EXPLICIT M/G/1 WAITING-TIME DISTRIBUTIONS FOR A CLASS OF LONG-TAIL SERVICE-TIME DISTRIBUTIONS

by

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Abstract

O. J. Boxma and J. W. Cohen recently obtained an explicit expression for the M/G/1 steady-state waiting-time distribution for a class of service-time distributions with power tails. We extend their explicit representation from a one-parameter family of service-time distributions to a two-parameter family. The complementary cumulative distribution function (ccdf's) of the service times all have the asymptotic form $F^c(t) \sim \alpha t^{-3/2}$ as $t \to \infty$, so that the associated waiting-time ccdf's have asymptotic form $W^c(t) \sim \beta t^{-1/2}$ as $t \to \infty$. Thus the second moment of the service time and the mean of the waiting time are infinite. Our result here also extends our own earlier explicit expression for the M/G/1 steady-state waiting-time distribution when the service-time distribution is an exponential mixture of inverse Gaussian distributions (EMIG). The EMIG distributions form a two-parameter family with ccdf having the asymptotic form $F^c(t) \sim \alpha t^{-3/2} e^{-\eta t}$ as $t \to \infty$. We now show that a variant of our previous argument applies when the service-time ccdf is an undamped EMIG, i.e., with ccdf $G^c(t) = e^{\eta t} F^c(t)$ for $F^c(t)$ above, which has the power tail $G^c(t) \sim \alpha t^{-3/2}$ as $t \to \infty$. The Boxma-Cohen long-tail service-time distribution is a special case of an undamped EMIG.

Keywords: M/G/1 queue, waiting-time distribution, Pollaczek-Khintchine formula, long-tail distributions, power-tail distributions, exponential mixture of inverse Gaussian distributions.

1. Introduction

The steady-state waiting-time distribution in the M/G/1 queue is available via the classical Pollaczek-Khintchine transform. It can be readily computed by numerical transform inversion, when the service-time Laplace transform is available, e.g., as shown in Abate and Whitt [1]. Nevertheless it is interesting to have explicit formulas. When the service-time distribution has a rational transform, so does the waiting-time distribution, and the transform can be inverted analytically. More generally, the transform can be inverted analytically, yielding the Beneš formula, which is an infinite series containing n-fold convolutions of the service-time stationary-excess distribution for all n; e.g., see 4.82 on p. 255 of Cohen [8]. Because of the complexity of the Beneš formula, however, it is natural to look for more explicit formulas.

A more explicit formula for a non-rational service-time distribution was evidently first obtained for the gamma service-time distribution with shape parameter 1/2 in (9.21) of Abate and Whitt [1]. This result was extended in Proposition 8.2 of Abate and Whitt [3] to all exponential mixtures of inverse Gaussian (EMIG) service-time distributions. These service-time distributions have probability densities with asymptotics of the form $f(t) \sim \alpha t^{-3/2} e^{-\eta t}$ as $t \to \infty$, where $f(t) \sim g(t)$ as $t \to \infty$ means that $f(t)/g(t) \to 1$. Because of the $e^{-\eta t}$ term, these EMIG distributions do not have a long (a heavy) tail. However, recently, Boxma and Cohen [7] obtained an explicit expression for the M/G/1 waiting-time distribution for a class of long-tail service-time distributions. In this paper, we extend Boxma and Cohen's result to a larger class of long-tail service-time distributions. In particular, we extend our result in [3] to undamped EMIGs, i.e., to distributions with complementary cumulative distribution functions (ccdf's) $G^c(t) \equiv 1 - G(t) = e^{\eta t} F^c(t)$, where $F^c(t)$ is an EMIG ccdf. The Boxma-Cohen service-time distributions are a subclass.

