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REPRESENTATION AND APPROXIMATION OF NONCOOPERATIVE SEQUENTIAL GAMES*

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Abstract. Noncooperative sequential games, including the noncooperative stochastic game of Rogers (1969) and Sobel (1971), are investigated in the monotone contraction operator framework of Denardo (1967). Sufficient conditions are determined for the existence of equilibrium points in this setting. Techniques for comparing and approximating dynamic programs previously developed by the author are then applied to these sequential games, yielding conditions for the existence of ε-equilibrium points.

1. Introduction and Summary. It is now widely recognized in economics and several other fields that there is a need for mathematical models which can represent the behavior of several competing decision makers interacting over time, possibly under uncertainty. A natural model for this purpose is the sequential game, which combines the dynamic properties of dynamic programming with the competitive properties of game theory. The purpose of the present paper is to provide a general framework for analyzing and approximating a large class of noncooperative sequential games. We focus on noncooperative equilibrium points in the sense of Nash (1951), i.e., we look for policies or strategies for all players with the property that no single player acting alone can do better by changing. We consider the important questions of existence and approximation. Approximation seems particularly worth studying because it opens the way to computation and existence proofs for larger games.

The framework we suggest is the monotone contraction operator model introduced by Denardo (1967). He showed that this model encompasses the two-person zero-sum discounted stochastic game of Shapley (1953) plus many dynamic programming models. In this paper, we consider N-person nonzero-sum noncooperative sequential games in the same framework. The motivating special case is the noncooperative discounted stochastic game studied by Rogers (1969), Sobel (1971), Parthasarathy (1973), Himmelberg, Parthasarathy, Raghavan and Van Vleck (1976) and Federgruen (1978). As with Denardo (1967), the generality and abstraction here is useful to identify the essential structure. The contraction operator framework is also very natural because it emphasizes the reduction of the initial dynamic sequential game to a static one-period game. The final payoff to all players associated with a specification of all strategies is the unique fixed point of the contraction operator; the static game involves the choice of the fixed point. However, the sequential game is not immediately covered by the existing theory of static one-period noncooperative games because, as will be developed, the payoff (fixed-point) is a function of the state.

The contraction assumption means that the criterion for evaluating a payoff stream is discounted present value. However, it is well known that in many instances the average cost criterion can be reduced to a discounting criterion, cf. p. 149 of Ross (1970). Moreover, as in Section 5 of Denardo (1967), we use the N-stage contraction assumption, which covers a larger class of models, including many finite-stage models, cf. Whitt (1977).

A primary purpose of this paper is to apply to noncooperative sequential games the approximation techniques developed for dynamic programs and two-person zero-sum stochastic games in Whitt (1978). The idea is to replace the original state and action

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spaces with smaller sets and define a new transition and reward structure to approximate the original. In this way, we show that the extension of an ε_1-equilibrium policy vector in the smaller model is an ε_2-equilibrium policy vector in the original model, where ε_2 is a function of ε_1 and an appropriate measure of oscillation, cf. Theorem 4.2. The approximation results are in turn used to provide conditions under which a noncooperative sequential game has an ε-equilibrium point for each $\varepsilon > 0$, cf. Theorem 5.1.

As special cases, we obtain new results for stochastic games. Of particular interest is the application of the approximation procedure to provide conditions for the existence of ε-equilibrium points for all $\varepsilon > 0$ in the noncooperative discounted stochastic game when the state space is uncountable, cf. Theorem 6.4. The only other results for uncountable state space seem to be in Himmelberg, Parthasarathy, Raghavan and Van Vleck (1976). We also suggest what appears to be a promising procedure for finding ε-equilibrium points in many large noncooperative stochastic games, namely combining the approximation procedure here with an algorithm for finding approximate fixed-points of a continuous function mapping a subset of R^n into itself, cf. Remark (3) at the end of § 6.

A good indication of possible economic applications can be obtained by looking at the specific stochastic game in Kirman and Sobel (1974). As noted by Federgruen (1978), earlier work by Sobel (1973) on discounted stochastic games with uncountable state space, which is applied in Kirman and Sobel (1974), is not valid. Our results can be applied to obtain conditions for the existence of ε-equilibria in the game studied by Kirman and Sobel (1974).

We now briefly indicate how this paper is organized. We begin in § 2 by defining á la Denardo (1967), noncooperative monotone contraction operator games. Following van Nunen (1976), Wessels (1977) and others, we allow for unbounded rewards. As in § 5 of Denardo (1967), we use the N-stage contraction assumption. In § 3 we apply the Glicksberg (1952)–Fan (1952) generalization of the Kakutani fixed-point theorem to obtain sufficient conditions for the existence of equilibrium points. In § 4 we show how two sequential games can be compared, which provides the basis for approximations. In § 5 the approximation scheme is applied to provide conditions for the existence of ε-equilibrium points for each $\varepsilon > 0$. Finally, the special case of a noncooperative stochastic game is investigated in § 6.

2. Noncooperative monotone contraction operator games. Our model of a noncooperative sequential game is a direct extension of Denardo (1967), with the representation of a noncooperative discounted stochastic game being very similar to the representation of Shapley's (1953) two-person zero-sum stochastic game in Example 2 of § 8 in Denardo (1967). Let the *state space* S and the *player space* I be nonempty sets. For each player $i \in I$ and each state $s \in S$, let the *action space* $A_i(s)$ be a nonempty set. To allow for randomized strategies, $A_i(s)$ is often $\mathcal{P}(B_i(s))$, i.e., the set of all probability measures on an underlying action space $B_i(s)$, but we do not stipulate this yet. Let the space of all possible actions for all players in state s be the product space $A(s) = X_{i\in I}A_i(s)$. For each $i \in I$, let the *policy space for player* i be $\Delta_i = X_{s\in S}A_i(s)$. An element δ_i in Δ_i is called a *stationary policy* for player i because it represents the policy that takes action $\delta_i(s)$ every time the system is in state s. Let $\Delta = X_{i\in I}\Delta_i$ represent the space of policies for all players. Throughout this paper, we consider only stationary policies, but the symmetry argument in § 7 of Denardo (1967) can be used to show that no one player acting alone can do better by employing a more general history-remembering policy. Hence, we show that there exist equilibrium points or ε-equilibrium points consisting of stationary policies within the class of all history-remembering policies. Of course, we do not exclude existence of other equilibrium points and ε-equilibrium points consisting of nonstationary policies. While Δ and Δ_i contain only stationary policies, more general policies such as history-remembering policies can be included in this scheme by enlarging the state space. For example, the stage should usually be included as part of the state description in representations of finite-stage sequential games via monotone contraction operator models, cf. Whitt (1977).

Let the space V of potential return functions be a subset of $R^{S\times I}$. In order to allow for unbounded rewards, let $\alpha : S \to (0, \infty)$ and $\beta : S \to R$ be two functions. (The common choice of α and β is $\alpha(s) = 1$ and $\beta(s) = 0$ for all $s \in S$, which yields bounded rewards.) For any $v_1, v_2 \in R^{S\times I}$, let

$$\|v_1\| = \sup\{|v_1(s, i)| : s \in S, i \in I\}$$

(1) and

$$d(v_1, v_2) = \|\alpha(v_1 - v_2)\|,$$

where we regard $\alpha(s)$ as a function of both s and i which is independent of i. Let the space of potential return functions be

$$V = \{v \in R^{S\times I} | d(v, \beta) < \infty\}. \tag{2}$$

It is easy to see that (V, d) is a complete metric space.

The basic ingredient in the model specification is the *local income function* $h(s, i, \mathbf{a}, v)$, which assigns a real number to each quadruple $(s, i, \mathbf{a}, v)$ with $s \in S$, $i \in I$, $\mathbf{a} \in A(s)$ and $v \in V$. The number $h(s, i, \mathbf{a}, v)$ represents the return to player i beginning in state s when player j uses action a_j for all $j \in I$ and all future returns are described by the function v in V. For each $\delta \in \Delta$, let $(H_\delta v)(s, i) = h(s, i, \delta(s), v)$. We make the following basic boundedness (B), monotonicity (M) and N-stage contraction (NC) assumptions about the collection of operators $\{H_\delta, \delta \in \Delta\}$:

(B) There exist constants K_1 and K_2 such that $\|\alpha(H_\delta v - \beta)\| \leqq K_1 + K_2\|\alpha(v - \beta)\|$ for all $\delta \in \Delta$ and $v \in V$.

