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QUEUES WITH SUPERPOSITION ARRIVAL PROCESSES IN HEAVY TRAFFIC

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To help provide a theoretical basis for approximating queues with superposition arrival processes, we prove limit theorems for the queue-length process in a $\sum GI_i/G/s$ model, in which the arrival process is the superposition of n independent and identically distributed stationary renewal processes each with rate n^{-1}. The traffic intensity ρ is allowed to approach the critical value one as n increases. If $n(1-\rho)^2 \to c$, $0<c<\infty$, then a limit is obtained that depends on c. The two iterated limits involving ρ and n, which do not agree, are obtained as $c\to 0$ and $c\to\infty$.

queues * heavy traffic * superposition * limit theorems * central limit theorem

1. Introduction and results

In order to analyze queues with superposition arrival processes and other complex congestion models such as non-Markov networks of queues, it is important to have approximations; see Albin [1, 2], Whitt [18, 19] and references there. Three limit theorems help in developing approximations for queues with superposition arrival processes. The first states that under appropriate regularity conditions a superposition arrival process approaches a Poisson process as the number n of component processes increases; see Çinlar [5]. The second states that the queueing model is continuous; the queue-length process associated with the superposition arrival process also converges to the queue length process associated with the Poisson process as $n\to\infty$; see Chapter 3 of Franken et al. [8]. The third states that under appropriate regularity conditions the queue-length process approaches a reflected Brownian motion diffusion process, which has an exponential stationary distribution, as the traffic intensity ρ approaches the critical value one from below with n held fixed; see Iglehart and Whitt [11, 12]. Moreover, if the n component processes being superposed are i.i.d. renewal processes, then the reflected Brownian motion and the mean of the equilibrium distribution obtained as $\rho\to 1$ depend on the individual renewal processes only through the first two moments of the renewal interval and are independent of n.

Unfortunately, these limit theorems do not tell the whole story even for large n and ρ, because the two iterated limits involving $n\to\infty$ and $\rho\to 1$ do not agree. The

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purpose of this paper is to prove limit theorems in which $n \to \infty$ and $\rho \to 1$ simultaneously. These heavy-traffic limit theorems with two parameters changing together are in the same spirit as the heavy-traffic limit theorems in Halfin and Whitt [10]; there the number of servers and the traffic intensity are allowed to change together. The joint limits are very helpful for determining regions of validity of different approximation schemes.

The joint limit considered here arises in many applications in which a queue is fed by many stochastically identical arrival streams. We give an example from packet-switched voice communication; see Decina and Vlack [7], Jenq [13] and Sriram and Whitt [16].

Example 1. We consider a model to describe the delays in a statistical multiplexer or concentrator that handles many separate voice lines. Each voice signal is sampled and represented digitally in packets of fixed length. With the use of silence detection, a typical voice signal can be viewed as a sequence of alternating talkspurts and silence periods. A simple model for the voice signal is an alternating renewal process in which the successive talkspurts and silence periods are exponentially distributed with different means. When the voice signal in talkspurts is packetized, this leads to a geometrically distributed number of packets of fixed length in each talk spurt and no packets at all during silence periods. As a consequence of the lack of memory property associated with the geometric distribution, the arrival process of packets into the multiplexer for each voice line can be modeled as a renewal process in which each renewal interval is of length d (the packet length) with probability p or of length $d+I$ with probability $1-p$, where I is the exponentially distributed silence period. Typical values of the parameters are $d=16$, $I=650$ and $p=21/22$. These parameter values make the renewal arrival process from each voice source highly bursty (variable); e.g. the squared coefficient of variation (variance divided by the square of the mean) of a packet interarrival time is $c^2=18.1$.

Assuming that the packet service times are essentially constant at the multiplexer (the packets are the same length and are transmitted from the multiplexer at a fixed rate), the delays of packets at the multiplexer from n identical active voice lines can be described using the $\sum_{i=1}^{n} GI_i/D/1$ queueing model, in which the arrival process is the superposition of n i.i.d. renewal processes of the kind above, the service times are deterministic, the queue discipline is FCFS (first-come first-served) and there is unlimited waiting room. In order to engineer the system, we wish to determine how the delays depend on the number of active voice lines.