Here is how the rest of this paper is organized. In Section 2 we give the explicit solution for the steady-state waiting-time distribution. In Section 3 we show that the service-time distributions used in Section 2 can be represented as undamped EMIGs. In Section 4 we show that both EMIGs and undamped EMIGs are completely monotone (mixtures of exponentials) and give their mixing densities. In Section 5 we give the asymptotic behavior of undamped EMIGs as $t \to 0$ and as $t \to \infty$. We apply that result to give the first two terms of the asymptotic expansion for the waiting-time ccdf in Section 2, which agrees with Boxma and Cohen [7]. In Section 6 we discuss the heavy-traffic approximation due to Boxma and Cohen [7]. For the service-time distributions considered here, we derive their limit from the explicit waiting-time ccdf. We conclude in Section 7 by discussing other service-time distributions for which explicit representations of the waiting-time distribution are possible, but the greater complexity make them of dubious value.

2. The Explicit Solution

Consider a service-time probability density function (pdf) $g(t)$ with Laplace transform

$$\hat{g}(s) \equiv \int_0^\infty e^{-st} g(t) dt = 1 - \frac{s}{(\mu + \sqrt{s})(1 + \sqrt{s})} , \tag{2.1}$$

which has mean $m_1(g) = \mu^{-1}$ and all higher moments infinite. The pdf g has two-parameters, the displayed μ and the scale, which has been omitted. Both can range over the positive reals.

The Pollaczek-Khintchine formula involves the associated stationary-excess pdf $g_e(t) \equiv \mu G(t)$, $t \geq 0$. Its Laplace transform has the nice form

$$\hat{g}_e(s) \equiv \frac{1 - g(s)}{s m_1(g)} = \frac{\mu}{(\mu + \sqrt{s})(1 + \sqrt{s})} . \tag{2.2}$$

For $\mu \neq 1$,

$$\hat{g}_e(s) = \left(\frac{\mu}{1-\mu}\right) \left(\frac{1}{\mu + \sqrt{s}} - \frac{1}{1 + \sqrt{s}}\right) , \tag{2.3}$$

so that, by 29.3.37 of Abramowitz and Stegun [6],

$$g_e(t) = \mu G^c(t) = \left(\frac{\mu}{1-\mu}\right) (\psi(t) - \mu\psi(\mu^2 t)), \quad t \geq 0 \tag{2.4}$$

where

$$\psi(t) \equiv e^t \, erfc(\sqrt{t}) \sim \frac{1}{\sqrt{\pi t}} \quad \text{as} \quad t \to \infty , \tag{2.5}$$

with $erfc$ being the complementary error function, i.e.,

$$erfc(t) \equiv \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du \equiv 2\Phi^c(\sqrt{2}t) , \tag{2.6}$$

where $\Phi^c(t) \equiv 1 - \Phi(t)$ is the standard (mean 0, variance 1) normal complementary cumulative distribution function (ccdf); see 7.1.1 and 26.2.29 of Abramowitz and Stegun [6]. We will establish further properties of G and G_e in the next section.

The case $\mu = 1$ was considered by Boxma and Cohen [7]. The case $\mu = 1$ also corresponds to a subclass of beta mixtures of exponential (BME) pdf's considered by Abate and Whitt [4]; we will discuss this connection further in the next section. Boxma and Cohen show that the service-time ccdf when $\mu = 1$ is

$$G^c(t) = (2t + 1)\psi(t) - 2\sqrt{t/\pi}, \quad t \geq 0 , \tag{2.7}$$

for ψ in (2.5). In the next section we will show that the associated stationary-excess ccdf is

$$G_e^c(t) = 2\sqrt{t/\pi} - (2t-1)\psi(t), \quad t \geq 0 \; . \tag{2.8}$$

We now consider the steady-state waiting-time distribution in the M/G/1 queue with arrival rate λ. It has an atom of $1-\rho$ at the origin, assuming that $\rho \equiv \lambda/\mu < 1$, but otherwise a pdf. The Laplace transform of the ccdf is

$$\hat{W}^c(s) = \frac{\rho}{s}(1 - \hat{w}_\rho(s)) \; , \tag{2.9}$$

where $\hat{w}_\rho(s)$ is the Laplace transform of the conditional waiting time pdf, given that there is a positive wait, i.e.,

$$\hat{w}_\rho(s) = \frac{(1-\rho)\hat{g}_e(s)}{1-\rho\hat{g}_e(s)} \; . \tag{2.10}$$

Paralleling Propoosition 8.2 of Abate and Whitt [3], we can find an explicit expression for $\hat{W}^c(s)$ and analytically invert it. From (2.2)–(2.10), we deduce the following.

