hence, for each $\varphi\in\mathcal{S}$, the sequence $(\langle \hat{\mathcal{G}}^n, \varphi\rangle - \langle \check{\mathcal{G}}^n, \varphi\rangle)_{n\geq 1}$ is tight in $\mathbb{D}([0,T],\mathbb{R})$ and so in order to show (A.8), it suffices to show that for each $0\leq t\leq T$,

$$\langle \hat{\mathcal{G}}_t^n, \varphi\rangle - \langle \check{\mathcal{G}}_t^n, \varphi\rangle \Rightarrow 0 \qquad \text{as } n\to\infty. \tag{A.10}$$

First note that we may write

$$\begin{aligned}
&\langle \hat{\mathcal{G}}_t^n, \varphi\rangle - \langle \check{\mathcal{G}}_t^n, \varphi\rangle \\
&\quad = \mathbf{1}_{\{E_t^n \geq \lfloor n\bar{E}_t\rfloor\}} \frac{1}{\sqrt{n}} \left(\sum_{i=\lfloor n\bar{E}_t\rfloor}^{E_t^n} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle)\right) \\
&\qquad - \mathbf{1}_{\{\lfloor n\bar{E}_t\rfloor \geq E_t^n\}} \frac{1}{\sqrt{n}} \left(\sum_{i=E_t^n}^{\lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle)\right).
\end{aligned}$$

Now squaring both sides of the above and using the basic identity $(x_1 + x_2)^2 \le 2(x_1^2 + x_2^2)$, it is straightforward to show that one may write

$$(\langle \hat{\mathcal{G}}_t^n, \varphi\rangle - \langle \check{\mathcal{G}}_t^n, \varphi\rangle)^2$$

$$\le \mathbf{1}_{\{\bar{E}_t^n \le 2\bar{E}_t\}} \frac{2}{n} \left(\sum_{i=E_t^n \wedge \lfloor n\bar{E}_t\rfloor}^{E_t^n \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle) \right)^2 \tag{A.11}$$

$$+ \mathbf{1}_{\{\bar{E}_t^n \ge 2\bar{E}_t\}} \frac{2}{n} \left(\sum_{i=E_t^n \wedge \lfloor n\bar{E}_t\rfloor}^{E_t^n \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle) \right)^2. \tag{A.12}$$

We now show that each of the terms (A.11) and (A.12) converges to 0 in probability as n tends to ∞, which implies (A.10) and completes the proof.

We begin with (A.11). First note that by the independence of $\{\eta_i, i \ge 1\}$ from the arrival process E^n, and the i.i.d. nature of the sequence $\{\eta_i, i \ge 1\}$, we have that

$$\begin{aligned}
&\mathbb{E}\left[\frac{2}{n}\mathbf{1}_{\{\bar{E}_t^n \le 2\bar{E}_t\}} \left(\sum_{i=E_t^n \wedge \lfloor n\bar{E}_t\rfloor}^{E_t^n \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle) \right)^2\right] \\
&\quad = \frac{2}{n}\mathbb{E}\left[\mathbf{1}_{\{\bar{E}_t^n \le 2\bar{E}_t\}} \sum_{i=E_t^n \wedge \lfloor n\bar{E}_t\rfloor}^{E_t^n \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle)^2\right] \\
&\quad \le \frac{2}{n}\mathbb{E}\left[\sum_{i=(E_t^n \wedge \lceil 2n\bar{E}_t\rceil) \wedge \lfloor n\bar{E}_t\rfloor}^{(E_t^n \wedge \lceil 2n\bar{E}_t\rceil) \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle)^2\right] \\
&\quad \le 8\|\varphi^2\|_\infty \mathbb{E}[|(\bar{E}_t^n \wedge 2\bar{E}_t) - \bar{E}_t|].
\end{aligned}$$

However, since $\bar{E}_t^n \Rightarrow \bar{E}_t$ as $n \to \infty$, it follows that

$$\mathbb{E}[|(\bar{E}_t^n \wedge 2\bar{E}_t) - \bar{E}_t|] \to 0 \qquad \text{as } n \to \infty.$$

This then implies that (A.11) converges to 0 in probability as n tends to ∞, as desired.

We next proceed to (A.12). Note that since $\bar{E}_t^n \Rightarrow \bar{E}_t$ as $n \to \infty$, it follows that for each $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbf{1}_{\{\bar{E}_t^n \ge 2\bar{E}_t\}} \frac{2}{n} \left(\sum_{i=E_t^n \wedge \lfloor n\bar{E}_t\rfloor}^{E_t^n \vee \lfloor n\bar{E}_t\rfloor} (\varphi(\eta_i) - \langle \mathcal{F}, \varphi\rangle) \right)^2 > \varepsilon \right) = 0.$$

This shows that (A.12) converges to 0 in probability as n tends to ∞, which completes the proof. $\square$

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