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USEFUL MARTINGALES FOR STOCHASTIC STORAGE PROCESSES WITH LÉVY INPUT

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Abstract

We apply the general theory of stochastic integration to identify a martingale associated with a Lévy process modified by the addition of a secondary process of bounded variation on every finite interval. This martingale can be applied to queues and related stochastic storage models driven by a Lévy process. For example, we have applied this martingale to derive the (non-product-form) steady-state distribution of a two-node tandem storage network with Lévy input and deterministic linear fluid flow out of the nodes.

STOCHASTIC INTEGRATION; QUEUEING THEORY; POLLACZEK–KHINCHINE FORMULA; STORAGE MODELS

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1. Introduction

In Kella and Whitt (1991) we established stochastic decomposition results for the steady-state distribution of queues with server vacations by characterizing the steady-state distribution of a Lévy process with secondary jump input. We obtained these results by identifying an appropriate martingale; see Theorem 3.1 of Kella and Whitt (1991). In this paper we extend the martingale representation and provide a shorter proof based on the general theory of stochastic integration, as in Dellacherie and Meyer (1978), (1982), Jacod and Shiryaev (1987), Lipster and Shiryaev (1986), Métivier (1982), Protter (1990), Rogers and Williams (1987) and others. This general theory gives us a convenient way to obtain (local) martingales from initial (local) martingales by integrating a 'nice enough' (predictable) process with respect to it and applying the generalized Itô and integration-by-parts formulae. The extension involves replacing the secondary jump input in Kella and Whitt (1991) by a right-continuous process of bounded variation on every finite interval. This extension was motivated by (and is

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applied to analyze) a tandem fluid network with Lévy input in Kella and Whitt (1992). In particular, in Kella and Whitt (1992) we apply the martingale derived here to determine the steady-state distribution of a two-node tandem network (which is not product-form). We believe that it will have other applications. Another example that goes beyond Kella and Whitt (1991) is a model of a dam in Section 4(e) here. For other applications of martingale theory to storage and queueing theory see, among others, Baccelli and Makowski (1989a, b), Brémaud (1981), Harrison (1985) and Rosenkrantz (1983).

The rest of this paper is organized as follows. We give background on Lévy processes in Section 2; we establish the main results in Section 3; and we discuss applications in Section 4.

2. Background on Lévy processes

In this section we review basic facts about Lévy processes and associated martingales. For additional background, see Bingham (1975), Breiman (1968), Fristedt (1974), Jacod and Shiryaev (1987), Prabhu (1980), Protter (1990) and others.

We begin with an underlying filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t \mid t \geqq 0\})$. The *filtration* $\{\mathscr{F}_t \mid t \geqq 0\}$ is an increasing family of σ-fields, i.e. it satisfies $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$ for all $0 \leqq s < t$. We assume that the filtration is *standard* or *satisfies the usual conditions*, i.e. it is right-continuous and augmented. By a Lévy process (understood to be with respect to $\{\mathscr{F}_t \mid t \geqq 0\}$), we mean a process $\{X_t \mid t \geqq 0\}$ which is continuous in probability, with $X_0 = 0$ and $X_{s+t} - X_t$ independent of $\mathscr{F}_t$ and distributed like X_s for every non-negative s, t (stationary independent increments). Without loss of generality (e.g. see p. 21 of Protter (1990) or Fristedt (1974)) we take the cadlag (right-continuous left-limit) strong Markov version; hence, from now on whenever the term *Lévy process* is mentioned, we mean the cadlag version. Given the centering function $h(x) = \operatorname{sgn}(x)(\min(|x|, 1))$, let $\mu(\cdot)$ denote the (unique) associated Lévy measure, i.e. $\mu(\cdot)$ is such that

$$\psi(\alpha) \equiv \log E \exp(i\alpha X_1) = ic\alpha - \frac{\sigma^2}{2}\alpha^2 + \int_{(-\infty,\infty)} (\exp(i\alpha x) - 1 - i\alpha h(x))\mu(dx),$$

$$\mu((-\infty, -1) \cup (1, \infty)) < \infty, \quad \mu(\{0\}) = 0 \quad \text{and} \quad \int_{[-1,1]} x^2 \mu(dx) < \infty.$$

The function $\psi(\cdot)$ is called the *exponent* of the Lévy process and we have $E \exp(i\alpha X_t) = \exp(t\psi(\alpha))$ for all $t \geqq 0$.

The following are some known results about Lévy processes:

(i) If $\{X_t \mid t \geqq 0\}$ is a Lévy process with bounded jumps, then $E|X_1|^n < \infty$ for every $n \geqq 0$.