Theorem 2.1. *For the service-time pdf $g(t)$ with Laplace transform $\hat{g}(s)$ in (2.1),*

$$\hat{w}_\rho(s) = \frac{(1-\rho)\mu}{\nu_1 - \nu_2}\left(\frac{1}{\nu_2 + \sqrt{s}} - \frac{1}{\nu_1 + \sqrt{s}}\right) \tag{2.11}$$

and

$$\hat{W}^c(s) = \frac{\rho}{\nu_1 - \nu_2}\left(\frac{\nu_1}{\sqrt{s}(\nu_2 + \sqrt{s})} - \frac{\nu_2}{\sqrt{s}(\nu_1 + \sqrt{s})}\right) \; , \tag{2.12}$$

so that

$$W^c(t) = \frac{\rho}{\nu_1 - \nu_2}\left(\nu_1\psi(\nu_2^2 t) - \nu_2\psi(\nu_1^2 t)\right) \; , \tag{2.13}$$

where ψ is given in (2.5) and

$$\nu_{1,2} = \frac{1+\mu}{2} \pm \sqrt{\left(\frac{1+\mu}{2}\right)^2 - (1-\rho)\mu} \; . \tag{2.14}$$

Proof. Algebra yields (2.11) and (2.12). The Laplace transform (2.12) is easy to invert using 29.3.43 of Abramowitz and Stegun [6]. ∎

The case $\mu = 1$ (with $v_1 = 1 + \sqrt{\rho}$ and $v_2 = 1 - \sqrt{\rho}$) was obtained by Boxma and Cohen [7]. They included an atom at the origin in the service-time distribution, which we could do as well. The atom at the origin simply gets absorbed in ρ, i.e., corresponds to changing the arrival rate λ. This property is most easily seen from the virtual waiting time, which has the same distribution as the actual waiting time in M/G/1. A customer with 0 service time causes no change in the virtual waiting-time process upon its arrival. By the Poisson thinning property, the arrival process of customers with positive service times is also a Poisson process but with reduced arrival rate $\lambda(1-\eta)$, where η is the atom at 0 in the service-time distribution. Hence, having an atom of mass η at 0 in the service-time distribution is equivalent to changing the arrival rate to $\lambda(1-\eta)$ and considering the service-time distribution without the atom, i.e., the conditional service-time distribution given that it is positive.

3. Undamped EMIGs

We obtain the service-time transform $\hat{g}(s)$ in (2.1) by undamping an *exponential mixture of inverse Gaussian* (EMIG) ccdf's. The EMIGs were discussed in Section 8 of [3].

Introducing a slight change of notation, we start with the Laplace transform of an EMIG pdf

$$\hat{f}(s) = \frac{\mu+1}{\mu+\sqrt{1+s}} \ . \tag{3.1}$$

Formula (3.1) is obtained from (8.9) of [3] by first replacing μ by $\mu+1$ and then introducing the scale parameter $\omega \equiv 1/2(\mu+1)$; i.e., $\hat{f}(s) = \hat{\rho}(s;\omega,\mu+1) \equiv \hat{\rho}(\omega s, 1, \mu+1)$ for that ω. Paralleling $\hat{g}(s)$ in (2.1), an extra scale parameter can be added to $\hat{f}(s)$ in (3.1).