Since n is typically large (about 100), it is tempting to invoke the superposition limit theorem in Çinlar [5] and use the much more tractable $M/D/1$ model. Indeed, from analytical calculations and statistical analysis of simulation output [16], we see that the distribution of an interarrival time in the superposition process is very nearly exponential and the correlations between successive interarraival times are very small. Moreover, the $M/D/1$ model works very well for small and moderate values of n. However, the Poisson approximation does not work well for large values of n. Comparisons of the $M/D/1$ model with simulation of the $\sum_{i=1}^{n} GI_i/D/1$ model show that the $M/D/1$ model grossly underestimates the expected delay at the multiplexer for large values of n.

At first glance, this phenomenon may seem to contradict the basic superposition limit theorem, but it actually does not. For the superposition limit theorem, the component processes should become sparse as n increases so that the total arrival rate remains fixed. In contrast, as n increases here, the arrival rate and traffic intensity increase too. If the service rate is multiplied by n too, so that the traffic intensity ρ remains fixed, then the Poisson approximation does indeed become better and better as n increases. However, from the point of view of the queue, the quality of the Poisson approximation for the superposition arrival process depends critically on the traffic intensity ρ. As ρ increases, the long-term dependence in the arrival process becomes more important, and in this example there are many small positive correlations that eventually have a significant cumulative impact over a large number of interarrival times. As we will prove, if n and ρ both increase, the limiting behavior depends on $n(1-\rho)^2$. In order for the Poisson limit to be appropriate, we should have $n(1-\rho)^2 \to \infty$ as $n \to \infty$ and $\rho \to 1$. However, here $n(1-\rho)^2 \to 0$ as $n \to \infty$ for $\rho < 1$.

When $n(1-\rho)^2 \to 0$ as $n \to \infty$, the heavy-traffic description in [11, 12] corresponding to $\rho \to 1$ with n fixed eventually becomes appropriate (as shown by Theorems 1-3 here). Since the variance of the renewal interval in each component renewal process is much greater than the variance of the exponential distribution, the observed behavior of the simulation for large n can be anticipated to differ dramatically from the $M/D/1$ model. Since the squared coefficient of variation of the renewal interval in each component process is 18.1, the ratio of the true mean queue length to the predicted $M/D/1$ value approaches 18.1 as n increases, just as it would if $\rho \to 1$ with n held fixed.

An approximation for the mean delay, based in part on the analysis in this paper, is contained in formulas (33) and (44) of [19]. This approximation describes the delays in this example reasonably well over the full range of n and does very well for large n; see [13] and [16].

This paper is closely related to previous papers by Albin [1] and Newell [14]. Albin [1] did other simulation experiments that show how the two limits involving n and ρ are related. She simulated queues with arrival processes that are superpositions of i.i.d. renewal processes. She considered several values of n ($n = 2^j$ for $j = 1, 2, \ldots, 10$) and ρ ($\rho = 0.5$, 0.8 and 0.9). As expected, the superposition process in isolation approaches a Poisson process as $n \to \infty$: The distribution of the interval between points rapidly approaches the exponential distribution and the correlation between successive intervals rapidly approaches zero. Moreover, the queue soon behaves as if the arrival process were Poisson for low values of ρ. However, it does not for high values of ρ. Newell [14] helped explain these results, showing by heuristic arguments that the behavior of the queue as $n \to \infty$ and $\rho \to 1$ depends on $n(1-\rho)^2$. In order for the queue to behave as if the arrival process is Poisson, it is necessary, not only for n to be large, but also for $n(1-\rho)^2$ to be large. Newell also indicated that the heavy-traffic approximations are appropriate when $n(1-\rho)^2$ is small.

We supplement Newell's analysis by proving limit theorems. When $n(1-\rho)^2 \to c$, $0 < c < \infty$, we show that the queue-length process, appropriately normalized, converges to a non-degenerate limit, which we can describe, but it is complicated. If, afterwards, we let $c \to 0$ or $c \to \infty$, then we obtain the same limiting behavior as the iterated limits involving $\rho \to 1$ and $n \to \infty$ separately.