(M) If $v_1 \leqq v_2$ in V, i.e., if $v_1(s, i) \leqq v_2(s, i)$ for all $s \in S$ and $i \in I$, then $H_\delta v_1 \leqq H_\delta v_2$ for all $\delta \in \Delta$.

(NC) There exists a positive integer N and nonnegative constants m and c, $0 \leqq c < 1$, such that

$$d(H_\delta v_1, H_\delta v_2) \leqq m\, d(v_1, v_2)$$

and

$$d(H_\delta^N v_1, H_\delta^N v_2) \leqq c\, d(v_1, v_2)$$

for all $\delta \in \Delta$ and $v_1, v_2 \in V$, where H_δ^N is the N-fold iterate of H_δ.

Obviously (B) implies that the range of H_δ is contained in V. Property (NC) is the N-stage contraction assumption, cf. § 5 of Denardo (1967). The ordinary contraction assumption occurs when $N = 1$. The contraction modulus c often arises as a discount factor. Properties (M) and (NC) imply that each operator H_δ has a unique fixed point v_δ in V which we call the *return function* associated with policy vector δ. Note that the monotone contraction operator model reduces a sequential game to a one-stage game; the set of strategies available to player i is Δ_i and the return to player i from a specification of strategies by all players, i.e., δ, is the fixed point $v_\delta(\cdot, i)$. This differs from the usual static noncooperative game, however, because the return to each player is not a real number, but a function of the state.

A slight modification of Theorem 4 in Denardo (1967) yields

$$d(v_\delta, v) \leqq (1+m+\cdots+m^{N-1})(1-c)^{-1}\, d(H_\delta v, v) \tag{3}$$

for all $\delta \in \Delta$ and $v \in V$. The N-stage contraction assumption covers many N-stage sequential games with $c=0$, cf. Whitt (1977).

It should be noted that it is often possible to transform an N-stage contraction into a 1-stage contraction by modifying the bounding function α. A transformation for Markov programs, which also applies to the stochastic games in § 6 here, was constructed in § 8 of van Nunen (1976). However, it appears that such a transformation is not always possible for the more general monotone contraction operator models here. Moreover, even when such a transformation is possible, the new distance d is different from the old one and may be difficult to compute. Hence, we keep the N-stage contraction assumption.

For any $\delta \in \Delta$ and $\gamma_i \in \Delta_i$, let $[\delta^{-i}, \gamma_i]$ represent the policy vector δ' in Δ with $\delta'_j = \delta_j$ for $j \neq i$ and $\delta'_i = \gamma_i$. Let f_δ represent the *optimal return function* given that the other players are using δ^{-i} for each i, defined by

$$f_\delta(s, i) = \sup\{v_{[\delta^{-i}, \gamma_i]}(s, i) : \gamma_i \in \Delta_i\}.$$

Let F_δ be the associated *maximal return operator*, defined by

$$(F_\delta v)(s, i) = \sup\{(H_{[\delta^{-i}, \gamma_i]} v)(s, i) : \gamma_i \in \Delta_i\}$$

for each $s \in S$, $i \in I$, $\delta \in \Delta$ and $v \in V$.

Note that property (B) insures that the range of F_δ is in V for each $\delta \in \Delta$. A slight modification of Theorem 4 in Denardo (1967) shows that f_δ is the unique fixed point of F_δ. It is natural to define a *disequilibrium function* $\eta : \Delta \times S \times I \to R$ as $\eta_\delta(s, i) = f_\delta(s, i) - v_\delta(s, i)$. Call a policy δ an *ε-equilibrium point* (ε-EP) if $\eta_\delta(s, i) \leqq \varepsilon/\alpha(s)$ for all i and s, i.e., if $d(f_\delta, v_\delta) \leqq \varepsilon$. Call a policy δ an *equilibrium point* (EP) if it is an ε-EP for $\varepsilon = 0$.

3. Existence of equilibria. The existence of equilibrium points in noncooperative sequential games can be established by applying classical fixed point theorems, following the original line of reasoning used by Nash (1951) to treat static games. This approach has been applied to stochastic games by Rogers (1969), Sobel (1971), Parthasarathy (1973), Himmelberg et al. (1976) and Federgruen (1978). In this paper, we indicate how to apply the Kakutani fixed-point theorem for point-to-set functions as generalized by Glicksberg (1952) and Fan (1952) to the monotone contraction operator games. An alternate approach would be to apply the Brouwer fixed point theorem as generalized by Schauder and Tychonoff, cf. Theorem 1 of Sobel (1971).

Let 2^Y represent the set of all nonempty closed subsets of a Hausdorff topological space Y. Let X be a Hausdorff topological space. A set-valued function $\Phi : X \to 2^Y$ is called upper-semicontinuous (u.s.c.) if $y \in \Phi(x)$ for each $x \in X$, net $\{x_j, j \in J\}$ in X and net $\{y_j, j \in J\}$ in Y such that $x_j \to x$, $y_j \to y$ and $y_j \in \Phi(x_j)$ for each j. (Since X and Y need not be first countable, we use nets instead of sequences, cf. Chapter X of Dugundji (1966).)

THEOREM 3.1 (Kakutani, Glicksberg and Fan). *If X is a convex compact subset of a Hausdorff locally convex topological vector space (LCTVS) and $\Phi : X \to 2^X$ is convex-valued and u.s.c., then $x \in \Phi(x)$ for some $x \in X$.*

For our application, we want $X = \Delta$ and $\Phi = \psi_\varepsilon$, where $\psi_\varepsilon(\delta) = X_{i \in I}\psi_\varepsilon(\delta)_i$ and

$$\psi_\varepsilon(\delta)_i = \{\gamma_i \in \Delta_i : f_\delta(s, i) \leqq v_{[\delta^{-i}, \gamma_i]}(s, i) + \varepsilon/\alpha(s) \quad \text{for all } s\}. \tag{4}$$

The rest of this section is devoted to providing conditions on the monotone contraction operator game in order for (Δ, ψ_0) to satisfy the conditions of Theorem 3.1. The obvious modification (to account for the metric d in (2)) of Corollary 1 together with Theorem 4 of Denardo (1967) shows that $\psi_\varepsilon(\delta)_i$ is nonempty for each $\varepsilon > 0$. Throughout this paper, let $\Delta = X_{i\in I}\Delta_i$ and $\Delta_i = X_{s\in S}A_i(s)$ be given the product topology, cf. p 98 of Dugandji (1966).

THEOREM 3.2. *There exists an EP if*

(i) *$A_i(s)$ is a convex compact subset of a LCTVS for each $i \in I$ and $s \in S$,*

(ii) *$h(s, i, \mathbf{a}, v)$ is a concave function of a_i for each s, i, $\mathbf{a}$, v, and*

(iii) *$v_\delta(s, i)$ and $f_\delta(s, i)$ are continuous functions of δ for each $s \in S$ and $i \in I$.*

Proof. Since the properties of convexity, Hausdorff, compactness, TVS and LCTVS are preserved under arbitrary products, cf. pp. 138 and 224 of Dugundji (1966) and pp. 19 and 52 of Schaefer (1966), the product spaces Δ_i and Δ are convex compact subsets of a LCTVS. Condition (ii) implies that ψ_ε is convex-valued. Conditions (i) and (ii) plus Corollary 2 of Denardo (1967) show that $\psi_\varepsilon(\delta)_i$ is nonempty for $\varepsilon = 0$ as well as $\varepsilon > 0$. To see that ψ_ε is u.s.c., suppose $\{\delta_j, j \in J\}$ and $\{\delta'_j, j \in J\}$ are nets in Δ with $\delta_j \to \delta$, $\delta'_j \to \delta'$ and $\delta'_j \in \psi_\varepsilon(\delta_j)$ for each $j \in J$. Let δ_{ji} and δ'_{ji} be the ith coordinate in Δ_i of δ_j and δ'_j in Δ. Apply the triangle inequality to obtain

$$\begin{aligned}|v_{[\delta^{-i},\delta'_i]}(s, i) - f_\delta(s, i)| \leqq\ & |v_{[\delta^{-i},\delta'_i]}(s, i) - v_{[\delta_j^{-i},\delta'_{ji}]}(s, i)| \\ & + |v_{[\delta_j^{-i},\delta'_{ji}]}(s, i) - f_{\delta_j}(s, i)| + |f_{\delta_j}(s, i) - f_\delta(s, i)|\end{aligned}$$

for each s and i. The first and third term converge to zero by condition (iii) and the second term is less than or equal to ε for each j because $\delta'_{ji} \in \psi_\varepsilon(\delta_j)$ for each j. Hence, $\delta' \in \psi_\varepsilon(\delta)$, so ψ_ε is u.s.c. and the conditions of Theorem 3.1 are satisfied with $\varepsilon = 0$.