The moments of the pdf with transform in (3.1) can be derived from the inverse Gaussian moments by using (8.3) and (8.10) of [3] (r should be n in (8.3)). They are

$$m_1(F) = \frac{1}{2(\mu+1)}, \quad m_{n+1}(F) = \frac{1}{(2+2\mu)^{n+1}} \sum_{k=0}^{n} \frac{(n+1-k)(n+k)!}{k!} \left(\frac{\mu+1}{2}\right)^k \tag{3.2}$$

and squared coefficient of variation (variance divided by the mean) $c^2 = \mu+2$. For the case $\mu = 1$, (3.1) is the BME transform $\hat{v}(1/2, 3/2; s)$ studied in [4] and the moments in this case are $m_n = n!\beta_n/(n+1)$ where $\beta_n = \binom{2n}{n}4^{-n}$.

Paralleling (8.13) and (8.14) of [3], the ccdf has the Laplace transform

$$\begin{aligned} \hat{F}^c(s) &= \frac{1-\hat{f}(s)}{s} = \frac{1}{(\mu+\sqrt{1+s})(1+\sqrt{1+s})} & (3.3) \\ &= \frac{1}{\mu-1}\left(\frac{1}{1+\sqrt{1+s}} - \frac{1}{\mu+\sqrt{1+s}}\right) , \quad \mu \neq 1 \ . & (3.4) \end{aligned}$$

From (3.4) we see that EMIG stationary-excess pdf is

$$f_e(t) = \frac{\mu+1}{\mu-1} v(1/2, 3/2; t) - \frac{2}{\mu-1} f(t) \ , \tag{3.5}$$

from which we obtain the simple moment recurrence for $\mu \neq 1$

$$m_{n+1}(F) = \frac{n!\beta_n}{2(\mu-1)} - \frac{n+1}{\mu^2-1} m_n(F) \ . \tag{3.6}$$

The recurrence formula (3.6) is recommended over (3.2) to calculate the moments. It is noteworthy that the moments $m_n(F)$ are always integer sequences when μ is an integer and they are scaled by the factor $(2+2\mu)^n$. Except for the cases $\mu = 0$ and 1, none of these integer sequences are found in Sloane and Plouffe [12]. For example, the moment sequence for $\mu = 2$ is 1, 5, 51, 807, 17445, 479565, ...

From (3.1) and 29.3.37 of Abramowitz and Stegun [6],

$$f(t) = (\mu+1)\left(\frac{e^{-t}}{\sqrt{\pi t}} - \mu e^{(\mu^2-1)t}\, erfc\,(\mu\sqrt{t})\right), \quad t \ge 0\ , \tag{3.7}$$

Going from (3.7) to (3.2) is surprisingly difficult. It can be done by applying the Gosper-Zeilberger algorithm, e.g., see Section 5.8, especially p. 236, of Graham, Knuth and Patashnik [10] or Petkovsek, Wilf and Zeilberger [11]. The associated EMIG pdf in [3], which unfortunately was inadvertently omitted from (8.10) of [3], is

$$\rho(t;1,\nu) = \frac{\nu e^{-t/2\nu}}{\sqrt{2\pi\nu t}} - 2^{-1}(\nu-1)e^{(\nu-2)t/2}\, erfc((\nu-1)\sqrt{t/2\nu})\ . \tag{3.8}$$

To obtain (3.7) and (3.8), first scale t by the factor $2v$, then let $\nu = \mu + 1$.

Similarly, from (3.4), we have for $\mu \ne 1$,

$$F^c(t) = \frac{1}{\mu-1}(\mu e^{(\mu^2-1)t}\, erfc\,(\mu\sqrt{t}) - erfc\,(\sqrt{t})), \quad t \ge 0\ , \tag{3.9}$$

whereas for $\mu = 1$, we invert $(1+\sqrt{1+s})^{-2}$ to get

$$F^c(t) = (1+2t)\, erfc\,(\sqrt{t}) - 2\sqrt{\pi/t}e^{-t}, \quad t \ge 0\ . \tag{3.10}$$

In the case $\mu = 1$, the pdf $f(t)$ in (3.7) coincides with the beta mixture of exponentials (BME) pdf $v(1/2, 3/2; t)$ in Abate and Whitt [4], which in turn coincides with the RBM first-moment pdf $h_1(t)$; see Table 3 in [4]. The associated cdf in (3.10) is $v(3/2, 3/2; t)/4$. (See the next section for further discussion.)