(ii) Any Lévy process $\{X_t \mid t \geqq 0\}$ can be decomposed into two independent Lévy processes $\{Y_t \mid t \geqq 0\}$ and $\{Z_t \mid t \geqq 0\}$, i.e. $X_t = Y_t + Z_t$ for all $t \geqq 0$, where the former has bounded jumps and the latter is an independent difference of compound Poisson processes.

(iii) Any Lévy process $\{X_t \mid t \geqq 0\}$ can be decomposed into an independent sum of a Brownian motion (possibly with drift) and another Lévy process (this is not saying much). When the Lévy measure satisfies the condition $\int_{[-1,1]} |x| \mu(dx) < \infty$, this second process can be taken to be a difference of driftless subordinators (non-decreasing pure-jump Lévy processes).

(iv) If $\mu((-\infty, 0)) = 0$, then the Lévy process has no negative jumps. In this case the Laplace–Stieltjes transform exists and is given by $E\exp(-\alpha X_t) = \exp(t\varphi(\alpha))$ where

$$\varphi(\alpha) \equiv \log E\exp(-\alpha X_1) = -c\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} (\exp(-\alpha x) - 1 + \alpha h(x))\mu(dx).$$

In this case $\varphi(\cdot)$ is called the *exponent* as well and it is easy to check that $\varphi(0) = 0$ and $\varphi(\cdot)$ is convex. If $\{X_t \mid t \geqq 0\}$ is not a subordinator then $P(X_1 < 0) > 0$, so that $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$. Whenever $EX_1 < 0$, $\varphi(\cdot)$ is strictly increasing on $[0, \infty)$.

It is easy to check that $\exp[i\alpha X_t - \psi(\alpha)t]$ and, for the case of no negative jumps, $\exp[-\alpha X_t - \varphi(\alpha)t]$ are martingales. Often these are referred to as Wald martingales. For a Lévy process with no negative jumps and $EX_1 < 0$, we let $\varphi^{-1}(\cdot)$ be the inverse function of the (Laplace–Stieltjes) exponent $\varphi(\cdot)$, then $\exp[-\varphi^{-1}(\beta)X_t - \beta t]$ is a martingale for every $\beta > 0$. Denote by $T^a = \inf\{t \mid X_t = -a\}$. Then Doob's optional sampling theorem implies that for every $t > 0$, $E\exp(-\varphi^{-1}(\beta)X_{T^a \wedge t} - \beta(T^a \wedge t)) = 1$. The strong law of large numbers implies that

$$\frac{X_n}{n} = \frac{1}{n}\sum_{k=1}^{n} (X_k - X_{k-1}) \to EX_1 \qquad \text{a.s. as } n \to \infty.$$

Since $EX_1 < 0$, $X_n \to -\infty$ a.s. as $n \to \infty$, which implies that $P(T^a < \infty) = 1$. Also note that

$$\exp(-\varphi^{-1}(\beta)X_{T^a \wedge t} - \beta(T^a \wedge t)) \leqq \exp(\varphi^{-1}(\beta)a)$$

for every $t > 0$. Hence, by the bounded convergence theorem and the fact that $X_{T^a} = -a$, we obtain the well-known result that the Laplace–Stieltjes transform of T^a is given by

$$E\exp(-\beta T^a) = \exp(-\varphi^{-1}(\beta)a); \tag{1}$$

see Bingham (1975) for a proof using the Wiener–Hopf factorization. By differentiation, (1) implies that

$$ET^a = a(\varphi^{-1})'(0) = \frac{a}{\varphi'(0)} = \frac{a}{-EX_1}. \tag{2}$$

(Although it is tempting to use the martingale $X_t - tEX_1$ to obtain Equation (2), it is not clear how to justify $EX_{T^a \wedge t} \to EX_{T^a} = -a$ without adding unnecessary extra conditions.)

If we take any non-negative random variable U which is independent of the Lévy process and define $T^U = \inf\{t \mid X_t = -U\}$, then by conditioning and unconditioning we obtain

$$E\exp(-\beta T^U) = E\exp(-\varphi^{-1}(\beta)U), \qquad ET^U = \frac{EU}{-EX_1}.$$

Note that for the $M/G/1$ queue, if U is distributed as a service time, then T^U is the busy period; moreover if the arrival rate is λ, then $\varphi(\alpha) = \alpha - \lambda(1 - E\exp(-\alpha U))$. (See Prabhu (1980) for the 'Lévy' approach to treating the $M/G/1$ queue, and see Rosenkrantz (1983) and Baccelli and Makowski (1989a, b) for related martingale arguments.)