LEMMA 3.1. *If*

(i) *$A_i(s)$ is a compact metric space for each $i \in I$ and $s \in S$,*

(ii) *S is countable, and*

(iii) *$v_\delta(s, i)$ is a continuous function of δ for each $s \in S$ and $i \in I$, then $f_\delta(s, i)$ is a continuous function of δ for each s and i.*

Proof. Suppose $\{\delta_j, j \in J\}$ is a net in Δ with $\delta_j \to \delta$. Let s and i be given. For any $\varepsilon_1, \varepsilon_2 > 0$ there is a $\gamma_1 \in \Delta_i$ and a j_0 such that

$$\begin{aligned}f_\delta(s, i) &\leqq v_{[\delta^{-i},\gamma_i]}(s, i) + \varepsilon_1 \\ &\leqq v_{[\delta_j^{-i},\gamma_i]}(s, i) + \varepsilon_1 + \varepsilon_2 \quad \text{for } j \geqq j_0 \\ &\leqq f_{\delta_j}(s, i) + \varepsilon_1 + \varepsilon_2 \qquad \text{for } j \geqq j_0.\end{aligned}$$

Moreover, there is a net $\{\gamma_{ji}, j \in J\}$ in Δ_i such that

$$f_{\delta_j}(s, i) \leqq v_{[\delta_j^{-i},\gamma_{ji}]}(s, i) + \varepsilon_1 \quad \text{for all } j,$$

so that

$$\limsup_{j\in J} f_{\delta_j}(s, i) \leqq \limsup_{j\in J} v_{[\delta_j^{-i},\gamma_{ji}]}(s, i) + \varepsilon_1.$$

Choose a countable totally ordered subset J' of the directed set J so that the lim sup is attained on the left. Then, using the fact that Δ_i is compact metric space, by virtue of conditions (i) and (ii), choose a convergent subsequence $\{\gamma_{j_k i}\}$ of $\{\gamma_{ji}, j \in J'\}$

with limit γ_i. Hence

$$\begin{aligned}\limsup_{j\in J} f_{\delta_j}(s,i) &\leqq \limsup_{k\to\infty} v_{[\delta^{-i}_{j_k},\gamma_{i_k}]}(s,i)+\varepsilon_1\\ &\leqq v_{[\delta^{-i},\gamma_i]}(s,i)+\varepsilon_1\\ &\leqq f_\delta(s,i)+\varepsilon_1.\end{aligned}$$

LEMMA 3.2. *If $H_\delta v:\Delta\to V$ is a continuous function of δ for each $v\in W$, where ′is a subset of V containing v_δ for all $\delta\in\Delta$, then $v_\delta:\Delta\to V$ is a continuous function of δ, that $v_\delta(s,i):\Delta\to R$ is a continuous function of δ for each $s\in S$ and $i\in I$.*

Proof. By (3),

$$d(v_{\delta_j}, v_\delta)\leqq(1+m+\cdots+m^{N-1})(1-c)^{-1}\,d(H_{\delta_j}v_\delta, v_\delta),$$

where $d(H_{\delta_j}v_\delta, v_\delta)=d(H_{\delta_j}v_\delta, H_\delta v_\delta)\to 0$ as $\delta_j\to\delta$.

The continuity condition in Lemma 3.2 is more likely to hold if W is a subs ɔf V with convenient properties. For example, if $H_\delta v$ is continuous (concave, monoto) for each δ and each continuous (concave, monotone) v in V, then H_δ maps the ɔsed subset of all continuous (concave, monotone) functions in V into itself, so th ıxed point v_δ is continuous (concave, monotone). However, even if W has con ient properties, the continuity condition in Lemma 3.2 is quite strong because it r ires

$$d(H_{\delta_n}v, H_\delta v)=\sup_{\substack{s\in S\\ i\in I}}|\alpha(s)(h(s,i,\delta_n(s),v)-h(s,i,\delta(s),v))|\to 0 \tag{5}$$

whenever $\delta_n\to\delta$. Since Δ has the product topology, the metric convergence i 5) is difficult to achieve unless S and I are finite. More useful conditions are contai d in

LEMMA 3.3. *Suppose $\{\delta_j, j\in J\}$ is a net in Δ converging to δ. If*

(i) *$h(s,i,\delta_j(s),v_j)\to h(s,i,\delta(s),v)$ whenever $v_j(s,i)\to v(s,i)$ for all $s\in S$, $i\in I$ d v_j, $v\in V$; and*

(ii) $\sup_{j\in J} d(H^k_{\delta_j}v_0, v_{\delta_j})\to 0$ as $k\to\infty$ *for some $v_0\in V$; then $v_{\delta_j}(s,i)\to v_\delta(s,i)$.*

Proof. By (i), $(H_{\delta_j}v_0)(s,i)\to(H_\delta v_0)(s,i)$ for all s, i.

By (i) again and mathematical induction,

$$(H^k_{\delta_j}v_0)(s,i)=[H_{\delta_j}(H^{k-1}_{\delta_j}v_0)](s,i)\to[H_\delta(H^{k-1}_\delta v_0)](s,i)=(H^k_\delta v_0)(s,i)$$

as $j\to\infty$ for each $k\geqq 1$. As a consequence of this and (ii), $v_{\delta_j}(s,i)\to v_\delta(s,i)$.

The standard way to make Δ convex and $h(s,i,\mathbf{a},v)$ concave in a_i is to int luce the *mixed extension*, i.e., let $A_i(s)=\mathcal{P}(B_i(s))$, the set of all probability measures ı an underlying action space $B_i(s)$, and let the local income function applied to prob ility measures be defined via expectation:

$$h(s,i,\mathbf{a},v)=\int h(s,i,\mathbf{b},v)\,d\mu_{\mathbf{a}}(\mathbf{b}), \tag{6}$$

where μ_a is the product probability measure on the product σ-field of $X_{i\in I}B_i(s$ with one-dimensional marginal probability distributions a_i and the integral is an ɔper integral if $h(s,i,\mathbf{b},v)$ is not measurable in $\mathbf{b}$, cf. Example 3 in § 8 of Denardo (67).

It is well known that if $B_i(s)$ is a topological space and $\mathcal{P}(B_i(s))$ is endowed w the topology of weak convergence, then $\mathcal{P}(B_i(s))$ tends to inherit the topological pro ties of $B_i(s)$. For completely regular spaces, the weak convergence topology is na ally characterized by the continuity of $\int f\,dP$ in P for each bounded continuous real-va ed f. The basic inheritance properties here can be found in § II.6 of Parthasarathy (67), Varadarajan (1958) and footnote 10 in Fan (1952). Call a measure μ regular [Ra n] if

$\mu(A)=\sup\{\mu(C): C\subseteq A\}$ for all measurable subsets A, where the supremum is over all closed [compact] subsets. Obviously regular and Radon are equivalent in compact Hausdorff spaces. The LCTVS that appears below is the space of finite signed measures.

LEMMA 3.4. *Let $A_i(s)=\mathcal{P}(B_i(s))$ with the topology of weak convergence.*

(a) *If $B_i(s)$ is a separable [compact] metric space, then $A_i(s)$ is a separable [compact] metrizable convex subset of a* LCTVS.

