THE TRANSIENT *BMAP/G/*1 QUEUE

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ABSTRACT

We derive the two-dimensional transforms of the transient workload and queue-length distributions in the single-server queue with general service times and a batch Markovian arrival process (*BMAP*). This arrival process includes the familiar phase-type renewal process and the Markov modulated Poisson process as special cases, and allows correlated interarrival times and batch sizes. Numerical results are obtained via two-dimensional transform inversion algorithms based on the Fourier-series method. From the numerical examples we see that predictions of system performance based on transient and stationary performance measures can be quite different.

Key Words: transient behavior of queues, transient waiting times, *N/G/*1, busy period, emptiness function, numerical inversion of transforms, two-dimensional transform inversion

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August 1, 2000

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1. Introduction

In this paper we consider the single-server queue with unlimited waiting space, a work-conserving service discipline and i.i.d. (independent and identically distributed) service times that are independent of a general arrival process. Our purpose is to obtain computable transient results for this general model.

In order to obtain computable results, we assume that the arrival process is a *batch Markovian arrival process* (*BMAP*), as in (Lucantoni, [1]). The *BMAP* is a convenient representation of the *versatile Markovian point process* (Neuts, [2] [3]) or *Neuts (N) process* (Ramaswami, [4]). The *BMAP* generalizes the *Markovian arrival process* (*MAP*), which was introduced by (Lucantoni, Meier-Hellstern and Neuts, [5]). The *MAP* includes as special cases both the phase-type renewal process (Neuts, [6]) and the Markov-modulated Poisson process (Heffes and Lucantoni, [7]). Indeed, stationary *MAP*s are dense in the family of all stationary point processes (Asmussen and Koole, [8]). An important property of *MAP*s and *BMAP*s is that superpositions of independent processes of these types are again processes of the same type; this property is exploited in (Choudhury, Lucantoni and Whitt, [9]) to study the effect of statistically multiplexing a large number of bursty sources.

Hence, we consider the *BMAP*/*G*/1 queue and derive the two-dimensional transforms of the workload (or virtual waiting time) distribution at time t and the queue-length distribution at time t. As usual with the *BMAP*/*G*/1 queue, these quantities are actually $m \times m$ matrices, with the $(i,j)^{\text{th}}$ element specifying that the auxiliary phase is j at time t, conditioned upon the phase at time 0 being i.

These transient results can be regarded as matrix generalizations of transient results for the *M*/*G*/1 queue, which can be found in (Takács, [10]), (Abate and Whitt, [11]) and references cited there. As in the *M*/*G*/1 special case, a key role here is played by the busy-period distribution and the emptiness function. These are discussed in Sections 2.4 and 3.1 here.

In fact, there is a long history of transient results for single-server queueing models generalizing *M*/*G*/1, as can be seen from the books by (Neuts, [6][2]), (Takács, [10]), and (Benes, [12]), and references therein. With regard to the present work, the 1967 papers by (Çinlar, [13], [14]) and the early papers of Neuts (cited in [2]) are notable.

A distinctive feature of our paper, in relation to previous papers on transient behavior for these *M*/*G*/1-type queues, is that *we demonstrate that our formulas are computable*. In particular, we calculate the time-dependent probability distributions by *numerically inverting the two-dimensional transforms*. For this purpose, we apply the two-dimensional transform inversion algorithms in (Choudhury, Lucantoni and Whitt, [15]). These algorithms are based on the Fourier-series method [16], exploiting the two-dimensional Poisson summation formula, as in (5.44)–(5.48) of [16]. For this purpose, we obtain the busy-period transform by iterating the characterizing functional equation, drawing upon (Choudhury, Lucantoni and Whitt, [17]).

The remainder of this paper is organized as follows. In §2 we review the definition and basic properties of the Batch Markovian Arrival Process and the single server queue with this arrival process. In particular, we review the transform of the duration of the busy period which plays a fundamental role in the transient solution of this model. In §3 we derive the Laplace transform for the probability that the system is empty at time *t*. Sections 4 and 5 contain the main results on the transient distributions of the workload and queue length, respectively. The algorithm for inverting multidimensional Laplace transforms is presented in §6 and this algorithm is used for computing the numerical examples in §7. All of the proofs are presented in §8.

2. The *BMAP/G/*1 Queue

2.1 The Batch Markovian Arrival Process

The *BMAP* is a natural generalization of the Poisson process (see (Lucantoni, [1])). It is constructed by considering a two-dimensional Markov process $\{N(t),\, J(t)\}$ on the state space $\{(i,j):\, i\geq 0,\ 1\leq j\leq m\}$ with an infinitesimal generator Q having the structure

$$Q = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ & D_0 & D_1 & D_2 & \cdots \\ & & D_0 & D_1 & \cdots \\ & & & D_0 & \cdots \\ & & & & \cdots \end{bmatrix}, \tag{1}$$

where $D_k,\ k\geq 0$, are $m\times m$ matrices; D_0 has negative diagonal elements and nonnegative off-diagonal elements; $D_k,\ k\geq 1$, are nonnegative and D, defined by

$$D = \sum_{k=0}^{\infty} D_k, \tag{2}$$

is an irreducible infinitesimal generator. We also assume that $D\neq D_0$, which assures that arrivals will occur.

