Appendix A

Regular Variation

Since we use regular variation at several places in this book, we give a brief account here without proofs, collecting together the properties we need. Bingham, Goldie and Teugels (1989) is the definitive treatment; it contains everything here. Nice accounts also appear in Feller (1971) and Resnick (1987).

We say that a real-valued function $f$ defined on the interval $(c, \infty)$ for some $c > 0$ is asymptotically equivalent to another such function $g$ (at infinity) and write

$$f(x) \sim g(x) \quad \text{as} \quad x \to \infty \quad \text{(A.1)}$$

if

$$f(x)/g(x) \to 1 \quad \text{as} \quad x \to \infty.$$ \hspace{1cm} \text{(A.2)}

We say that the real-valued function $f$ has a power tail of index $\alpha$ (at infinity) if

$$f(x) \sim Ax^\alpha \quad \text{as} \quad x \to \infty \quad \text{(A.3)}$$

for a non-zero constant $A$. Regular variation is a generalization of the power-tail property that captures just what is needed in many mathematical settings.

A positive, Lebesgue measurable real-valued function (on some interval $(c, \infty)$) $L$ is said to be slowly varying (at infinity) if

$$L(\lambda x) \sim L(x) \quad \text{as} \quad x \to \infty \quad \text{for each} \quad \lambda > 0.$$ \hspace{1cm} \text{(A.4)}

Examples of slowly varying functions are positive constants, functions that converge to positive constants, logarithms and iterated logarithms. (Note that $\log x$ is positive on $(1, \infty)$, while $L_2x \equiv \log \log x$ is positive on $(e, \infty)$.)

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The positive functions \(2 + \sin x\) and \(e^{-x}\) are not slowly varying. It can happen that a slowly varying function experiences infinite oscillation in the sense that
\[
\liminf_{x \to \infty} L(x) = 0 \quad \text{and} \quad \limsup_{x \to \infty} L(x) = \infty ;
\]
an example is
\[
L(x) = \exp\left( (\ln(1 + x))^{1/2} \cos((\ln(1 + x))^{1/2}) \right) . \tag{A.5}
\]
It is significant that there is local uniform convergence in (A.4).

**Theorem A.1.** (local uniform convergence) *If \(L\) is slowly varying, then \(L(\lambda x)/L(x) \to 1\) as \(x \to \infty\) uniformly over compact \(\lambda\) sets.*

**Theorem A.2.** (representation theorem) *The function \(L\) is locally varying if and only if*
\[
L(x) = c(x) \exp \left( \int_a^x \frac{b(u)}{u} du \right), \quad x \geq a, \tag{A.6}
\]
*for some \(a > 0\), where \(c\) and \(b\) are measurable functions, \(c(x) \to c(\infty) \in (0, \infty)\) and \(b(x) \to 0\) as \(x \to \infty\).*

In general a slowly varying function need not be smooth, but it is always asymptotically equivalent to a smooth slowly varying function; see p. 15 of Bingham et al. (1989).

**Theorem A.3.** *If \(L\) is slowly varying function, then there exists a slowly varying function \(L_0\) with continuous derivatives of all orders \((in C^\infty)\) such that*
\[
L(x) \sim L_0(x) \quad \text{as} \quad x \to \infty .
\]
*If \(L\) is eventually monotone, then so is \(L_0\).*

We now turn to regular variation. A positive, Lebesgue measurable function \((on some interval \((c, \infty))\) is said to be *regularly varying* of index \(\alpha\), and we write \(h \in \mathcal{R}(\alpha)\), if
\[
h(\lambda x) \sim \lambda^\alpha h(x) \quad \text{as} \quad x \to \infty \quad \text{for all} \quad \lambda > 0 . \tag{A.7}
\]
Of course, (A.7) holds whenever \(h\) has a power tail with index \(\alpha\), but it also holds more generally.