(b) *If $B_i(s)$ is a compact Hausdorff space, then the subset of regular probability measures in $\mathcal{P}(B_i(s))$ is a compact convex subset of a LCTVS.*

There is still a major stumbling block—the integral in (6). There is no problem if the set I is countable and the set $B_i(s)$ has a countable base (i.e., is second countable, which is true if $B_i(s)$ is a separable metric space); then the product σ-field on $X_{i\in I}B_i(s)$ will coincide with the Borel σ-field with respect to the product topology. However, if either I is uncountable or if $B_i(s)$ does not have a countable base, then there can be complications. Henceforth, we make the assumptions to avoid the complications. We can combine this observation with Theorem 3.2 and Lemmas 3.1–3.4 to obtain the following result for the mixed extension.

THEOREM 3.3. *If*

(i) *S and I are countable,*

(ii) *$A_i(s)=\mathcal{P}(B_i(s))$ with the topology of weak convergence, where $B_i(s)$ is a compact metric space,*

(iii) *$h(s, i, \mathbf{b}_n, v_n)\to h(s, i, \mathbf{b}, v)$ whenever $b_{ni}\to b_i$ and $v_n(s, i)\to v(s, i)$ for each $s\in S$ and $i\in I$,*

(iv) *$h(s, i, \mathbf{a}, v)=\int h(s, i, \mathbf{b}, v)\, d\mu_{\mathbf{a}}(\mathbf{b})$, where $\mu_{\mathbf{a}}$ is the product measure on $X_{i\in I}B_i(s)$ with marginal measures $a_i\in A_i(s)$,*

(v) $\sup_n d(H^k_{\delta_n}v_0, v_{\delta_n})\to 0$ as $k\to\infty$ *for some v_0 in V and any convergent sequence $\{\delta_n\}$ in Δ,*

then there exists an EP, i.e., there exists $\delta^*\in\Delta$ such that $\delta^*\in\psi_0(\delta^*)$.

Proof. By conditions (i) and (ii) and Lemma 3.4(a), Δ is a convex compact metrizable subset of a LCTVS. By (i) and (ii), the Borel σ-field on $X_{i\in I}B_i(s)$ with the product topology coincides with the product σ-field. By (iii), the integral in (iv) is well defined. By (iii) and the almost-surely convergent representation of weak convergence, cf. Dudley (1968), $h(s, i, \delta_n(s), v_n)\to h(s, i, \delta(s), v)$ whenever $\delta_n(s)\to\delta(s)$ and $v_n(s, i)\to v(s, i)$ for each (s, i). This and (v) plus Lemma 3.3 imply that $v_\delta(s, i)$ is continuous in δ for each (s, i). Lemma 3.1 implies that $f_\delta(s, i)$ is continuous in δ for each (s, i). By (iv), ψ_0 is convex-valued. Hence, all conditions of Theorem 3.2 are satisfied.

Remark. The difficult condition in Theorem 3.3 is (iii). Since the convergence $\mathbf{b}_n\to\mathbf{b}$ and $v_n\to v$ is pointwise in s and i, in order to satisfy (iii) it will often be convenient to have I and/or S finite.

4. Comparing sequential games. Following Whitt (1978), let $(S, I, \{A_i(s), s\in S, i\in I\}, h, \alpha, \beta, c)$ and $(\tilde{S}, \tilde{I}, \{\tilde{A}_i(s), s\in\tilde{S}, i\in\tilde{I}\}, \tilde{h}, \tilde{\alpha}, \tilde{\beta}, \tilde{c})$ be two sequential games as defined in § 2. In order to compare these games, we require that several comparison functions be defined. These comparison functions arise naturally in deliberate approximations, which can be constructed by selecting partitions of subsets of the sets S, I and $A_i(s)$ for each $i\in I$ and $s\in S$, with one point selected in each partition subset, cf. Section 4 of Whitt (1978). In that setting the mappings below correspond to projections and extensions, which is the motivation for the notation. The comparison functions are:

(i) a mapping p of S *onto* $\tilde{S}$;

(ii) a *one-to-one* mapping p of I *onto* $\tilde{I}$;

(iii) a mapping p of $A_i(s)$ *onto* $\tilde{A}_{p(i)}(p(s))$ for each $i\in I$ and $s\in S$;

(iv) a mapping e of $\tilde{S}$ *into* S such that $p(e[\tilde{s}])=\tilde{s}$ for each $\tilde{s}\in\tilde{S}$;

(v) a mapping $e_{s,i}$ of $\tilde{A}_{p(i)}(p(s))$ *into* $A_i(s)$ such that $p(e_{s,i}[\tilde{a}])=\tilde{a}$ for each $\tilde{a}\in\tilde{A}_{p(i)}(p(s))$, $i\in I$ and $s\in S$.

(vi) $e:\tilde{V}\to R^{S\times I}$ with $e(\tilde{v})(s,i)=\tilde{v}(p(s),p(i))$ for each $S\in S$ and $i\in I$;

(vii) $p: V\to\tilde{V}$ with $p(v)(\tilde{s},\tilde{i})=v(e(\tilde{s}),e(\tilde{i}))$ for each $\tilde{s}\in\tilde{S}$ and $\tilde{i}\in\tilde{I}$;

(viii) $e:\tilde{I}\to I$ with $p[e(\tilde{i})]=\tilde{i}$ for all $\tilde{i}\in\tilde{I}$;

(ix) $e:\tilde{\Delta}_{p(i)}\to\Delta_i$ with $e(\tilde{\delta}_{p(i)})(s)=e_{s,i}(\tilde{\delta}_{p(i)}[p(s)])$ for each $s\in S$ and $i\in I$, and

(x) $p:\Delta_i\to\tilde{\Delta}_{p(i)}$ with $p(\delta_i)(\tilde{s})=p(\delta_i[e(\tilde{s})])$ for each $\tilde{s}\in\tilde{S}$ and $\tilde{i}\in\tilde{I}$.

Let e and p also map product spaces onto product spaces in the obvious way, e.g., $e:\tilde{\Delta}\to\Delta$ with $e(\tilde{\delta})_i=e(\tilde{\delta}_{p(i)})$ and $p:X_{i\in I}A_i(s)\to X_{p(i)\in\tilde{I}}\tilde{A}_{p(i)}(p(s))$ with $p(\{a_i(s)\})_{p(i)}=p(a_i(s))$ for $a_i(s)\in A_i(s)$ for each $i\in I$ and $s\in S$. Note that $e(\tilde{\delta})\in\tilde{\Delta}$ for each $\tilde{\delta}\in\tilde{\Delta}$.

Assume that $e(\tilde{v})\in V$ for each $\tilde{v}\in\tilde{V}$. Note that this is automatic if $\alpha(s)\leqq\tilde{\alpha}(p(s))$ and $\beta(s)-\tilde{\beta}(p(s))=0$ for all $s\in S$, but might fail in general.

We expect interest to be focused on approximating the action spaces $A_i(s)$, because these spaces—usually being sets of probability measures—are often large. Thus the map $p: S\to\tilde{S}$ might often be one-to-one as is the map $p: I\to\tilde{I}$, but we do not require it. The "distance" between these models can be expressed in terms of the measure of oscillation

$$\begin{aligned} K(\tilde{v})&=\sup_{\delta\in\Delta} d(H_\delta e(\tilde{v}), e(\tilde{H}_{p(\delta)}\tilde{v}))\\ &=\sup_{\substack{s\in S\\ \delta\in\Delta\\ i\in I}} |\alpha(s)[h(s,i,\delta(s),e(\tilde{v}))-\tilde{h}(p(s),p(i),p[\delta(s)],\tilde{v})]|. \end{aligned}\tag{7}$$

Obviously $p: I\to\tilde{I}$ should usually be one-to-one, as already assumed, in order for $K(\tilde{v})$ to have any chance of being small, but the following results hold even if $p: I\to\tilde{I}$ were not required to be one-to-one.

THEOREM 4.1. *For any* $\tilde{\delta}\in\tilde{\Delta}$,

$$d(e(\tilde{v}_{\tilde{\delta}}), v_{e(\tilde{\delta})})\leqq(1+m+\cdots+m^{N-1})(1-c)^{-1}K(\tilde{v}_{\tilde{\delta}}).$$

Proof. Just as in Theorem 3.2 of Whitt (1978), substitute $e(\tilde{\delta})$ for δ and $\tilde{v}_{\tilde{\delta}}$ for $\tilde{v}$ in (7) to obtain

$$d(H_{e(\tilde{\delta})}e(\tilde{v}_{\tilde{\delta}}), e(\tilde{v}_{\tilde{\delta}}))\leqq K(\tilde{v}_{\tilde{\delta}}).$$

Finally, apply formula (3) recalling that we have assumed that $e(\tilde{v})\in V$ for each $\tilde{v}\in\tilde{V}$.