The variable $N(t)$ counts the number of arrivals in the interval $(0,t]$, and the variable $J(t)$ represents an auxiliary state or phase. Transitions from a state (i,j) to a state $(i+k,l),\ k\geq 1,\ 1\leq j,l\leq m$, correspond to batch arrivals of size *k*, and thus the batch size can depend on *j* and *l*. The matrix D_0 is a stable matrix (see e.g., pg. 251 of Bellman [18]), which implies that it is nonsingular and the sojourn time in the set of states $\{(i,j):\, 1\leq j\leq m\}$ is finite with probability one, for all *i*; see Lemma 2.2.1 of (Neuts, [6]). This

implies that the arrival process does not terminate.

Let $\boldsymbol{\pi}$ be the stationary probability vector of the Markov process with generator D, i.e., $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} D = \mathbf{0}, \qquad \boldsymbol{\pi} \boldsymbol{e} = 1, \tag{3}$$

where $\boldsymbol{e}$ is a column vector of 1's. Then the component π_j is the stationary probability that the arrival process is in state j. The arrival rate of the process is then

$$\lambda = \boldsymbol{\pi} \sum_{k=1}^{\infty} k D_k \boldsymbol{e} = \boldsymbol{\pi} \boldsymbol{d}, \tag{4}$$

where $\boldsymbol{d} = \sum k D_k \boldsymbol{e}$.

Intuitively, we think of D_0 as governing transitions in the phase process which do not generate arrivals and D_k as the rate of arrivals of size k (with the appropriate phase change). For other examples and further properties of the *BMAP* see [1].

A key quantity for analyzing the *BMAP*/*G*/1 queue is the matrix generating function

$$D(z) = \sum_{k=0}^{\infty} D_k z^k, \quad \text{for } |z| \le 1.$$

Let $P_{ij}(n,t) = P(\, N(t) = n,\, J(t) = j \mid N(0) = 0,\, J(0) = i\,)$ be the (i,j) element of a matrix $P(n,t)$. That is, $P(n,t)$ represents the probability of n arrivals in $(0,t]$ plus the phase transition. Then the matrix generating function $P^*(z,t)$ defined by

$$P^*(z,t) = \sum_{n=0}^{\infty} P(n,t) z^n, \quad \text{for } |z| \le 1,$$

is given explicitly by

$$P^*(z,t) = e^{D(z)t}, \qquad \text{for } |z| \le 1,\ t \ge 0, \tag{5}$$

where $e^{D(z)t}$ is an exponential matrix (see e.g., pg. 169 of Bellman, [18]). Note that for Poisson arrivals, $m = 1$, $D_0 = -\lambda$, $D_1 = \lambda$, and $D_k = 0$, $k \ge 2$, so that (5) reduces to $P^*(z,t) = e^{-\lambda(1-z)t}$ which is the

familiar generating function of the Poisson counting process.

2.2 The Queueing Model

Consider a single-server queue with a *BMAP* arrival process specified by the sequence $\{D_k,\ k \geq 0\}$. Let the service times be i.i.d. and independent of the arrival process; let the service time have an arbitrary distribution function H with Laplace-Stieltjes transform (*LST*) h and n^{th} moment α_n. We assume that the mean $\alpha \equiv \alpha_1$ is finite. Let the *traffic intensity*, $\rho \equiv \lambda\alpha$.

2.3 The Embedded Markov Renewal Process at Departures

The embedded Markov renewal process at departure epochs is defined as follows. Define $X(t)$ and $J(t)$ to be the number of customers in the system (including in service, if any) and the phase of the arrival process at time t, respectively. Let τ_k be the epoch of the k^{th} departure from the queue, with $\tau_0 = 0$. (We understand that the sample paths of these processes are right continuous and that there is a departure at $\tau_0 = 0$.) Then $(X(\tau_k),\ J(\tau_k),\ \tau_{k+1} - \tau_k)$ is a semi-Markov process on the state space $\{\ (i,j):\ i \geq 0,\ 1 \leq j \leq m\ \}$. The semi-Markov process is *positive recurrent* when $\rho < 1$. The transition probability matrix of the semi-Markov process is given by

$$Q(x) = \begin{bmatrix} \hat{B}_0(x) & \hat{B}_1(x) & \hat{B}_2(x) & \cdots \\ \hat{A}_0(x) & \hat{A}_1(x) & \hat{A}_2(x) & \cdots \\ 0 & \hat{A}_0(x) & \hat{A}_1(x) & \cdots \\ 0 & 0 & \hat{A}_0(x) & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}, \qquad x \geq 0, \tag{6}$$

where, for $n \geq 0$, $\hat{A}_n(x)$ and $\hat{B}_n(x)$ are the $m \times m$ matrices of mass functions with elements defined by

$[\hat{A}_n(x)]_{ij}$ = $P($ Given a departure at time 0, which left at least one customer in the system and the arrival process in phase i, the next departure occurs no later than time x with the arrival process in phase j, and during that service there were n arrivals),

$[\hat{B}_n(x)]_{ij}$ = $P($ Given a departure at time 0, which left the system empty and the arrival process in phase i, the next departure occurs no later than time x with the arrival process in phase j, leaving n customers in the system).