For all $\mu > 0$, the asymptotic expansion for $F^c(t)$ is

$$F^c(t) \sim \frac{e^{-t}}{\sqrt{\pi t}} \sum_{n=1}^{\infty} (-1)^{n+1} k_n(\mu) n! \beta_n t^{-n} \quad \text{as} \quad t \to \infty\ , \tag{3.11}$$

where $\beta_n = \binom{2n}{n} 4^{-n}$ is the moment sequence of the gamma pdf $\gamma(t) = e^{-t}/\sqrt{\pi t}$ as in Table 3 of [4] and

$$k_n(\mu) = \sum_{k=0}^{2n-1} \mu^k = \frac{1}{\mu-1}\left(1 - \frac{1}{\mu^{2n}}\right)\ , \tag{3.12}$$

drawing on 7.1.23 of Abramowitz and Stegun [6]. Note that $k_n(1) = 2n$.

As in our construction of B_2ME ccdf's from BME ccdf's in [4], we define the ccdf G^c associated with $\hat{g}(s)$ in (2.1) by undamping the ccdf $F^c(t)$, i.e., by letting

$$G^c(t) = e^t F^c(t), \quad t \geq 0 \; . \tag{3.13}$$

Combining (3.3) and (3.13), we obtain

$$\hat{G}^c(s) = \hat{F}^c(s-1) = \frac{1}{(\mu+\sqrt{s})(1+\sqrt{s})} \tag{3.14}$$

and

$$\hat{g}(s) = 1 - s\hat{G}^c(s) = 1 - \frac{s}{(\mu+\sqrt{s})(1+\sqrt{s})} \; , \tag{3.15}$$

just as in (2.1). Moreover,

$$\hat{G}^c_e(s) \equiv \frac{1-\hat{g}_e(s)}{s} = \left(\frac{\mu+1}{\mu}\right)\frac{1}{\sqrt{s}(1+\sqrt{s})} + \left(\frac{1}{\mu(1-\mu)}\right)\frac{1}{1+\sqrt{s}} - \left(\frac{1}{\mu(1-\mu)}\right)\frac{1}{\mu+\sqrt{s}} \; , \tag{3.16}$$

so that, by 29.3.37 and 29.3.43 of Abramowitz and Stegun [6],

$$G^c_e(t) = \frac{\mu}{1-\mu}(\mu^{-1}\psi(\mu^2 t) - \psi(t)), \quad t \geq 0 \; , \tag{3.17}$$

for ψ in (2.5).

In the case $\mu = 1$, we can apply the BME and B_2ME calculus in [4], in particular, (1.20), (1.7) and Table 3, to get

$$\begin{aligned} g_e(t) = G^c(t) &= V^c_2(1/2, 3/2; t) = e^t V(1/2, 3/2; t) \\ &= (1/4)e^t v(3/2, 3/2; t) \\ &= (2t+1)\psi(t) - 2\sqrt{t/\pi} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} G^c_e(t) &= V^c_2(3/2, 1/2; t) = e^t V^c(3/2, 1/2; t) \\ &= (3/4)e^t v(5/2, 1/2; t) \\ &= 2\sqrt{t/\pi} - (2t-1)\psi(t) \; , \end{aligned} \tag{3.19}$$

as given in (2.8).

4. Representation as a Mixture of Exponentials

We now show that EMIGs and undamped EMIGs are both completely monotone; i.e., can be expressed as mixtures of exponentials. As a consequence, they can be approximated arbitrarily closely by hyperexponential (finite mixtures of exponential) distributions; see Feldmann and Whitt [9]. Of course, the hyperexponential approximations never match the asymptotic tail behavior. Nevertheless, the associated M/G/1 waiting-time distributions are also matched arbitrarily closely; see [9].