THEOREM 4.2. *If* $\tilde{\delta}$ *is an* ε-*EP*, *then* $e(\tilde{\delta})$ *is an* $(1+m+\cdots+m^{N-1})(1-c)^{-1}(\varepsilon+2K(\tilde{v}_{\tilde{\delta}}))$-*EP*.

Proof. As in the proof of Theorem 3.1 of Whitt (1978),

$$\begin{aligned} \alpha(s)[H_{[e(\tilde{\delta})^{-i},\gamma_i]}e(\tilde{v}_{\tilde{\delta}})](s,i)&=\alpha(s)h(s,i,[e(\tilde{\delta})^{-i},\gamma_i](s),e(\tilde{v}_{\tilde{\delta}}))\\ &\leqq\alpha(s)\tilde{h}(p(s),p(i),p([e(\tilde{\delta})^{-i},\gamma_i](s)),\tilde{v}_{\tilde{\delta}})+K(\tilde{v}_{\tilde{\delta}})\\ &\leqq\alpha(s)\tilde{h}(p(s),p(i),p([e(\tilde{\delta})^{-i},\gamma_i](s)),\tilde{f}_{\tilde{\delta}})+K(\tilde{v}_{\tilde{\delta}})\\ &\leqq\alpha(s)\tilde{f}_{\tilde{\delta}}(p(s),p(i))+K(\tilde{v}_{\tilde{\delta}})\\ &\leqq\alpha(s)e(\tilde{v}_{\tilde{\delta}})(s,i)+(K(\tilde{v}_{\tilde{\delta}})+\varepsilon) \end{aligned}$$

for each $s\in S$, $\gamma_i\in\Delta_i$ and $i\in I$. As a consequence of properties (M) and (NC),

$$\begin{aligned} &\alpha(s)[H^N_{[e(\tilde{\delta})^{-i},\gamma_i]}e(\tilde{v}_{\tilde{\delta}})](s,i)\\ &\qquad\leqq\alpha(s)\,e(\tilde{v}_{\tilde{\delta}})(s,i)+(1+m+\cdots+m^{N-1})(K(\tilde{v}_{\tilde{\delta}})+\varepsilon) \end{aligned}$$

and, by induction,

$$\alpha(s)[H^{Nk}_{[e(\tilde{\delta})^{-i},\gamma_i]}\, e(\tilde{v}_{\tilde{\delta}})](s, i)$$
$$\leqq \alpha(s)\, e(\tilde{v}_{\tilde{\delta}})(s, i) + (1 + c + \cdots + c^{k-1})(1 + m + \cdots + m^{N-1})(K(\tilde{v}_{\tilde{\delta}}) + \varepsilon)$$

for all $k \geqq 1$. Since $d(H^{Nk}_{\delta} v, v_{\delta}) \to 0$ as $k \to \infty$,

$$\alpha(s) v_{[e(\tilde{\delta})^{-i},\gamma_i]}(s, i) \leqq \alpha(s)\, e(\tilde{v}_{\tilde{\delta}})(s, i) + (1-c)^{-1}(1 + m + \cdots + m^{N-1})(K(\tilde{v}_{\tilde{\delta}}) + \varepsilon)$$

for all $\gamma_i \in \Delta_i$, so that

$$\alpha(s) f_{e(\tilde{\delta})}(s, i) \leqq \alpha(s) e(\tilde{v}_{\tilde{\delta}})(s, i) + (1-c)^{-1}(1 + m + \cdots + m^{N-1})(K(\tilde{v}_{\tilde{\delta}}) + \varepsilon).$$

Apply Theorem 4.1 to obtain

$$a(s) f_{e(\tilde{\delta})}(s, i) \leqq \alpha(s) v_{e(\tilde{\delta})}(s, i) + (1-c)^{-1}(1 + m + \cdots + m^{N-1})(2K(\tilde{v}_{\tilde{\delta}}) + \varepsilon)$$

or

$$d(f_{e(\tilde{\delta})}, v_{e(\tilde{\delta})}) \leqq (1-c)^{-1}(1 + m + \cdots + m^{N-1})(2K(\tilde{v}_{\tilde{\delta}}) + \varepsilon).$$

5. Existence of ε-equilibria. We now combine Theorems 3.3 and 4.2 to obtain sufficient conditions for the existence of ε-EP's in sequential games with uncountable state spaces and noncompact action spaces. The ε-EPs obtained are also mixtures of only finitely many actions for each player in each state. Throughout this section, let m represent several different metrics and let the set I be countable. For any subset C in a metric space (B, m), let

$$C^{\varepsilon} = \{b \in B : m(b, b') < \varepsilon \text{ for some } b' \in C\}.$$

THEOREM 5.1. *If*

(i) *S is a separable metric space;*

(ii) *$\beta(s) = 0$ and $\alpha(s)$ is bounded over any finite sphere $\{s\}^{\varepsilon}$ in S;*

(iii) *for each (i, s), $A_i(s) = \mathcal{P}(B_i(s))$ with the topology of weak convergence, where $B_i(s)$ is a subset with compact closure in a metric space B;*

(iv) *for each i, the set-valued function mapping s into $B_i(s)$ is uniformly continuous: for each $\varepsilon_1 > 0$ there is an $\varepsilon_2 > 0$ such that $B_i(s_1) \subseteq B_i(s_2)^{\varepsilon_1}$ whenever $m(s_1, s_2) \leqq \varepsilon_2$;*

(v) *for each (i, s), $h(s, i, \mathbf{a}, v) = \int h(s, i, \mathbf{b}, v)\, d\mu_{\mathbf{a}}(\mathbf{b})$, where $\mu_{\mathbf{a}}$ is the product measure on the product σ-field on $X_{i \in I} B_i(s)$ with marginal measures $a_i \in A_i(s)$ and the integral is an upper integral if $h(s, i, \mathbf{b}, v)$ is not measurable in $\mathbf{b}$;*

(vi) *for any $\varepsilon_1 > 0$, there is an $\varepsilon_2 > 0$ such that*

$$\sup_{\substack{v \in V \\ i \in I}} |\alpha(s')[h(s', i, \mathbf{b}', v) - h(s'', i, \mathbf{b}'', v)]| < \varepsilon_1$$

if $m(s', s'') \leqq \varepsilon_2$ and $m(b_i', b_i'') \leqq \varepsilon_2$ for all i;

(vii) *$h(s, i, \mathbf{b}, v_n) \to h(s, i, \mathbf{b}, v)$ whenever $b_{ni} \to b_i$ and $v_n(s, i) \to v(s, i)$ for all s, i;*

(viii) *$\sup_n d(H^k_{\delta_n} v_0, v_{\delta_n}) \to 0$ for some v_0 in V and any convergent sequence $\{\delta_n\}$ in Δ; then, for any $\varepsilon > 0$, there exists an ε-EP δ^* with $\delta_i^*(s)$ being a probability measure on a finite subset of $B_i(s)$ for each s and i.*

Proof. We construct a sequence of approximate models

$$\{(\tilde{S}_n, \tilde{I}_n, \{\tilde{A}_{ni}(s), s \in \tilde{S}_n, i \in \tilde{I}_n\}, h_n, \alpha_n, \beta_n, c), n \geqq 1\}$$

according to the scheme in § 4, each of which satisfies the conditions of Theorem 3.3. Let $\tilde{I}_n = I$ for each $n \geqq 1$. Let $\{s_k\}$ be a countable dense subset of S, which exists by virtue of condition (i). For each $n \geqq 1$, form a countable partition of subsets of S by setting

$$S_{n1} = \{s \in S : m(s, s_1) \leqq n^{-1}\}$$

and

$$S_{nk} = \{s \in S : m(s, s_k) \leqq n^{-1}, s \notin \bigcup_{j=1}^{k-1} S_{nj}\}, \qquad k \geqq 2.$$

For each $n \geqq 1$, let $\tilde{S}_n$ be obtained by selecting one point s_{nk} from each nonempty subset in the partition $\{S_{nk}\}$. (Henceforth, omit all empty partition subsets.) Select the point s_{nk} so that $\alpha(s_{nk}) \geqq \alpha(s)/2$ for all $s \in S_{nk}$. This can be done by condition (ii).