An embedded Markov renewal process with a transition probability matrix having the structure in (6) is called ''*M*/*G*/1-type'' (Neuts, [2]) since it has matrix generalizations of the skip-free-to-the-left and spatial homogeneity properties of the ordinary *M*/*G*/1 queue.

Following the treatment of the *M*/*G*/1 queue in §1.5 of [2] we define the Markov renewal function $M(t)$ whose elements $M_{(i,j),(k,l)}(t)$ are the expected number of visits to state (k,l) in $[0,t]$, starting in state (i,j) at time $t=0$. Let $U(t)$ be the infinite diagonal matrix where the diagonal entries are 0 for $t<0$ and 1 for $t\geq 0$. Then the matrix $M(t)$ satisfies

$$M(t) \;=\; U(t) \;+\; (Q*M)(t) \;=\; U(t) \;+\; (M*Q)(t), \qquad \text{for} \quad t\geq 0, \tag{7}$$

where $*$ denotes matrix convolution. For background on Markov renewal processes, see Chapter 10 of (Çinlar, [19]).

It is well known that $M(t)$ is given by the Neumann series

$$M(t) \;=\; \sum_{n=0}^{\infty} Q^{(n)}(t),$$

where $Q^{(n)}(t)$, $n\geq 1$, is the n-fold matrix convolution of $Q(\cdot)$ with itself and $Q^{(0)}(\cdot) \;=\; U(\cdot)$. If we partition $M(t)$ and $U(t)$ into $m\times m$ blocks then we see from (6) and (7) that the blocks satisfy

$$M_{i_0 i}(t) \;=\; \delta_{i_0 i} U_{ii}(t) \;+\; (M_{i_0 0} * \hat{B}_i)(t) \;+\; \sum_{n=1}^{i+1} (M_{i_0 n} * \hat{A}_{i-n+1})(t), \tag{8}$$

where $\delta_{ii}=1$, and $\delta_{ij}=0$ for $i\neq j$.

We introduce the transform matrices

$$A_n(s) \;=\; \int_0^{\infty} e^{-sx}\, d\hat{A}_n(x), \qquad B_n(s) \;=\; \int_0^{\infty} e^{-sx} d\hat{B}_n(x), \qquad m_{i_0 i}(s) \;=\; \int_0^{\infty} e^{-st} dM_{i_0 i}(t),$$

$$\tilde{A}(z,s) \;=\; \sum_{n=0}^{\infty} A_n(s) z^n, \qquad \tilde{B}(z,s) \;=\; \sum_{n=0}^{\infty} B_n(s) z^n, \qquad \tilde{m}_{i_0}(z,s) \;=\; \sum_{i=0}^{\infty} m_{i_0 i}(s) z^i, \tag{9}$$

where $\mathrm{Re}(s)\geq 0$ and $|z|\leq 1$. It was shown in (Lucantoni, [1]) that

$$\tilde{A}(z,s) \;=\; \int_0^\infty e^{-sx} e^{D(z)x} dH(x) \;\equiv\; h(sI - D(z)), \tag{10}$$

and

$$\tilde{B}(z,s) \;=\; z^{-1}[sI - D_0]^{-1}[D(z) - D_0]\tilde{A}(z,s). \tag{11}$$

The definition in (10) above is consistent with the usual definition of a scalar function evaluated at a matrix argument (see Theorem 2, pg. 113 of Gantmacher, [20]). In particular, since h is analytic in the right half-plane, the above function is defined by using the matrix argument in the power series expansion of h. This is well defined as long as the spectrum of the matrix argument also lies in the right half plane. Note that from (10) we see that $\tilde{A}(z,s)$ is a power series in $D(z)$. Thus, $\tilde{A}(z,s)$ and $D(z)$ commute. This property is used repeatedly in the proofs.

Using (8)-(11), we have

$$\begin{aligned} \tilde{m}_{i_0}(z,s)[zI - \tilde{A}(z,s)] \;&=\; z^{i_0+1} I \;+\; m_{i_0 0}(s)[z\tilde{B}(z,s) \;-\; \tilde{A}(z,s)], \\ &=\; z^{i_0+1} I \;+\; m_{i_0 0}(s)(sI - D_0)^{-1}[D(z) - sI]\tilde{A}(z,s), \end{aligned} \tag{12}$$

since

$$(sI - D_0)^{-1}[D(z) - D_0] \;-\; I \;=\; (sI - D_0)^{-1}[D(z) - D_0 - (sI - D_0)] \;=\; (sI - D_0)^{-1}[D(z) - sI].$$

2.4 The Busy Period

Following the general treatment of Markov chains of $M/G/1$-type in [2], we define $\hat{G}_{jj'}{}^{[r]}(x)$, $x \geq 0$, as the probability that the first passage from the state $(i+r,\, j)$ to the state (i, j'), $i \geq 1$, $1 \leq j,\, j' \leq m,\, r \geq 1$, occurs no later than time x, and that (i, j') is the first state visited in level i. The matrix with elements $\hat{G}_{jj'}{}^{[r]}(x)$ is $\hat{G}^{[r]}(x)$.