3. The main results

Here we identify a useful martingale and exhibit some of the consequences. First, for an adapted cadlag process $\{Y_t \mid t \geqq 0\}$, let $\Delta Y_t = Y_t - Y_{t-}$ where $Y_{t-} = \lim_{s\uparrow t} Y_s$, with the convention that $\Delta Y_0 = Y_0$. Also we use the notation (recalling that a cadlag process can have only countably many points of discontinuity), $Y_t^c = Y_t - \Sigma_{0\leqq s\leqq t}\Delta Y_s$, i.e. $\{Y_t^c \mid t \geqq 0\}$ is a continuous adapted process with $Y_0^c = 0$ and is of bounded variation on finite intervals.

Theorem 1. Let $\{X_t \mid t \geqq 0\}$ be a Lévy process with (Fourier–Stieltjes) exponent $\psi(\cdot)$, let $\{Y_t \mid t \geqq 0\}$ be an adapted cadlag process of bounded variation on finite intervals, and let $Z_t = X_t + Y_t$. Then

$$\begin{aligned} M_t \equiv \psi(\alpha) \int_0^t \exp(i\alpha Z_s)ds + \exp(i\alpha Y_0) - \exp(i\alpha Z_t) + i\alpha \int_0^t \exp(i\alpha Z_s)dY_s^c \\ + \sum_{0<s\leqq t} \exp(i\alpha Z_s)(1 - \exp(-i\alpha \Delta Y_s)) \end{aligned} \tag{3}$$

is a local martingale. If, in addition, the expected variation of $\{Y_t^c \mid t \geqq 0\}$ and the expected number of jumps of $\{Y_t \mid t \geqq 0\}$ are finite on every finite interval, then $\{M_t \mid t \geqq 0\}$ is in fact a martingale.

Proof. Consider the (complex-valued) martingale $N_t = \exp(i\alpha X_t - \psi(\alpha)t)$ and the process $B_t = \exp(i\alpha Y_t + \psi(\alpha)t)$. Applying the integration-by-parts formula (see Protter (1990), p. 60, Corollary 2) to the real and imaginary parts gives

$$N_t B_t = N_0 B_0 + \int_{(0,t]} N_{s-}dB_s + \int_{(0,t]} B_{s-}dN_s + \sum_{0<s\leqq t} \Delta N_s \Delta B_s.$$

(The rightmost expression on the right side is valid since the real and imaginary parts of $\{B_t \mid t \geqq 0\}$ are of bounded variation on bounded intervals.) Since $\int_{(0,t]} \Delta N_s dB_s = \Sigma_{0<s\leqq t} \Delta N_s \Delta B_s$, we have that

$$-\int_{(0,t]} B_{s-}dN_s = \int_{(0,t]} N_s dB_s + N_0 B_0 - N_t B_t \tag{4}$$

(note N_s, rather than N_{s-}, on the right side). Since $\{N_t \mid t \geqq 0\}$ is a martingale, the left and, hence, the right side of Equation (4) is a local martingale (apply Theorem 29 of Protter (1990), p. 142, to the real and imaginary parts separately). It suffices to identify the right side of Equation (4) with M_t (Equation (3)). This is done by observing (again, treating the real and imaginary parts separately) that, for $0 < s \leqq t$,

$$dB_s = \psi(\alpha)B_s ds + i\alpha B_s dY_s^c + B_s(1 - \exp(-i\alpha\Delta Y_s)). \tag{5}$$

(Note that the differential in (5) is of the Lebesgue–Stieltjes type and is defined path by path.) If, on every finite interval, $\{Y_t^c \mid t \geqq 0\}$ has finite expected variation and $\{Y_t \mid t \geqq 0\}$ has finite expected number of jumps, then $E \sup_{0 \leqq s \leqq t} |M_s| < \infty$ for every finite t. Hence, by dominated convergence (e.g. Protter (1990), p. 35, Theorem 47), M_t is a martingale.

Remark. As pointed out by a referee, it is also possible to obtain this result using the theory of semimartingale exponential characteristics (see Jacod and Shiryaev (1987), Lipster and Shiryaev (1986), Métivier (1982) and more). For those who are familiar with this theory, it should be noted that the local martingale in Equation (3) is not quite the exponential local martingale associated with the predictable characteristic of $Z = X + Y$. The reason is that the rightmost term is an expression involving the random measure associated with the jumps of the process, rather than its predictable compensator.