Theorem 4.1. *An EMIG is completely monotone; in particular, the ccdf can be expressed as*

$$F^c(t) = \int_0^1 e^{-t/y} w(y) dy \ , \tag{4.1}$$

where

$$w(y) = \frac{\mu+1}{\pi\sqrt{y}} \left(\frac{\sqrt{1-y}}{1+(\mu^2-1)y} \right), \quad 0 \le y \le 1 \ . \tag{4.2}$$

Proof. We regard the Laplace transform $\hat{F}^c(s)$ in (3.4) as the Stieltjes transform of the spectral density; i.e., initially assuming that

$$F^c(t) = \int_0^\infty e^{-xt} \phi(x) dy \ , \tag{4.3}$$

we obtain

$$\hat{F}^c(s) = \int_0^\infty \frac{1}{s+x} \phi(x) dx \ . \tag{4.4}$$

We can then calculate the alleged spectral density $\phi(x)$ by inverting its Stieltjes transform, p. 126 of Widder [14]; i.e.,

$$\phi(x) = -\frac{\operatorname{Im} \hat{F}^c(-x)}{\pi} = \frac{1}{\pi(\mu-1)} \left(\frac{\sqrt{x-1}}{x} - \frac{\sqrt{x-1}}{x+\mu^2-1} \right) = \frac{(\mu+1)\sqrt{x-1}}{\pi x (x+\mu^2-1)}, \quad x > 1 \ . \tag{4.5}$$

The mixing density $w(y)$ is related to the spectral density $\phi(x)$ by $w(y) = y^{-2}\phi(y^{-1})$. Hence, from (4.5) we obtain (4.2). ■

We can combine (3.13) and Theorem 4.1 to obtain a corresponding result for undamped EMIGS.

Corollary 1. *An undamped EMIG is also completely monotone, i.e.,*

$$\begin{aligned} G^c(t) &= \int_0^1 e^{-t(1-y)/y} w(y) dy & (4.6) \\ &= \int_0^\infty e^{-t/z} w(z/(z+1))(1+z)^{-2} dz & (4.7) \end{aligned}$$

for $w(y)$ in (4.2).

In two special cases the EMIG is a beta mixture of exponentials (BME), as considered in [4]. Recall that the beta density is

$$b(p,q;y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 \le y \le 1 \ . \tag{4.8}$$

Corollary 2. *For $\mu = 0$, $w(y) = b(1/2, 1/2; y)$; for $\mu = 1$, $w(y) = b(1/2, 3/2; y)$.*

Hence, in the notation of [4], the EMIG in (3.1) is $\nu(1/2, 1/2; t)$ when $\mu = 0$ and $\nu(1/2, 3/2; t)$ when $\mu = 1$. For those cases additional properties are given in [4]. Recall that the special case considered by Boxma and Cohen [7] is $\mu = 1$. Thus their case is the B$_2$ME pdf $\nu_2(1/2, 3/2; t)$. By Theorem 8 of [4], it can also be expressed as a gamma mixture of Pareto distributions.

More generally, we can express the mixing pdf $w(y)$ in (4.2) as a linear combination of beta pdf's. To do so, we expand $(1 + (\mu^2 - 1)y)^{-1}$ in (4.2) in a power series.

Theorem 4.2. *For $\mu > 0$ with $\mu \neq 1$,*

$$w(y) = \frac{\mu + 1}{2} \sum_{n=0}^{\infty} (1 - \mu^2)^n \frac{\beta_n}{n + 1} b\left(\frac{2n + 1}{2}, 3/2; y\right) \quad . \tag{4.9}$$

where $\beta_n \equiv \binom{2n}{n} 4^{-n}$, the moments of $b(1/2, 1/2; y)$.