By a first passage argument, it was shown in (Neuts, [21]) that the transform matrix $G(s)$, defined by

$$G(s) \;=\; \int_0^\infty e^{-sx} d\hat{G}^{[1]}(x), \qquad \text{for} \;\; \mathrm{Re}(s) \geq 0,$$

satisfies the nonlinear matrix equation

$$G(s) = \sum_{n=0}^{\infty} A_n(s)\, G(s)^n. \tag{13}$$

In the context of the *BMAP/G/*1 queue, $G(s)$ governs the duration of the busy period. It was also shown in [21] that the transform matrix governing the duration of a busy period starting with r customers, is given by $G(s)^r$. Equation (13) is the key equation in the matrix analytic solution to queues of the *M/G/*1 type.

It was shown in (Lucantoni, [1]) that $G(s)$ is also the solution to

$$G(s) = \int_0^{\infty} e^{-sx} e^{D[G(s)]x} dH(x) \equiv h(sI - D[G(s)]), \tag{14}$$

where $D[G(s)] \equiv \sum_{k=0}^{\infty} D_k G(s)^k$. Equation (14) is the matrix analogue of the *Kendall functional equation,* (see (59) in (Kendall, [22]), and the discussion of I. J. Good on pg. 182 there). In particular, if $m = 1$ then the *BMAP* is a Poisson process with $D_0 = -\lambda$, $D_1 = \lambda$, and $D_k = 0$ for $k \geq 2$, so that (14) reduces to $G(s) = h(s + \lambda - \lambda G(s))$ which is (59) in [22].

The matrix $D[G]$, where $G \equiv G(0)$ has a nice probabilistic interpretation which was originally pointed out in (Lucantoni, Meier-Hellstern and Neuts, [5]). Since G is strictly positive, it follows that the off-diagonal entries of $D[G]$ are nonnegative. When the queue is stable, G is stochastic so that $D[G]\boldsymbol{e} = \mathbf{0}$; that is, $D[G]$ is the infinitesimal generator of a finite-state, irreducible Markov process. From the structure of the matrix we see that starting in some state i, there will be an exponential sojourn time with rate $|(D_0)_{ii}|$. Then there will either be a transition to state j, with rate $(D_0)_{ij}$ (i.e., without an arrival), or a transition to state j with rate $(\sum_{k=1}^{\infty} D_k G^k)_{ij}$. That is, a batch of size k arrives followed by k busy periods which end in phase j, corresponding to an instantaneous phase change from i to j in this process. It is clear that this process is the phase of the arrival process observed only during idle periods, i.e., the time during the busy periods are *excised.* In the unstable case, i.e., $\rho > 1$, G is strictly substochastic so that $D[G]$ is a stable matrix. In other words, in this case the total amount of idle time observed before the last busy period (that never ends) is *phase – type* with representation $(\boldsymbol{\alpha}, D[G])$, where $\boldsymbol{\alpha}$ is the vector of initial phase probabilities at time 0; (see, e.g., [6]).

The matrix G is the key ingredient in the solution of the stationary version of this system. An efficient algorithm for computing this matrix based on uniformization is given in [1]. For the transient solution, we need to compute the matrix $G(s)$ for complex s. It is shown in (Choudhury, Lucantoni and Whitt, [23]) that $G(s)$ may be computed by iterating in (14). Convergence is guaranteed if the iteration is started with either $G_0 = 0$ or $G_0 = G$ and, in fact, if both of these iterations are carried out, then by stopping the iteration at any point the matrices obtained correspond to the transforms of distributions which bound the true distribution. This extends results for the *M/G/*1 queue in (Abate and Whitt, [24]).

In order to compute the right hand side of (14) in each iteration, two cases are considered in [23]. If the service-time distribution has a rational Laplace transform (e.g., phase-type or other distributions in the Coxian family), then the right hand side may be computed exactly with one matrix inversion and a few matrix multiplications. If the service time distribution is not rational, then a procedure similar to uniformization is used.

2.5 Simplifications for the $M^X/G/1$ Queue

We end this and the next three sections by displaying the main results for the special case in which there are batch Poisson arrivals. In this case all the matrix equations reduce to scalar equations. In particular, if the arrival rate of batches is δ and the batch-size probability mass function is $\{\gamma_n,\ n \geq 1\}$, with probability generating function $\gamma(z)$ and mean $\overline{\gamma}$, then $\lambda = \delta\overline{\gamma}$ and $D(z) = -\delta + \delta\gamma(z)$. Therefore, from (14), we have

$$G(s) = h(s+\delta-\delta\gamma(G(s))), \qquad \mathrm{Re}(s) \geq 0. \tag{15}$$

If the batch size distribution is identically equal to 1 then $\gamma(z) = z$, $\lambda = \delta$, and these results agree with those in [10] for the ordinary $M/G/1$ queue.