Here is our model. Let $N(t)$ be a renewal counting process having renewal intervals distributed according to the nonnegative random variable X with cdf F where $EX = 1$. Let $N_e(t)$ be the associated stationary or equilibrium renewal counting process, i.e., the delayed renewal counting process in which the first interval has density $[1 - F(t)]$ and all subsequent intervals have cdf F. For each n, let the arrival process be the superposition of n i.i.d. copies of $\{N_e(t/n),\ t \geq 0\}$. Notice that the component processes have been scaled so that the total arrival rate is 1 for all n. This is tantamount to having the renewal interval in each component renewal process be distributed as nX for each n.

There also are s homogeneous servers in parallel with unlimited waiting room and the FCFS discipline. (Of course, the queue-length process is the same for many other disciplines.) The service times are i.i.d. and independent of the arrival process (the $\sum GI_i/G/s$ model). Each service time is distributed as $s\rho Y$ where Y is a nonnegative random variable with $EY = 1$ and finite variance σ_Y^2. Hence, for each $\rho < 1$ the queue has traffic intensity ρ.

Let $Q_{\rho n}(t)$ be the queue length (number of customers in the system) at time t as a function of ρ and n and let $\boldsymbol{Q}_{\rho n}$ be the normalized process defined by

$$\boldsymbol{Q}_{\rho n} \equiv \boldsymbol{Q}_{\rho n}(t) = (1-\rho) Q_{\rho n}(t(1-\rho)^{-2}), \quad t \geq 0. \tag{1}$$

We use the notion of convergence in distribution (weak convergence) of random elements of the function space $D[0, \infty)$, denoted here by $\Rightarrow$; see Billingsley [3], Whitt [17] and references there. Our limit process involves the usual reflecting barrier function f, defined by

$$f(x)(t) = x(t) - \inf\{x(u) \colon 0 \leq u \leq t\}, \quad t \geq 0, \tag{2}$$

for any $x \in D[0, \infty)$.

We also impose a regularity condition on the basic renewal-interval cdf F.

Condition F. $\limsup_{t \to 0} [F(t) - F(0)]/t < \infty$.

Condition F is satisfied, for example, if F has an atom at 0 but otherwise is absolutely continuous in a neighborhood of 0. Condition F is necessary for our method of proof; see Theorem 5 in Section 2.

Theorem 1. *Assume Condition F. If $n \to \infty$, $\rho \to 1$ and $n(1-\rho)^2 \to c$, $0 < c < \infty$, then $\boldsymbol{Q}_{\rho n} \Rightarrow f(\boldsymbol{A} - \boldsymbol{S} + \boldsymbol{M})$ in $D[0, \infty)$, where $\boldsymbol{M}(t) = -t$, $t \geq 0$; $\boldsymbol{S}$ is a Brownian motion* *independent of* $\boldsymbol{A}$ *having* 0 *drift and diffusion coefficient* $s^{-2}\sigma_Y^2$; f *is the reflecting barrier function in* (2), *and* $\boldsymbol{A}$ *is a centered Gaussian process with stationary increments, continuous paths and covariance function*

$$K(t, u) = EA(t)A(u) = cEN_e(t/c)N_e(u/c) - tu. \tag{3}$$

In order to prove Theorem 1, we apply Theorem 1(a) of Iglehart and Whitt [12]. With this previous result, it suffices to prove a weak consequence theorem for the superposition arrival process in isolation. For each n, let

$$A_n(t) = N_{n1}(t) + \cdots + N_{nn}(t), \quad t \geq 0, \tag{4}$$

and

$$\boldsymbol{A}_n \equiv \boldsymbol{A}_n(t) = (A_n(nt) - nt)/n^{1/2}, \quad t \geq 0, \tag{5}$$

where $\{N_{ni}(t), t \geq 0\}$ are independent for different i and distributed as $\{N_e(t/n), t \geq 0\}$.