The connection to slowly varying function is provided by the characterization theorem.
Theorem A.4. (characterization theorem) If

\[ h(\lambda x) \sim g(\lambda)h(x) \quad \text{as} \quad x \to \infty \quad (A.8) \]

for all \( \lambda \) in a set of positive measure, then

(i) \( (A.8) \) holds for all \( \lambda > 0 \),

(ii) there exists a number \( \alpha \) such that \( g(\lambda) = \lambda^\alpha \) for all \( \lambda \), and

(iii) \( h(x) = x^\alpha L(x) \) for some slowly varying function \( L \).

From Theorem A.4 (iii) we see that we could have defined a regularly varying function in terms of a slowly varying function \( L \). On the other hand, \( (A.8) \) is an appealing alternative starting point, implying \( (A.7) \) and the representation in terms of slowly varying functions. As a consequence of Theorem A.4 (iii), we write \( h \in \mathcal{R}(0) \) when \( h \) is slowly varying.

We now indicate how the local-uniform-convergence property of \( \mathcal{R}(0) \) in Theorem A.1 extends to \( \mathcal{R} \).

Theorem A.5. (local uniform convergence) If \( h \in \mathcal{R}(\alpha) \) and \( h \) is bounded on each interval \( (0, \epsilon] \), then

\[ h(\lambda x)/h(x) \to \lambda^\alpha \quad \text{as} \quad x \to \infty \]

uniformly in \( \lambda \):

on each \( [a, b] \), \( 0 < a < b < \infty \), if \( \alpha = 0 \)

on each \( (0, b] \), \( 0 < b < \infty \), if \( \alpha > 0 \)

on each \( [a, \infty) \), \( 0 < a < \infty \), if \( \alpha < 0 \).

The representation theorem for slowly varying functions in Theorem A.2 also extends to regularly varying functions.

Theorem A.6. (representation theorem) We have \( h \in \mathcal{R}(\alpha) \) if and only if

\[ h(x) = c(x) \exp \left\{ \int_a^x (b(u)/u)du \right\}, \quad x \geq a, \]

for some \( a > 0 \), where \( c \) and \( b \) are measurable functions, \( c(x) \to c(\infty) \in (0, \infty) \) and \( b(x) \to \alpha \) as \( x \to \infty \).

We now present some closure properties.
**Theorem A.7.** (closure properties) Suppose that $h_1 \in \mathcal{R}(\alpha_1)$ and $h_2 \in \mathcal{R}(\alpha_2)$. Then:

(i) $h_1^0 \in \mathcal{R}(\alpha_1)$.

(ii) $h_1 + h_2 \in \mathcal{R}(\alpha)$ for $\alpha = \max\{\alpha_1, \alpha_2\}$.

(iii) $h_1 h_2 \in \mathcal{R}(\alpha_1 + \alpha_2)$.

(iv) If, in addition, $h_2(x) \to \infty$, then $(h_1 \circ h_2)(x) \equiv h_1(h_2(x)) \in \mathcal{R}(\alpha_1 \alpha_2)$.

Regular variation turns out to imply local regularity conditions; see p. 18 of Bingham et al. (1989).

**Theorem A.8.** (local integrability theorem) If $h \in \mathcal{R}(\alpha)$ for some $\alpha$, then $h$ and $1/h$ are both bounded and integrable over compact subintervals of $(c, \infty)$ for suitably large $c$.

It is significant that integrals of regularly varying functions are again regularly varying with the same slowly varying function (i.e., the slowly varying function passes through the integral).

**Theorem A.9.** (Karamata’s integral theorem) Suppose that $L \in \mathcal{R}(0)$ and that $L$ is bounded on every compact subset of $[c, \infty)$ for some $c \geq 0$. Then

(a) for $\alpha > -1$,

$$\int_c^x \ell^\alpha L(t) dt \sim \frac{x^{\alpha+1}}{\alpha + 1} L(x) \quad \text{as} \quad x \to \infty ;$$

(b) for $\alpha < -1$,

$$\int_x^\infty \ell^\alpha L(t) dt \sim -\frac{x^{\alpha+1}}{\alpha + 1} L(x) \quad \text{as} \quad x \to \infty .$$

It is also possible to deduce regular variation of functions from the regular-variation properties of the integrals, i.e., the converse half of the Karamata integral theorem; see p. 30 of Bingham et al. (1989).