For each $n \geqq 1$, $k \geqq 1$ and $s \in S_{nk}$, form finite partitions $\{B_{nkij}(s), 1 \leqq j \leqq K_{nk}\}$ of nonempty measurable subsets of $B_i(s)$ of common cardinality K_{nk} such that $m(b_1, b_2) \leqq \nu(n)$ if $b_1 \in B_{nkil}(s_1)$ and $b_2 \in B_{nkil}(s_2)$, where $s_1, s_2 \in S_{nk}$ and $\nu(n) \to 0$ as $n \to \infty$. These properties can be satisfied because of conditions (iii) and (iv).

For each $i \in I$, $n \geqq 1$, $k \geqq 1$ and $s \in S_{nk}$, let $B_{ni}(s)$ be a finite subset of $B_i(s)$ obtained by selecting one point from each partition subset $B_{nkij}(s)$, $1 \leqq j \leqq K_{nk}$. Let $\tilde{A}_{nk}(s) = \mathcal{P}(B_{ni}(s))$ for each $i \in I$, $s \in \tilde{S}_n$ and $n \geqq 1$.

We now define the five basic comparison functions. Let $p_n : I \to \tilde{I}_n$ be defined by $p_n(i) = i$. Let $p_n : S \to \tilde{S}_n$ be defined by $p_n(s) = s_{nk}$ if $s, s_{nk} \in S_{nk}$ and $s_{nk} \in \tilde{S}_n$. This obviously yields $m(s, p_n(s)) \leqq n^{-1}$ for all s and n. Let $p_n : A_i(s) \to \tilde{A}_{n,p_n(i)}(p_n(s))$ be defined by

$$p_n(a_i(s))(\{b\}) = a_i(s)(B_{nkij}(s)),$$

where $b \in B_{n,p_n(i)}(p_n(s))$ and $b \in B_{nkp_n(i)j}(p_n(s))$, which requires that $s \in S_{nk}$. This obviously means that $p_n(a_i(s))$ is the probability measure in $\tilde{A}_{n,p_n(i)}(p_n(s))$ assigning mass to each point in $B_{n,p_n(i)}(p_n(s))$ equal to the mass the probability measure $a_i(s)$ assigns to the corresponding partition subset $B_{nkij}(s)$ in $B_i(s)$.

Let the mapping $e_n : \tilde{S}_n \to S$ be defined in the obvious way: $e_n(\tilde{s}_n) = \tilde{s}_n$. Let the mappings $e_{nsi} : \tilde{A}_{n,p_n(i)}(p_n(s)) \to A_i(s)$ be defined by setting

$$e_{nsi}(\tilde{a}_{n,p_n(i)}(p_n(s)))(\{b\}) = \tilde{a}_{n,p_n(i)}(p_n(s))(\{b'\})$$

for $b \in B_{ni}(s)$, $b \in B_{nkij}(s)$ and $b' \in B_{nkp_n(i)j}(p_n(s))$. This implies that $e_n(\tilde{\delta}_{np_n(i)})(s)$ is a probability measure on a finite set for each i, s, n and $\tilde{\delta}_{ni} \in \tilde{\Delta}_{ni}$.

Let the approximate local income functions be defined as

$$h_n(\tilde{s}_n, \tilde{i}_n, \tilde{a}_n, \tilde{v}_n) = h(\tilde{s}_n, \tilde{i}_n, \tilde{a}_n, e_n(\tilde{v}_n))$$

for all n, $\tilde{s}_n \in \tilde{S}_n$, $\tilde{i}_n \in \tilde{I}_n$, $\tilde{a}_{n\tilde{i}_n} \in \tilde{A}_{n\tilde{i}_n}(\tilde{s}_n)$ and $\tilde{v}_n \in \tilde{V}_n$, just as in (4.1) of Whitt (1978). Then, by condition (vi), the measure of oscillation $K_n(\tilde{v}_n)$ in (7) is

$$K_n(\tilde{v}_n) = \sup_{\substack{s \in S \\ i \in I \\ \delta \in \Delta}} |\alpha(s)[h(s, i, \delta(s), e_n(\tilde{v}_n)) - h(p_n(s), i, p_n[\delta(s)], e_n(\tilde{v}_n))]|$$

$$\leqq \sup |\alpha(s')[h(s', i, \mathbf{b}', v) - h(s'', i, \mathbf{b}'', v)]|$$

where the second supremum is over all $v \in V$, all s', $s'' \in S$ with $m(s', s'') \leqq n^{-1}$, $b_i' \in B_i(s')$ and $b_i'' \in B_i(s'')$ with $m(b_i', b_i'') \leqq \nu(n) \to 0$ as $n \to \infty$ and all $i \in I$. Hence, condition (vi) implies that $\sup_{\tilde{\delta}_n \in \tilde{\Delta}_n} K(\tilde{v}_{\tilde{\delta}_n}) \to 0$ as $n \to \infty$.

The construction above plus conditions (vii) and (viii) imply that the conditions in Theorem 3.2 are satisfied in each approximate model, so there exists an EP in each approximate model. Theorem 4.2 then implies that, for each $\varepsilon > 0$, there is an n_0 such that the extension of each EP in the nth approximate model is an ε-EP in the original model for all $n \geqq n_0$.

Remarks. (1) Note that conditions (vii) and (viii) are only applied to establish the existence of an EP in each approximate model. If the existence is already known, these conditions can be omitted. The conditions could also be stated for each approximate model. For example, if I is finite, then (vi) can be replaced with $h(s, i, \mathbf{b}, v_n) \to h(s, i, \mathbf{b}, v)$ whenever $v_n(s, i) \to v(s, i)$.

(2) If S is a subset with compact closure in a metric space, then $\tilde{S}_n$ can be finite for each n.

6. Stochastic games. We now consider the special case of a noncooperative stochastic game. As before, let the set I of players be finite or countably infinite. Let the sets S and $B_i(s)$ be separable metric spaces endowed with their Borel σ-fields. Let $A_i(s)$ be the space $\mathcal{P}(B_i(s))$ with the topology of weak convergence. A stochastic game is obtained by letting the local income function be

$$h(s, i, \mathbf{b}, v) = r(s, i, \mathbf{b}) + \int_S v(x, i) q(dx|s, \mathbf{b}), \tag{8}$$

where $r(s, i, \mathbf{b})$ is a measurable real-valued function of $s \in S$, $i \in I$ and $\mathbf{b} \in X_{i\in I} B_i(s)$, $q(C|s, \mathbf{b})$ is a subprobability measure on S for each $s \in S$ and $\mathbf{b} \in X_{i \in I} B_i(s)$ and a measurable function of $(s, \mathbf{b})$ for each measurable subset C, and the integral in (8) is an abstract Lebesgue integral if v is measurable and an upper integral otherwise. (We assume the integral is well defined, i.e., the integral of $|v|$ is finite.)

Also let

$$r_\delta(s, i) = r(s, i, \delta(s)) = \int r(s, i, \mathbf{b})\, d\mu_{\delta(s)}(\mathbf{b}),$$

and

$$q_\delta(C|s) = q(C|s, \delta(s)) = \int q(C|s, \mathbf{b})\, d\mu_{\delta(s)}(\mathbf{b}),$$

where $\mu_{\delta(s)}$ is the probability measure on the product space $X_{i \in I} B_i(s)$ with marginal measures $\delta_i(s)$ for each i.