The following is the 'Laplace–Stieltjes' version of Theorem 1. Since the derivation is identical, we state it without proof.

Theorem 2. Let $\{X_t \mid t \geqq 0\}$ be a Lévy process with no negative jumps with (Laplace–Stieltjes) exponent $\varphi(\cdot)$; let $\{Y_t \mid t \geqq 0\}$ be an adapted cadlag process of bounded variation on finite intervals, and let $Z_t = X_t + Y_t$. Then

$$M_t = \varphi(\alpha) \int_0^t \exp(-\alpha Z_s)ds + \exp(-\alpha Y_0) - \exp(-\alpha Z_t) - \alpha \int_0^t \exp(-\alpha Z_s)dY_s^c$$
$$+ \sum_{0<s\leqq t} \exp(-\alpha Z_s)(1 - \exp(\alpha\Delta Y_s))$$

is a local martingale. If, in addition, the expected variation of $\{Y_t^c \mid t \geqq 0\}$ and the expected number of jumps of $\{Y_t \mid t \geqq 0\}$ are finite on every finite interval and $\{Z_t \mid t \geqq 0\}$ is a non-negative process (alternatively, bounded below), then $\{M_t \mid t \geqq 0\}$ is in fact a martingale.

Remark. It should be observed that the assumption in Theorem 2 that $\{Z_t \mid t \geqq 0\}$ is a non-negative process implies that

$$Z_s - \Delta Y_s \geqq Z_s - \Delta Z_s = Z_{s-} \geqq 0. \tag{6}$$

The first inequality in (6) holds because $\{Z_t \mid t \geqq 0\}$ has no negative jumps.

4. Applications

In this section we give a few examples of some known results which are simple consequences of Theorems 1 and 2.

(a) Consider a generalization of the virtual waiting time in an $M/G/1$ queue, which is a reflected Lévy process with no negative jumps. More precisely, if $\{X_t \mid t \geqq 0\}$ is a Lévy process with no negative jumps and $I_t = -\inf\{X_s \mid 0 \leqq s \leqq t\}$, then $Z_t = X_t + I_t$ is our reflected Lévy process. If we let $T_n^a = \inf\{t \mid X_t = -na\}$ for some $a > 0$ and $N_t^a = \sup\{n \mid T_n^a \leqq t\}$, then $\{N_t^a \mid t \geqq 0\}$ is a renewal counting process, so that $EN_t^a < \infty$ for every $t \geqq 0$ (e.g. pp. 181–182 of Karlin and Taylor (1975)). Since $I_t \leqq a(N_t^a + 1)$, $EI_t < \infty$. Since the points of increase of I_t are contained in the (random) set $\{t \mid Z_t = 0\}$ and $\{Z_t \mid t \geqq 0\}$ is a non-negative process, we have from Theorem 2 ($Y^c = Y = I$) that

$$M_t = \varphi(\alpha) \int_0^t \exp(-\alpha Z_s)ds + 1 - \exp(-\alpha Z_t) - \alpha I_t$$

is a martingale. For the case $EX_1 < 0$, the process $\{Z_t \mid t \geqq 0\}$ is regenerative when T_1^a is a regeneration epoch. Unless X_t is non-random, the distribution of T_1^a is non-arithmetic (as may be verified directly from its characteristic function which may be obtained through analytic continuation of Equation (1)). Applying Doob's optional sampling theorem to the bounded stopping times $\min(T_1^a, t)$, letting $t \to \infty$ and applying monotone and bounded convergence theorems, we immediately obtain that

$$\frac{1}{ET^a} E \int_0^{T^a} \exp(-\alpha Z_s)ds = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}$$

(note that $\varphi'(0) = -EX_1$). Hence the Laplace–Stieltjes transform of the limiting (and stationary) distribution is given by the above generalized Pollaczek–Khinchine formula (for the non-random case this result is a triviality). A special case is, of course, the $M/G/1$ queue with arrival rate λ, service times S_i distributed as S and $\rho = \lambda ES$; then $\varphi(\alpha) = \alpha - \lambda(1 - E\exp(-\alpha S))$ and $\varphi'(0) = 1 - \rho$.