Theorem 2. *If condition F holds, then* $\boldsymbol{A}_n \Rightarrow \boldsymbol{A}'$ *in* $D[0, \infty)$ *as* $n \to \infty$, *where* $\boldsymbol{A}'$ *is a centered Gaussian process with stationary increments, continuous paths and covariance function*

$$EA'(t)A'(u) = EN_e(t)N_e(u) - tu.$$

Theorem 2 in turn follows rather directly from Theorem 2 of Hahn [9], which establishes a central limit theorem for partial sums of stochastic processes in $D[0, \infty)$.

Example 2. If the basic renewal-interval cdf F is the mixture of an exponential random variable and a mass at 0, then the renewal counting process $N(t)$, the stationary version $N_e(t)$ and the superposition process are all batch Poisson processes with geometrically distributed batches. Then $\boldsymbol{A}$ in Theorem 1 and $\boldsymbol{A}'$ in Theorem 2 are standard Brownian motions. As a consequence, $f(\boldsymbol{A} - \boldsymbol{S} + \boldsymbol{M})$ in Theorem 1 is simply reflected Brownian motion with negative drift. Since the superposition arrival process is a renewal process for each n in this case, Theorem 1 can be deduced directly from Theorem 1(a) and Example 3(1) of [12].

Example 3. If $P(X = 1) = 1$, then the interval to the first point in the stationary renewal counting process $N_e(t)$ is uniform in $[0, 1]$. Then the process $\boldsymbol{A}'(u) - \boldsymbol{A}'(t)$ in Theorem 2 is a Brownian bridge diffusion process on $[t, t+1]$ for each t. Moreover, the Brownian bridges on $[t+k, t+k+1]$ for different k are identical (totally dependent). Since $\boldsymbol{A}(t)$ is distributed as $\sqrt{c}\boldsymbol{A}'(t/c)$, $\boldsymbol{A}$ is a Brownian bridge on $[t, t+c^{-1}]$. In this case, Theorem 2 is a minor modification of the functional central limit theorem for empirical cdf's, Theorems 13.1 and 16.4 of Billingsley [3].

In general, the limit process $\boldsymbol{A}'$ in Theorem 2 is complicated, so that the limit process $f(\boldsymbol{A}-\boldsymbol{S}+\boldsymbol{M})$ in Theorem 1 is complicated as well. As in Example 2, the process $f(\boldsymbol{A}-\boldsymbol{S}+\boldsymbol{M})$ is relatively simple when $\boldsymbol{A}$ is Brownian motion. This occurs asymptotically as c approaches 0 or ∞. Let σ_X^2 be the variance of X, which we now assume is finite.

Theorem 3. *As $c\to 0$, the f.d.d.'s (finite-dimensional distributions) of $\boldsymbol{A}$ converge to the f.d.d.'s of $\sigma_X \boldsymbol{B}$ where $\boldsymbol{B}$ is standard Brownian motion with zero drift and unit diffusion coefficient.*

Theorem 4. *Assume Condition F. As $c\to\infty$, the f.d.d.'s of $\boldsymbol{A}$ converge to the f.d.d.'s of the standard Brownian motion $\boldsymbol{B}$.*

Remarks. (1) The heavy-traffic limit in Theorem 4 is the same as if the component renewal processes are Poisson. This remains true if the points occur in batches, i.e., if $F(0)>0$.

(2) We do not know if Theorems 3 and 4 can be extended to the stronger weak convergence in $D[0,\infty)$ that was established in Theorems 1 and 2. The stronger weak convergence was essential in Theorem 2 to obtain even convergence of the f.d.d.'s for $\boldsymbol{Q}_{\rho n}$ in Theorem 1 via the continuous mapping argument in [12].

(3) The general approach in this paper can be applied to more general arrival processes than the superposition of n i.i.d. stationary renewal processes. To extend Theorems 1 and 2 to superpositions of i.i.d. non-renewal point processes, it suffices to verify the sufficient conditions for tightness in Theorem 2 of Hahn [9]. For example, these conditions are easily verified for general stationary point processes in which the interval between successive points is bounded below by $\delta>0$. This case covers many generalizations of Example 1.