3. Preliminary Results

3.1 The Emptiness Functions

In this section we characterize the probability that the system is empty at time t. The key role of this function for general systems was demonstrated by (Benes, [12]). We distinguish several cases depending on what information is available at $t=0$. In particular, we consider starting with an empty system; starting with a fixed number of customers, i_0, where $t=0$ is an epoch of departure; starting with a fixed amount of work x; and starting with an amount of work which is distributed according to an arbitrary distribution F.

Let $V(t)$ be the amount of work in the system at time t; let

$$P_{x0}^{ij}(t) = P(\ V(t)=0,\ J(t)=j \mid V(0)=x,\ J(0)=i\ );$$

and let the $m \times m$ matrix $P_{x0}(t)$ have (i,j)-entry $P_{x0}^{ij}(t)$. Also, let $p_{x0}(s) = \int_0^\infty e^{-st} P_{x0}(t)\, dt$, for $\mathrm{Re}(s) > 0$. Then we have the following generalization of the $M/G/1$ formula. (See (9) on pg. 52 of [10] and (34) and (36) in [11]).

Theorem 1: The matrix $p_{x0}(s)$ is given by

$$p_{x0}(s) = e^{-(sI-D[G(s)])x}(sI-D[G(s)])^{-1}, \quad \text{for } \mathrm{Re}(s) > 0. \tag{16}$$

Note that the exponential disappears when $x=0$. Since the components of the vector $G(s)\boldsymbol{e}$ are Laplace-Stieltjes transforms and $|G(s)\boldsymbol{e}| < 1$, for $\mathrm{Re}(s) > 0$, the eigenvalues of $D[G(s)]$ are in the left half-plane. Therefore, for $\mathrm{Re}(s) > 0$, the eigenvalues of $sI-D[G(s)]$ are in the right half-plane and the inverse appearing in (16) is well defined.

Let $\hat{P}_{i_0 0}(t)$ be the $m\times m$ matrix with (j,k) entry

$$\hat{P}^{jk}_{i_0 0}(t) = P(\, V(t)=0,\, J(t)=j \mid X(0)=i_0,\, J(0)=j\,,\, \tau_0=0\,).$$

As a consequence of Theorem 1, we immediately have

$$\hat{p}_{i_0 0}(s) \equiv \int_0^\infty e^{-st}\hat{P}_{i_0 0}(t)\,dt = G(s)^{i_0} p_{00}(s) = G(s)^{i_0}(sI-D[G(s)])^{-1}. \tag{17}$$

For later use, we note that, by conditioning on the last departure before time *t*, we can write

$$\hat{P}_{i_0 0}(t) = \int_0^t dM_{i_0 0}(u)\, e^{D_0(t-u)}.$$

Taking Laplace transforms leads to

$$\hat{p}_{i_0 0}(s) = m_{i_0 0}(s)(sI-D_0)^{-1.} \tag{18}$$

The unconditional emptiness function, starting with initial workload distributed according to cdf *F*, defined by

$$P_0(t) \equiv \int_0^\infty P_{x0}(t)\,dF(x), \tag{19}$$

has Laplace transform

$$p_0(s) \equiv \int_0^\infty e^{-st} P_0(t)\,dt = f(sI - D[G(s)])(sI - D[G(s)])^{-1}, \tag{20}$$

where f is the *LST* of F.

We now apply (20) to derive known steady state results. Recall that $\lim_{t\to\infty} P_0(t) = \lim_{s\to 0} sp_0(s)$. Let $R(s) = sp_0(s)$. Multiplying both sides of (20) by $s(sI - D[G(s)])$, we have

$$sR(s) - R(s)D[G(s)] = sR(s) - D[G(s)]R(s) = sf(sI - D[G(s)]), \tag{21}$$

since $(sI - D[G(s)])$ commutes with $f(sI - D[G(s)])$ as seen by expanding f in a power series. Letting $s \to 0$ in (21), we have $R(0)D[G] = D[G]R(0) = 0$. Therefore the columns of $R(0)$ are right eigenvectors of $D[G]$ corresponding to the eigenvalue 0. Similarly, the rows of $R(0)$ are left eigenvectors of $D[G]$ corresponding to the eigenvalue 0. Since $D[G]$ is the infinitesimal generator of an irreducible, finite state Markov process, its left and right eigenvectors corresponding to the eigenvalue 0 are unique up to a scalar constant and are proportional to $\boldsymbol{g}$ and $\boldsymbol{e}$, respectively, by the Perron-Frobenious theorem; see e.g., Theorem 2, pg. 53 of Gantmacher, [25]), That is, $R(0) = c\boldsymbol{eg}$, for some constant c. However, since we know that for any $G/G/1$ queue with $\rho \le 1$ the stationary probability that the system is empty is $1 - \rho$, we have

$$\lim_{t\to\infty} P_0(t) = \begin{cases} (1-\rho)\boldsymbol{eg} & \text{for } \rho \le 1, \\ 0 & \text{for } \rho > 1. \end{cases} \tag{22}$$