5. Time Asymptotics

Combining (3.9) and (3.13), we obtain the undamped EMIG ccdf $G^c(t)$. From that form, we can obtain the asymptotics as $t \to 0$ and as $t \to \infty$. In particular, from (3.11),

Theorem 5.1. *For the undamped EMIG distribution,*

$$G^c(t) \sim 1 - 2(\mu + 1)\sqrt{t/\pi} \quad \text{as} \quad t \to 0 \ , \tag{5.1}$$

$$G^c(t) \sim \left(\frac{\mu + 1}{2\mu^2}\right) \frac{1}{\sqrt{\pi t^3}} \quad \text{as} \quad t \to \infty \ , \tag{5.2}$$

and

$$G_e^c(t) \sim \left(\frac{\mu + 1}{\mu}\right) \frac{1}{\sqrt{\pi t}} \quad \text{as} \quad t \to \infty \ . \tag{5.3}$$

Similarly, we obtain the large-time asymptotics for $W^c(t)$ from (2.13). For other M/G/1 waiting-time asymptotics, see Willekens and Teugels [15], Abate, Choudhury and Whitt [5] and Boxma and Cohen [7].

Theorem 5.2. *with the undamped EMIG service-time pdf transform $\hat{g}(s)$ in (2.1),*

$$W^c(t) \sim \frac{\rho}{1 - \rho} G_e^c(t) \left[1 - \frac{(1 + \mu)^2 - 2(1 - \rho)\mu}{2(1 - \rho)^2 \mu^2 t}\right] \quad \text{as} \quad t \to \infty \ . \tag{5.4}$$

Formula (5.4) here agrees with formula (3.12) of Boxma and Cohen [7] for the case $\mu = 1$.

6. Heavy-Traffic Asymptotics

Boxma and Cohen [7] establish general heavy-traffic limits and approximations as $\rho \to 1$. We obtain their result for our special case directly from the explicit representation in Section 2.

Theorem 6.1. *If $\rho \to 1$, then $\nu_1 \to 1 + \mu$, $\nu_2/(1-\rho) \to \mu/(1+\mu)$ and*

$$W^c(t/\alpha)\psi(t) \tag{6.1}$$

for $\psi(t)$ in (2.5), where

$$\alpha = \frac{(1-\rho)^2}{\rho^2}\left(\frac{\mu}{1+\mu}\right)^2 . \tag{6.2}$$

Based on (6.1), we would use the approximation

$$W^c(t) \approx \psi(\alpha t) = e^{\alpha t}\, erfc(\sqrt{\alpha t}) \tag{6.3}$$

for α in (6.2). Since $\rho^2 \to 1$ as $\rho \to 1$, the factor ρ^2 in (6.2) plays no role in the heavy-traffic limit. However, it makes the heavy-traffic approximation (6.3) asymptotically correct as $t \to \infty$ for each ρ as well. We could further simplify the right side of (6.3) by replacing $erfc(\sqrt{\alpha t})$ by its asymptotic form as $\alpha \to 0$, but the numerics performed by Boxma and Cohen [7] show that it is better to keep the error function. This phenomenon very closely parallels our asymptotic normal approximation for the M/G/1 busy-period distribution in Abate and Whitt [2]. Indeed, the same approximating functions are involved.

7. Other Explicit Expressions

Smith [13] first observed that if the service-time distribution has rational Laplace transform, then so does the M/G/1 steady-state waiting-time distribution, so that at least in principle it can be inverted analytically. This is easy to see in two steps: (1) going from the service-time cdf G to its associated stationary-excess cdf G_e and (2) going from G_e to the waiting-time cdf exploiting the Pollaczek-Khintchine formula. The other explicit representations obtained so far can be viewed as generalizations of this result. If the service-time distribution has a Laplace transform that is a rational function of $s^{1/n}$, then it is easy to see that so does the M/G/1 steady-state waiting-time distribution. For general n, this property seems difficult to exploit, but for $n = 2$, we can exploit it, because we can relate the transform involving $\sqrt{s}$ to the error function.

For example, at least in principle, we can obtain the explicit $M/G/1$ waiting-time distribution when the service-time distribution is a mixture of k undamped EMIGs. By the usual partial fraction expansion (assuming no multiple roots), we can represent the waiting-time distribution as a linear combination of undamped EMIGs. However, the additional complexity seems to make this approach unattractive.

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