(4) Condition F is necessary for Hahn's [9] sufficient conditions for Theorem 2; see Theorem 5 in Section 2. We do not know if Condition F is necessary for Theorem 2 itself.

(5) Theorem 1 characterizes the limit process for the queue, but we do not know much about it. We do not even know the mean of the marginal distribution. We have nevertheless been able to apply Theorem 1 to develop approximations for networks of queues in [19]; also see [2], [13] and [16]. We use the fact that the congestion for large n and ρ depends on $n(1-\rho)^2$.

2. Proofs

Proof of Theorem 1. Apply Theorem 1(a) of [12] together with Theorem 2 here. We must also show that the standard $\sum GI_i/G/s$ system of interest is asymptotically equivalent to the modified system treated in [12]. As indicated in [12], this follows by applying the argument of Section 3 in [11]. To apply Theorem 2, we need to put Theorem 2 in the same framework as (1). Hence, let

$$A_{\rho n} \equiv A_{\rho n}(t) = (1-\rho)[A_n(t(1-\rho)^{-2}) - t(1-\rho)^{-2}], \quad t \geq 0. \tag{6}$$

Note that $\{A_{\rho n}(t),\ t \geq 0\} = \{\sqrt{c_{\rho n}} A_n(t/c_{\rho n}),\ t \geq 0\}$ for A_n in (5) and $c_{\rho n} = n(1-\rho)^2$. By Theorem 2 here and Theorem 5.5 of [3], $A_{\rho n} \Rightarrow A$ as $n \to \infty$, $\rho \to 1$ and $c_{\rho n} = n(1-\rho)^2 \to c$, where $A(t) = \sqrt{c} A'(t/c)$, $t \geq 0$. □

We prove Theorem 2 by applying Theorem 2 in Hahn [9]. In order to verify Hahn's sufficient conditions for tightness, we apply three elementary lemmas. One random variable Y_1 is *stochastically less than or equal to* another Y_2, denoted by $Y_1 \leq_{st} Y_2$, if $P(Y_1 > t) \leq P(Y_2 > t)$ for all t or, equivalently, if $Eg(Y_1) \leq Eg(Y_2)$ for all nondecreasing real-valued functions g for which the expectations are well defined.

Lemma 1. *For all* $t > 0$, $N_e(t) \leq_{st} 1 + N(t)$.

Proof. Make each sample path of $N_e(t)$ larger by shifting all points to the left the distance to the first point to the right of 0. The process so constructed is distributed as $N(t)$. Add 1 to account for the point moved to the origin. □

Lemma 2. *For* $0 < t < u$, $\{N(u) - N(t) \mid N(s),\ s \leq t\} \leq_{st} 1 + N(u-t)$ *with probability* 1.

Proof. Make each sample path of $\{N(u) - N(t) \mid N(s),\ s \leq t\}$ larger by doing the same construction as in Lemma 1. □

Lemma 3. *For all positive* t *and* k, $E\{N(t)^k\} < \infty$.

Proof. p. 155 of Prabhu [15]. □

Proof of Theorem 2. Since the basic renewal intervals associated with $\{N_{n1}(t),\ t \geq 0\}$ in (4) are distributed as nX, the processes $\{N_{n1}(nt),\ t \geq 0\}$ have the same f.d.d.'s for all n. Hence, $\{A_n(t),\ t \geq 0\}$ in (5) is distributed the same as $\{n^{-1/2} \sum_{i=1}^{n} X_i(t),\ t \geq 0\}$ where $X_i(t) = N_{1i}(t) - t$. Convergence of the f.d.d.'s thus follows immediately from the multivariate central limit theorem. Since the limit distribution is multivariate normal in each case, the limit process A' is Gaussian. Since $EX = 1$, $EX_i(t) = EN_{1i}(t) - t = 0$. Since $N_{1i}(t)$ has stationary increments, so does A'.