3.2 Simplifications for the $M^X/G/1$ Queue

For the $M^X/G/1$ queue with the notation in §2.5, we have

$$p_{x0}(s) = \frac{e^{-(s+\delta-\delta\gamma(G(s)))x}}{s+\delta-\delta\gamma(G(s))}, \qquad m_{x0}(s) = (s+\lambda)p_{x0}(s), \tag{23}$$

$$p_0(s) = \frac{f(s+\delta-\delta\gamma(G(s)))}{s+\delta-\delta\gamma(G(s))}, \qquad \hat{p}_{i_0 0}(s) = \frac{G(s)^{i_0}}{s+\delta-\delta\gamma(G(s))}, \quad \text{for } \mathrm{Re}(s)>0. \tag{24}$$

4. The Workload

4.1 The Transient Results

In this section we derive the transform of the workload (work in the system in uncompleted service time) at time *t*. We accomplish this in two steps. First, we assume a departure at time $t=0$ and derive the distribution of the work in the system at some fixed time *t*, conditioned on the number of customers left in the system after that departure. Using this result, we derive the more general distribution of the work in the system at time *t*, conditioned on the amount of work at time $t=0$, where this is not necessarily an epoch of departure. Although the second result is more general, from a practical viewpoint the first might be more useful. In particular, in a real system it might be easier to measure the number of customers, packets, etc., at departure times than to know the exact amount of work in the system.

Define the $m\times m$ matrix $W_{i_0}(t,x)$, whose (i,j) entry is

$$\left[W_{i_0}(t,x) \right]_{ij} = P(V(t)\le x,\ J(t)=j \mid X(0)=i_0, J(0)=i, \tau_0=0);$$

i.e., $W_{i_0}(t,x)$ is the conditional delay distribution at time *t* given the number of customers in the system following the departure at time $t=0$. Let the transform matrices be

$$w_{i_0}(t,s) = \int_0^\infty e^{-sx}\, d_x W_{i_0}(t,x), \qquad \text{and} \qquad \tilde{w}_{i_0}(\xi,s) = \int_0^\infty e^{-\xi t} w_{i_0}(t,s)\, dt,$$

where Re $(s)\ge 0$ and Re $(\xi)>0$. In the following theorem, the inverse need not exist for all argument pairs (ξ,s); at these points the left side is defined by continuity.

Theorem 2: The matrix $\tilde{w}_{i_0}(\xi,s)$ is given explicitly by

$$\tilde{w}_{i_0}(\xi,s) = (h(s)^{i_0} I - s\hat{p}_{i_0 0}(\xi))[\xi I - sI - D(h(s))]^{-1}, \tag{25}$$

and the matrix $w_{i_0}(t,s)$ is given by

$$w_{i_0}(t,s) = \left[h(s)^{i_0} I - s\int_0^t \hat{P}_{i_0 0}(u)\, e^{-(sI+D(h(s)))u}\, du \right] e^{(sI+D(h(s)))t}, \tag{26}$$

where Re $(s) \geq 0$, Re $(\xi) > 0$, and $\hat{P}_{i_0 0}(u)$ is defined above (17).

Although we are able to express the transform of the delay explicitly in terms of t in (26), we note that this expression is not trivial to evaluate numerically. It involves numerically inverting a Laplace transform where the evaluation of the transform at a value of s requires the numerical integration of the emptiness function times an exponential matrix where the values of the emptiness function are themselves obtained by inverting a Laplace transform. The corresponding expression for the ordinary $M/G/1$ queue also suffers from the same difficulty. This may partly explain why the known formulas for that case have not been widely used for practical computations.

In contrast, however, the transform expression in (25) is relatively simple to evaluate, so that with an inversion algorithm for 2-dimensional Laplace transforms, we have a practical method for obtaining numerical results. We describe such an algorithm in §6.

It can be shown using Rouché's theorem that for each s, $\mathrm{Re}(s) \geq 0$, the determinant of the matrix $X(s,\xi) \equiv [\xi I - sI - D(h(s))]$ appearing in the inverse in (25) has exactly m roots in the region $\mathrm{Re}(\xi) > 0$. (For similar arguments see (Çinlar, [13]) and (Neuts, [26] [27]).) Since $\tilde{w}_{i_0}$ is a transform and is therefore analytic in the interior of the above region, see p.26 of (Deutsch, [28]), these pairs of (ξ, s) must also be zeros of the first matrix on the right in (25). That is, they are removable singularities. The classical approach to this type of problem would then assume that the roots are distinct to obtain m independent linear equations for the rows of the matrix on the left. In practice, the roots may not be distinct, or if they are close, there may be numerical difficulties in locating these roots. These technical problems are circumvented in the present case since we derived explicit results for the matrices in (25).