Let the associated return operator H_δ be defined by

$$\begin{aligned}(H_\delta v)(s, i) &= h(s, i, \delta(s), v) \\ &= \int h(s, i, \mathbf{b}, v)\, d\mu_{\delta(s)}(\mathbf{b}) \\ &= r_\delta(s, i) + \int_S v(x, i) q_\delta(dx|s).\end{aligned}$$

Let the space (V, d) of potential return functions be as in (1) and (2). Let $(q_\delta w)(s) = \int w(x) q_\delta(dx|s)$ for any function w for which the integral is defined. Following van Nunen (1976), with the obvious modification to include N-stage contractions, we make the following assumptions:

$$\|\alpha(r_\delta - (1-c)\beta)\| \leqq M_1, \tag{9}$$

$$\|\alpha(q_\delta \beta - c\beta)\| \leqq M_2, \tag{10}$$

$$\begin{aligned}&\|\alpha q_\delta \alpha^{-1}\| = \sup_{s \in S} \left| \alpha(s) \int [1/\alpha(x)] q_\delta(dx|s) \right| \leqq m, \\ &\|\alpha q_\delta^N \alpha^{-1}\| \leqq c < 1\end{aligned} \tag{11}$$

for all $\delta \in \Delta$, where q_δ^N is the N-step transition kernel associated with q_δ, defined as usual by

$$q_\delta^N(C|s) = \int_S q_\delta^{N-1}(C|s')q_\delta(ds'|s).$$

Conditions under which (11) hold with $N=1$ are discussed by van Hee and Wessels (1977). If $\alpha(s)=1$ for all s, then (11) holds with $N=1$ if $q_\delta(S|s) \leqq c$, which arises naturally if a discount factor c has been incorporated into the probability transition function. As a straightforward extension of Lemma 3.2.2 of van Nunen (1976), we have

THEOREM 6.1. *Under* (9)–(11), *the return operators* H_δ, $\delta \in \Delta$, *satisfy properties* (B), (M) *and* (NC). *Moreover,* H_δ^N *maps* V_0 *into itself, where*

$$V_0 = \{v \in V : \|\alpha(v-\beta)\| \leqq (1+m+\cdots+m^{N-1})(1-c)^{-1}(M_1+M_2)\}. \tag{12}$$

Proof. (B) Note that

$$\begin{aligned}\|\alpha(H_\delta v - \beta)\| &= \|\alpha(r_\delta + q_\delta v - \beta)\| \\ &= \|\alpha(r_\delta - (1-c)\beta) + \alpha(q_\delta(v-\beta)) + \alpha(q_\delta\beta - c\beta)\| \\ &\leqq \|\alpha(r_\delta - (1-c)\beta)\| + \|\alpha q_\delta(v-\beta)\| + \|\alpha(q_\delta\beta - c\beta)\| \\ &\leqq M_1 + \|\alpha q_\delta \alpha^{-1}\| \cdot \|\alpha(v-\beta)\| + M_2 \\ &\leqq M_1 + M_2 + m\|\alpha(v-\beta)\|.\end{aligned}$$

(M) This is straightforward.

(NC) For any $v_1, v_2 \in V$,

$$\begin{aligned}d(H_\delta v_1, H_\delta v_2) &= \|\alpha q_\delta(v_1 - v_2)\| \\ &\leqq \|\alpha q_\delta \alpha^{-1}\| \cdot \|\alpha(v_1 - v_2)\| \leqq m d(v_1, v_2)\end{aligned}$$

and

$$\begin{aligned}d(H_\delta^N v_1, H_\delta^N v_2) &= \|\alpha q_\delta^N(v_1 - v_2)\| \\ &\leqq \|\alpha q_\delta^N \alpha^{-1}\| \cdot \|\alpha(v_1 - v_2)\| \leqq c d(v_1, v_2).\end{aligned}$$

(V_0) First note that

$$\begin{aligned}\|\alpha q_\delta^k w\| &\leqq \|\alpha q_\delta^k \alpha^{-1}\| \cdot \|\alpha w\| \\ &\leqq \|\alpha q_\delta(q_\delta^{k-1}\alpha^{-1})\| \cdot \|\alpha w\| \\ &\leqq \|\alpha q_\delta \alpha^{-1}\| \cdot \|\alpha q_\delta^{k-1}\alpha^{-1}\| \cdot \|\alpha w\| \leqq m^k \|\alpha w\|.\end{aligned}$$

For any $v \in V$,

$$\begin{aligned}\alpha(H_\delta^N v - \beta) &= \alpha[r_\delta + q_\delta r_\delta + q_\delta^2 r_\delta + \cdots + q_\delta^{N-1} r_\delta + q_\delta^N v - \beta] \\ &= \alpha[r_\delta - (1-c)\beta] + \alpha q_\delta[r_\delta - (1-c)\beta] \\ &\quad + \cdots + \alpha q_\delta^{N-1}[r_\delta - (1-c)\beta] + \alpha q_\delta^N[v-\beta] \\ &\quad + \alpha[q_\delta\beta - c\beta] + \alpha q_\delta[q_\delta\beta - c\beta] \\ &\quad + \cdots + \alpha q_\delta^{N-1}[q_\delta\beta - c\beta];\end{aligned}$$

so that

$$\|\alpha(H_\delta^N v-\beta)\| \leqq (1+m+\cdots+m^{N-1})(M_1+M_2)+\|\alpha q_\delta^N \alpha^{-1}\|\cdot\|\alpha(v-\beta)\|$$
$$\leqq (1+m+\cdots+m^{N-1})(M_1+M_2)+c\|\alpha(v-\beta)\|,$$

which implies that

$$\|\alpha(H_\delta^N v-\beta)\| \leqq (1+m+\cdots+m^{N-1})(1-c)^{-1}(M_1+M_2)$$

if

$$\|\alpha(v-\beta)\| \leqq (1+m+\cdots+m^{N-1})(1-c)^{-1}(M_1+M_2).$$

We now determine sufficient conditions for the existence of an EP by applying Theorem 3.3.

THEOREM 6.2. *The stochastic game defined above has an EP if*

(i) *S is countable*;

(ii) *$B_i(s)$ is a compact metric space for each i and s*;

(iii) *$r(s, i, \mathbf{b})$ and $q(\{s'\}|s, \mathbf{b})$ are continuous functions of $\mathbf{b}$ in $X_{i\in I}B_i(s)$ for each s, s' and i; and*

(iv) *for any $\varepsilon>0$ and convergent sequence $\{\mathbf{b}_n\}$, there exists a finite subset C of S such that*

$$\sum_{s'\in S-C}(|\beta(s')|+\alpha^{-1}(s'))q(\{s'\}|s, \mathbf{b}_n)<\varepsilon \quad \textit{for all } n.$$

Remarks. (1) Condition (iv) follows from condition (iii) if $\beta(s)=0$ and $\alpha(s)=1$ because the convergence $q(\{s'\}|s, \mathbf{b}_n)\to q(\{s'\}|s, \mathbf{b})$ implies uniform tightness, cf. p. 47 of Parthasarathy (1967).

(2) Conditions (iii) and (iv) are both satisfied automatically if I is finite and $B_1(s)$ is countable and discrete for each i and s.

(3) Theorem 6.2 reduces to Theorem 1 of Federgruen (1976) when $N=1$, $\beta(s)=0$, $\alpha(s)=1$ and I is finite—which in turn reduces to Theorem 1 of Sobel (1971)—when, in addition, S and $B_i(s)$ for each i and s are finite.

Proof. We show that the five conditions of Theorem 3.3 are satisfied. By direct assumption, conditions (i), (ii) and (iv) hold here. By condition (iii) and (iv) here

$$h(s, i, \mathbf{b}_n, v_n)=r(s, i, \mathbf{b}_n)+\sum_{s'\in S}v_n(s', i)q(\{s'\}|s, \mathbf{b}_n)$$

$$\to r(s, i, \mathbf{b})+\sum_{s'\in S}v(s', i)q(\{s'\}|s, \mathbf{b})=h(s, i, \mathbf{b}, v),$$

which is condition (iii) of Theorem 3.3. Finally, condition (v) holds because, for any $\delta\in\Delta$,

$$d(H_\delta^{Nk}v_0, v_\delta)=d(H_\delta^{Nk}v_0, H_\delta^{Nk}v_\delta)$$
$$\leqq c^k d(v_0, v_\delta)$$
$$\leqq c^k(\|\alpha(v_0-\beta)\|+\|\alpha(v_\delta-\beta)\|)$$
$$\leqq 2c^k(1+m+\cdots+m^{N-1})(1-c)^{-1}(M_1+M_2).$$

We now consider comparisons between the stochastic game model $(S, I, \{B_i(s), i\in I, s\in S\}, h, \alpha, \beta, c)$ and a "smaller" stochastic game model $(\tilde{S}, \tilde{I}, \{\tilde{B}_i(s), i\in\tilde{I}, s\in\tilde{S}\}, \tilde{h}, \tilde{\alpha}, \tilde{\beta}, \tilde{c})$ which are both assumed to satisfy (8)–(11). Assume that the comparison functions in § 4 have been defined. Let $\tilde{S}$, $\tilde{B}_i(s)$ for each $i \in \tilde{I}$ and $s \in \tilde{S}$, and $\tilde{I}$ be countable sets. Assume that

$$S_n = p^{-1}(\tilde{s}_n) = \{s \in S : p(s) = \tilde{s}_n\}, \qquad \tilde{s}_n \in \tilde{S}$$

and

$$\begin{aligned} B_{ni}(s) &= p^{-1}(\tilde{b}) \cap B_i(s) \\ &= \{b \in B_i(s) : p(i) = \tilde{i}, p(s) = \tilde{s}, p(b) = \tilde{b}\}, \qquad \tilde{b} \in \tilde{B}_i(\tilde{s}), \end{aligned}$$

are measurable subsets for each n, i and s.