The stronger weak convergence in $D[0, \infty)$ and the sample path continuity follow from Theorem 2 of Hahn [9]. The component processes $X_i(t)$ here are stochastically continuous as required by Hahn because $N_{1i}(t)$ is a stationary renewal process: The interval forward or backward from t to the next point has the density $1 - F(x)$. It remains to establish two moment inequalities in Hahn's Theorem 2, namely,

$$E[X_i(u) - X_i(t)]^2 = \mathrm{Var}[N_e(u-t)] \leq K(u-t) \tag{7}$$

and

$$E[(X_i(u)-X_i(t))^2(X_i(t)-X_i(s))^2]\leqslant K(u-s)^2 \tag{8}$$

for $0\leqslant s\leqslant t\leqslant u\leqslant x$ for some $x>0$ and $K<\infty$. Hahn's result was stated for $D[0, 1]$, but it supplies immediately to $D[t, t+x]$ for any positive t and x, which in turn implies weak convergence in $D[0,\infty)$ by Theorem 2.8 of [17].

First, since

$$\operatorname{Var} N_e(t)=2\int_0^t [EN(s)-s+0.5]\,\mathrm{d}s, \tag{9}$$

for all $t\geqslant 0$, by (7) on p. 57 of Cox [6], (7) is satisfied: to obtain the bound, substitute $EN(x)$ for $EN(s)$ in (9).

Second, to establish (8) let x be such that $x<1$ and $F(t)-F(0)<K_3 t$, $0\leqslant t\leqslant x$, for some constant K_3, which can be done by Condition F. Then define events

$$A_{ij}=\{N_e(t)-N_e(s)=i,\ N_e(u)-N_e(t)=j\} \tag{10}$$

and

$$\bar{A}_{11}=(A_{00}\cup A_{01}\cup A_{10})^c. \tag{11}$$

Let $Z=(X_i(t)-X_i(s))^2(X_i(u)-X_i(t))^2$ and write

$$EZ\leqslant E(Z|A_{00})+E(Z|A_{01})+E(Z|A_{10})+E(Z|\bar{A}_{11})P(\bar{A}_{11}). \tag{12}$$

We verify (8) by bounding each term in (12). First, since $u-s<1$,

$$E(Z|A_{00})=(t-s)^2(u-t)^2\leqslant(u-s)^4\leqslant(u-s)^2.$$

Second,

$$\begin{aligned}E(Z|A_{01})&\leqslant(t-s)^2E\{(N_e(u)-N_e(t))^2|A_{01}\}\\&\leqslant(t-s)^2E\{(1+N(u-t))^2|N(u-t)\geqslant 1\}\\&\leqslant(t-s)^2E\{(2+N(u-t))^2\}\leqslant K_1(t-s)^2\leqslant K_1(u-s)^2\end{aligned}$$

for some constant K_1, with lines 2-3 following from Lemmas 1-3, respectively. Also $E(Z|A_{10})\leqslant K_2(u-s)^2$ for some constant K_2 by the same argument in reverse time. Next,

$$\begin{aligned}E(Z|\bar{A}_{11})&\leqslant E\{(N_e(t)-N_e(s))^2(N_e(u)-N_e(t))^2|\bar{A}_{11})\}\\&\leqslant E\{E[(N_e(t)-N_e(s))^2(N_e(u)-N_e(t))^2|N_e(t)\\&\quad -N_e(s),\bar{A}_{11}]|\bar{A}_{11}\}\\&\leqslant E\{(N_e(t)-N_e(s))^2E[(2+N(u-t))^2]|\bar{A}_{11}\}\\&\leqslant E\{(2+N(t-s))^2\}E\{(2+N(u-t))^2\}\\&\leqslant(E\{(2+N(u-s))^2\})^2<\infty,\end{aligned}$$

with Lemmas 1 and 2 being used in both lines 3 and 4 and Lemma 3 being used to establish the final inequality. Finally, conditioning on the position of the last point before t, we have

$$P(\bar{A}_{11}) = \int_0^{t-s} (1-F(x))(F(u-t+x)-F(x))\,dx \leq \int_0^{t-s} F(y-t+x)\,dx.$$

Since x has been chosen so that $F(t)-F(0) \leq K_3 t$ for $0 \leq t \leq x$,

$$P(\bar{A}_{11}) \leq K_3 \int_0^{t-s} (u-t+x)\,dx \leq 2K_3(u-s)^2.$$

We have thus bounded each term in (12) appropriately, so that we have established (8). □

We now show that Condition F is necessary for Hahn's second moment condition (8). We use the following lemma.