As a consequence of Theorem 2, we can easily treat the workload at time t given that a departure occurred at time 0 and a random number of customers are present. Let $\tilde{w}^*(\xi,s)$, be the double Laplace transform

$$\tilde{w}^*(\xi,s) = \sum_{i=0}^{\infty} \phi_i \tilde{w}_i(\xi,s),$$

where $\{\phi_i\}$ is the probability mass function of the number of customers in the system after the departure at $t = 0$. If $\Phi(z)$ is the probability generating function of $\{\phi_i\}$, then

$$\tilde{w}^{*}(\xi,s) = \left[\Phi(h(s))I - s\Phi[G(\xi)]\left[\xi I - D[G(\xi)]\right]^{-1}\right][\xi I - sI - D(h(s))]^{-1}, \qquad (27)$$

where Re $(s) \geq 0$ and Re $(\xi) > 0$.

Let F be the cdf of the initial work at time 0 (where $t = 0$ need not be an epoch of departure) and let f be its Laplace-Stieltjes transform. Let $W(t,x)$ be the matrix whose $(i,j)^{\text{th}}$ element is the probability that the work in the system is less than x and the phase is j at time t, given that at time 0 the phase was i and the initial workload (including the customer in service, if any) was distributed according to F. Let $w(t,s)$ and $\tilde{w}(\xi,s)$ be the Laplace transforms

$$w(t,s) = \int_0^\infty e^{-sx} d_x W(t,x) \qquad \text{and} \qquad \tilde{w}(\xi,s) = \int_0^\infty e^{-\xi t} w(t,s)\, dt.$$

Then we have the following theorem.

Theorem 3: The Laplace transform $\tilde{w}(\xi,s)$ is given by

$$\tilde{w}(\xi,s) = (f(s)I - sp_0(\xi))[\xi I - sI - D(h(s))]^{-1}, \qquad (28)$$

and

$$w(t,s) = \left[f(s)I - s\int_0^t P_0(u)\, e^{-[sI + D(h(s))]u}\, du\right] e^{[sI + D(h(s))]t}, \qquad (29)$$

for Re$(s) \geq 0$), Re$(\xi) > 0$, where $P_0(u)$ and $p_0(\xi)$ are given in (19) and (20), respectively.

Note that Theorem 2 is a special case of Theorem 3 where $f(s) = h(s)^{i_0}$. However, (28) is more suitable for numerical inversion by the algorithm presented in §6. Note also, that (29) is the direct analogue of Equation (8) on pg. 51 of [10].

4.2 The Limiting Distribution of the Waiting Time

Differentiating with respect to t in (29), we have

$$\frac{\partial}{\partial t} w(t,s) \;=\; w(t,s)[sI + D(h(s))] \;-\; sP_0(t).$$

Therefore, using (22) and assuming that the partial derivative approaches 0 as $t \to \infty$, we see that the transform of the limiting distribution of the workload is given by

$$w(s) \;\equiv\; \lim_{t\to\infty} w(t,s) \;=\; \begin{cases} s(1-\rho)\,\boldsymbol{eg}[sI + D(h(s))]^{-1}, & \text{for } \rho < 1, \\ 0, & \text{for } \rho \geq 1, \end{cases}$$

which agrees with (44) in [1]. Hence, by [1], the partial derivative does indeed approach 0 as $t \to \infty$.

4.3 The First Moment Function

Let the first moment function be the $m \times m$ matrix

$$m_1(t) \;\equiv\; -\frac{\partial}{\partial s} w(t,s)\bigg|_{s=0},$$

where the (i,j) component is $E[\; V(t)\; I_{\{J(t)=j\}} \;|\; J(0)=i \;]$ with I_A being the indicator function of the set A. Let $\beta \equiv -f'(0)$ be the expected work in the system at time $t=0$, and let $D^{(1)} \;=\; \sum_{k=1}^{\infty} kD_k$. Recall from (4) that $\boldsymbol{d} \;=\; D^{(1)}\,\boldsymbol{e}$. Then we have the following theorem.

Theorem 4: Assume that $\alpha < \infty$ and $\beta < \infty$. Then the matrix $m_1(t)$ is explicitly given by

$$m_1(t) \;=\; \alpha \int_0^t e^{Du} D^{(1)} e^{D(t-u)}\, du \;+\; (\beta - t)e^{Dt} \;+\; \int_0^t P_0(u)\, e^{D(t-u)}\, du. \tag{30}$$

Equivalently, $m_1(t)$ satisfies the following differential equation

$$m_1'(t) \;=\; \alpha e^{Dt} D^{(1)} \;-\; e^{Dt} \;+\; P_0(t) \;+\; m_1(t)D, \qquad m_1(0) \;=\; \beta I. \tag{31}$$

The row sums of $m_1(t)$, i.e., $m_1(t)\,\boldsymbol{e}$, satisfy the differential equations

$$m_1'(t)\boldsymbol{e} = \alpha e^{Dt}\boldsymbol{d} - \boldsymbol{e} + P_0(t)\boldsymbol{e}, \qquad m_1(0)\boldsymbol{e} = \beta\boldsymbol{e}. \tag{32}$$

Note that the expression for $m_1(1)$ in (30) is more complex than the corresponding $M/G/1$ case since the matrices D and $D^{(1)}$ do not commute in general. Assuming that $m_1'(t) \to 0$ as $t \to \infty$, if we solve for $m_1(t)$ in (31) and let $t \to \infty$ we obtain expression (47) of [1] for the mean workload in the stationary $BMAP/G/1$ queue. Note that (31) is the matrix analogue of Equation (20) on pg. 55 of [10]. In the batch-Poisson case, the matrices become scalars and $D = 0$, so that the last term in (31) does not appear.