As in § 4, assume that $e(\tilde{v}) \in V$ for each $\tilde{v} \in \tilde{V}$. In this setting, the comparison results in § 4 can be expressed in terms of the measures of oscillation

$$K_r = \sup_{\substack{s \in S \\ i \in I \\ \mathbf{b} \in X B_i(s) \\ i \in I}} |\alpha(s)[r(s, i, \mathbf{b}) - \tilde{r}(p(s), p(i), p(\mathbf{b}))]|$$

$$K_q(v) = \sup_{\substack{s \in S \\ \mathbf{b} \in X \in B_i(s) \\ i \in I}} \left\{ \alpha(s) \sum_{n=1}^{\infty} (|\tilde{v}(s_n)|) \left| q(S_n | s, \mathbf{b}) - \tilde{q}(\{s_n\} | p(s), p(\mathbf{b})) \right| \right\}$$

and $K_q = K_q(\tilde{v}^*)$, where

$$\tilde{v}^*(s_n) = \sup_{s \in S_n} \{\tilde{\beta}(s_n) + \tilde{\alpha}^{-1}(s_n)(1 + m + \cdots + m^{N-1})(1 - \tilde{c})^{-1}(\tilde{M}_1 + \tilde{M}_2)\}, \qquad n \geqq 1. \tag{14}$$

THEOREM 6.3. *For any* $\tilde{\delta} \in \tilde{\Delta}$, $K(\tilde{v}_{\tilde{\delta}}) \leqq K_r + K_q(\tilde{v}_{\tilde{\delta}}) \leqq K_r + K_q$.

Proof. By (7) and the triangle inequality,

$$\begin{aligned} K(\tilde{v}_{\tilde{\delta}}) &= \sup_{\substack{s \in S \\ i \in I \\ \delta \in \Delta}} |\alpha(s)[h(s, i, \delta(s), e(\tilde{v}_{\tilde{\delta}})) - \tilde{h}(p(s), p(i), p[\delta(s)], \tilde{v}_{\tilde{\delta}})]| \\ &\leqq \sup_{\substack{s \in S \\ \delta \in \Delta \\ i \in I}} |\alpha(s)[r(s, i, \delta(s)) - \tilde{r}(p(s), p(i), p[\delta(s)])]| \\ &\quad + \sup_{\substack{s \in S \\ \delta \in \Delta}} \alpha(s) \sum_{n=1}^{\infty} (|\tilde{v}_{\tilde{\delta}}(s_n)|) |q(S_n | s, \delta(s)) - \tilde{q}(\{s_n\} | p(s), p[\delta(s)])| \\ &\leqq K_r + K_q(\tilde{v}_{\tilde{\delta}}) \leqq K_r + K_q, \end{aligned}$$

where the last step follows because $|\tilde{v}_{\tilde{\delta}}(s_n)| \leqq \tilde{v}^*(s_n)$ for all n by (12).

Remarks. When $\alpha(s) = \tilde{\alpha}(p[s]) = 1$ and $\beta(s) = \tilde{\beta}(p[s]) = 0$ for all s, Theorem 6.3 reduces to Theorem 6.1(a) of Whitt (1978). For further refinements, see § 6 of Whitt (1978).

We now present sufficient conditions for the stochastic game to have an ε-EP for each $\varepsilon > 0$. For simplicity, we assume I is finite and $\alpha(s) = 1$ and $\beta(s) = 0$ for all s.

THEOREM 6.4. *The stochastic game has an ε-EP for each $\varepsilon > 0$ if*

(i) *I is finite;*

(ii) *$B_i(s)$ is a subset with compact closure in a separable metric space for each i and s;*

(iii) *the point-to-set function mapping s into $B_i(s)$ is uniformly continuous for each i: for each $\varepsilon_1 > 0$, there exists an $\varepsilon_2 > 0$ such that $B_i(s_1) \subseteq B_i(s_2)^{\varepsilon_1}$ if $m(s_1, s_2) < \varepsilon_2$;*

(iv) *$\alpha(s) = 1$ and $\beta(s) = 0$ for all s;*

(v) *$r(s, i, \mathbf{b})$ and $q(C|s, b)$ are uniformly continuous in s and* $\mathbf{b}$, *uniformly in C.*

Proof. Construct a sequence of approximating models as in the proof of Theorem 5.1. Note that conditions (i)–(v) of Theorem 5.1 have been assumed again here and condition (vi) of Theorem 5.1 holds because of conditions (iv) and (v) here. For this purpose, it suffices to consider only those v with $|v(s, i)| \leqq (1 + \cdots + m^{N-1})(1-c)^{-1}M_1$. Alternatively, it is easy to see that $K_n(\tilde{v}_{\bar{s}_n}) \to 0$ by applying Theorem 6.3. Theorem 6.2 implies that each approximate game has an EP.

Remarks. (1) The transition kernel q satisfies condition (5) in Theorem 6.4 if $q(C|s, \mathbf{b}) = \int_C f(x|s, \mathbf{b})\lambda(dx)$, for all measurable subsets C, where λ is a finite measure on S and $f(x|s, \mathbf{b})$ is uniformly continuous in s and $\mathbf{b}$, uniformly in x.

(2) To see that it is not sufficient in Theorem 6.4 to have $q(\cdot\,|s, \mathbf{b})$ be uniformly continuous in the space of probability measures on S with the topology of weak convergence, let S be the unit circle, i.e., $S = [0, 1)$ with the metric $m(s_1, s_2) = \min\{s_2 - s_1, 1 - s_2 + s_1\}$ for $s_1 \leqq s_2$. Let $T: S \to S$ be defined by $T(s) = s + \lambda \pmod 1$ where λ is a fixed irrational number. Let $q(\{T_s\}|s, \mathbf{b}) = c$ and $q(S - \{T_s\}s, \mathbf{b}) = 0$ for all $s, \mathbf{b}$. Then $q(\cdot\,|s, \mathbf{b})$ is a uniformly continuous function of $(s, \mathbf{b})$ into the space of probability measures on S with the weak convergence topology. However, since the transformation T is ergodic, it is impossible to have $K_q < c$ for K_q in (6.9) and any countable partition of S.

(3) If, in addition to the assumptions of Theorem 6.4, S is a subset of a compact metric space, then there is a natural algorithm to find an ε-EP. Since each approximate model then can have S as well as I and $B_i(s)$ finite, the EP's in each approximate model can be found by applying Brouwer's fixed point theorem, as shown in Theorem 1 of Sobel (1971). Hence, it suffices to apply one of the algorithms for finding an approximate fixed point of a continuous function mapping a subset of R^n into itself, cf. Karmardian (1976).

(4) We have yet to determine interesting sufficient conditions for the existence of an EP (rather than an ε-EP) when S is uncountable. For example, suppose $S = [0, 1]$, $I = \{1, 2\}$, $B_i(s) = \{1, 2\}$ and $A_i(s) = \mathcal{P}(B_i(s))$ for each i and s. For simplicity consider either (1)$N = 1$ or (2) $N = 2$ and $c = 0$.

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