(b) Part (a) can be modified to cover the case in which the process does not start at 0. If we let $I_t^U = \max(I_t - U, 0)$ and $Z_t^U = U + X_t + I_t^U = X_t + \max(I_t, U)$ where $U \geqq 0$ and is independent of $\{X_t \mid t \geqq 0\}$ (alternatively, $U \in \mathscr{F}_0$), then

$$M_t \equiv \varphi(\alpha) \int_0^t \exp(-(\alpha Z_s^U + \beta I_t^U))ds + \exp(-\alpha U) - \exp(-(\alpha Z_t^U + \beta I_t^U))$$
$$-(\alpha + \beta)\frac{1 - \exp(-\beta I_t^U)}{\beta}$$

is a martingale. Applying expected values one obtains a (deterministic) integral equation, which upon inversion gives

$$E \exp(-(\alpha Z_t^U + \beta I_t^U))$$
$$= \frac{E\exp(-\alpha U) - (\alpha+\beta)\int_0^t \exp(-\varphi(\alpha)s)\, d\left(\frac{1 - E\exp(-\beta I_s^U)}{\beta}\right)}{\exp(-\varphi(\alpha)t)},$$

which immediately leads to the results on pp. 76–77 of Prabhu (1980). (There the proof is restricted to the case in which I_t has a density, which is the case if there is no Brownian component and $\int_{(0,1]} x\mu(dx) < \infty$, a restriction which we do not make.)

(c) Let T_n be a strictly increasing sequence of stopping times with $T_0 = 0$ and $T_n \to \infty$ with probability 1. Let U_n be $\mathscr{F}_{T_n}$-measurable (knowledge of U_n is determined only by the information gathered up to the stopping time T_n). Let $N_t = \sup\{n \mid T_n \leqq t\}$. Finally, let $Y_t = \Sigma_{n=0}^{N_t} U_n$ and $Z_t = X_t + Y_t$. Then $\{Y_t \mid t \geqq 0\}$ is adapted, of bounded variation on every finite interval, and $Y_t^c = 0$ for all $t \geqq 0$. If $EN_t < \infty$, then by Theorem 1 (and a trivial manipulation),

$$M_t = \psi(\alpha)\int_0^t \exp(i\alpha Z_s)ds + 1 - \exp(i\alpha Z_t) - \sum_{n=0}^{N_t} (\exp(i\alpha(Z_{T_n} - U_n)) - \exp(i\alpha Z_{T_n}))$$

is a martingale. This was demonstrated directly, without the use of stochastic calculus, in Kella and Whitt (1991). This martingale was used to characterize the steady-state distribution of $\{Z_t \mid t \geqq 0\}$.

(d) In addition to the setup in (c), assume that $\{X_t \mid t \geqq 0\}$ has no negative jumps, $U_n \geqq 0$ for all $n \geqq 0$ and $Z_t = X_t + Y_t + L_t$, where $L_t = \max(-\inf_{0 \leqq s \leqq t}(X_s + Y_s), 0)$. Since $L_t \leqq I_t$ (where I_t is defined in part (a)), $EL_t < \infty$. If in addition we assume once again that $EN_t < \infty$ for all $t \geqq 0$, we have that

$$M_t = \varphi(\alpha)\int_0^t \exp(-\alpha Z_s)ds + 1 - \exp(-\alpha Z_t) - \alpha L_t$$
$$- \sum_{n=0}^{N_t} (\exp(-\alpha(Z_{T_n} - U_n)) - \exp(-\alpha Z_{T_n}))$$

is a martingale. This case was also treated in Kella and Whitt (1991) directly.

(e) The following is a dam model. Let X_t be a subordinator (non-decreasing Lévy process), $r(\cdot)$ a non-negative function and $Z_t = Z_0 + X_t - \int_0^t r(Z_s)ds$, where $Z_0 \in \mathscr{F}_0$. If $E\sup\{r(Z_s) \mid 0 \leqq s \leqq t\} < \infty$ for every $t \geqq 0$ (e.g. $r(\cdot)$ is bounded), then

$$M_t = \varphi(\alpha)\int_0^t \exp(-\alpha Z_s)ds + \exp(-\alpha Z_0) - \exp(-\alpha Z_t)$$
$$+ \alpha\int_0^t r(Z_s)\exp(-\alpha Z_s)ds$$

is a martingale. Note that in this case it is more natural to take $\tilde{\varphi}(\cdot) = -\varphi(\cdot)$ as the exponent, rather than $\varphi(\cdot)$ itself. For example, one consequence is that if the (Markov) process is stationary, then taking expected values immediately gives

$$\tilde{\varphi}(\alpha)E\exp(-\alpha Z)=\alpha Er(Z)\exp(-\alpha Z) \tag{7}$$

where Z has the stationary distribution. In fact, Equation (7) characterizes the stationary distribution if it exists (see for example Equations (3.7) and (3.8) on p. 291 of Asmussen (1987) and discussion there).

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