Lemma 4. *If $F(t)-F(0) > Kt$, then $F(s)-F(s/3) \geq 2Ks/3$ for some s, $0 < s \leq t$.*

Proof. Since

$$F(t)-F(0) = \sum_{n=0}^{\infty} F(t3^{-n}) - F(t3^{-n-1}) \geq Kt = (2Kt/3) \sum_{n=0}^{\infty} 3^{-n},$$

for some n,

$$F(t3^{-n}) - F(t3^{-n-1}) \geq (2Kt/3)3^{-n}. \qquad \square$$

Theorem 5. *If Condition F fails, then* (8) *fails.*

Proof. We show that it is not possible to bound the last term in (12) appropriately. First, for $x < \frac{1}{2}$, $E(Z|\bar{A}_{11}) \geq \frac{1}{16}$. For $P(\bar{A}_{11})$, it suffices to consider only $t-s = u-t = \delta$. Choose δ small enough that $F(\delta) < 1-\varepsilon$. For this special case,

$$\begin{aligned} P(\bar{A}_{11}) &= \int_0^{\delta} (1-F(x))(F(\delta+x)-F(x))\,dx \geq \varepsilon \int_0^{\delta} (F(\delta+x)-F(x))\,dx \\ &\geq (\varepsilon\,\delta/2)(F(3\delta/2)-F(\delta/2)). \end{aligned}$$

By Lemma 4 and Condition F, for any K there is a $\delta' < \delta$ such that $F(3\delta'/2) - F(\delta'/2) > K\delta'$. Hence, for any K and δ, there is a δ' such that $0 < \delta' < \delta$ and $P(\bar{A}_{11}) \geq K\delta'^2$, contradicting (8). □

Proof of Theorem 3. Since t^{-1} Var $N_e(t) \to \sigma_x^2$ as $t \to \infty$ by (18) on p. 58 of Cox [6], $cK(t/c, t/c) \to \sigma_x^2$ as $c \to 0$. As $c \to 0$, $N_e(t/c)$ and $N_e(u/c) - N_e(t/c)$ are asymptotically independent, so that they are asymptotically uncorrelated. Hence, $c\,Cov(N_e(t/c), N_e(u/c) - N_e(t/c)) \to 0$ as $c \to 0$. □

Proof of Theorem 4. We first apply (9) to show that $t^{-1}\operatorname{Var} N_e(t)\to 1$ as $t\to\infty$. For this purpose, it suffices to assume that $F(0)=0$; having $F(0)>0$ only causes $EN(t)$ to be multiplied by a scalar. By Condition F, there are $t_0>0$ and K such that $F(t)\leqslant Kt<1$ for all $t\leqslant t_0$. Hence, $EN(t)\leqslant Kt/(1-Kt_0)$, $0<t<t_0$. With this bound, we can apply (9) to get the desired limit on $t^{-1}\operatorname{Var} N_e(t)$. Hence $cK(t/c, t/c)\to t$ as $c\to\infty$. It is also easy to see that $cE\{N_e(t/c)[N_e(u/c)-N_e(t/c)]\}\to 0$ as $c\to\infty$: Use the argument to treat the last term in (12) when verifying (8) for Theorem 2. However, here we do not need Condition F. Using the event $\bar{A}_{11}$ with $s=0$; we have

$$P(\bar{A}_{11})=\int_0^{t/c}(1-F(x))(F(x+(u-t)/c)-F(x))\,\mathrm{d}x$$
$$\leqslant\int_0^{t/c}(F(x+(u-t)/c)-F(x))\,\mathrm{d}x.$$

Since

$$\int_{-\infty}^{\infty}F(x+\delta)-F(x))\,\mathrm{d}x=\delta$$

for all δ,

$$\lim_{c\to\infty}c\int_0^{t/c}(F(x+(u-t)/c)-F(x))\,\mathrm{d}x=0;$$

see Exercises 2 on p. 43 and 16 on p. 49 of Chung [4]. □

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