If we pick the initial phase of the arrival process at time $t = 0$ according to the stationary distribution $\boldsymbol{\pi}$, then we have

$$\boldsymbol{\pi} m_1'(t)\boldsymbol{e} = \rho - 1 + \boldsymbol{\pi} P_0(t)\boldsymbol{e}, \tag{33}$$

which is a generalization of Equation (17) in (Abate and Whitt, [11]). See (Miyazawa, [29]) for more results related to (33).

4.4 Higher-Order Moment Functions

Along the lines of (Abate and Whitt, [11]), we can derive differential equations for the higher order moments of the delay at time t. In particular, let $V(s) = D(h(s))$ and let the i^{th} derivatives be $V^{(i)} = (-1)^i V^{(i)}(0)$, and $D^{(i)} = D^{(i)}(1)$ for $i \geq 1$. Then, by successively differentiating $V(s)$, we get

$$V^{(1)} = \alpha D^{(1)},$$

$$V^{(2)} = \alpha^2 D^{(2)} + \alpha_2 D^{(1)},$$

$$V^{(3)} = \alpha^3 D^{(3)} + 3\alpha\alpha_2 D^{(2)} + \alpha_3 D^{(1)},$$

etc. The expression for the n^{th} moment can be obtained by Fáa di Bruno's formula for the n^{th} derivative of a composite function, e.g., see p.36 of (Riordan, [30]), Ch.5 of (Riordan, [31]), and (Klimko and Neuts, [32]). Let the k^{th} moment function be defined by

$$m_k(t) \equiv (-1)^k \frac{\partial^k}{\partial s^k} W(t,s)\bigg|_{s=0}, \tag{34}$$

and let β_k be the k^{th} moment of the workload at time 0.

Theorem 5: If $\alpha_k < \infty$ and $\beta_k < \infty$, then the k^{th} moment function in (34) can be expressed as

$$m_k(t) = -k\int_0^t m_{k-1}(u)\,e^{D(t-u)}\,du + \sum_{j=0}^{k-1}\begin{bmatrix}k\\ j\end{bmatrix}\int_0^t m_j(u)\,V^{(k-j)}\,e^{D(t-u)}\,du + \beta_k e^{Dt}. \tag{35}$$

Equivalently, $m_k(t)$ satisfies the system of differential equations

$$m_k'(t) = -km_{k-1}(t) + \sum_{j=0}^{k-1}\begin{bmatrix}k\\ j\end{bmatrix} m_j(t)\,V^{(k-j)} + m_k(t)D, \qquad m_k(0) = \beta_k I. \tag{36}$$

Once again, simpler equations result if we are only interested in the marginal moments, i.e., the row sums of $m_k(t)$. Equation (36) is the matrix analogue of (19) in [11]. Also, assuming that $m_k'(t) \to 0$ as $t\to\infty$, if we solve (36) for $m_k(t)$ and let $t\to\infty$, we obtain expressions (A.1.3) and (A.1.4) in (Lucantoni and Neuts, [33]) for the n^{th} moments of the workload in the stationary version of the *BMAP/G/*1 queue.

4.5 Simplifications for the $M^X/G/1$ Queue

For the $M^X/G/1$ queue, with the notation in §2.5, we have

$$\tilde{w}_{i_0}(\xi,s) = \frac{h(s)^{i_0} - s\hat{p}_{i_0 0}(\xi)}{\xi - s + \delta - \delta\gamma(h(s))}, \tag{37}$$

$$w_{i_0}(t,s) = e^{(s-\delta+\delta\gamma(h(s)))t}\left[h(s)^{i_0} - s\int_0^t \hat{P}_{i_0 0}(u)\,e^{-(s-\delta+\delta\gamma(h(s)))u}du\right], \tag{38}$$

$$\tilde{w}(\xi,s) = \frac{f(s) - sp_0(\xi)}{\xi - s + \delta - \delta\gamma(h(s))}, \tag{39}$$

$$w(t,s) = e^{(s-\delta+\delta\gamma(h(s)))t}\left[f(s) - s\int_0^t P_0(u)\,e^{-(s-\delta+\delta\gamma(h(s)))u}du\right], \tag{40}$$

where $\mathrm{Re}(s)\geq 0$, $\mathrm{Re}(\xi) > 0$ and $\hat{p}_{i_0 0}(\xi)$ and $p_0(\xi)$ are given in (17) and (20), respectively. Note that (39) and (40) generalize (15) on p.53 and (8) on p.51 of [10] to batch arrivals, respectively.

5. The Queue Length

5.1 The Transient Results

Let $Y_{i_0 i}^{jk}(t) = P(\, X(t)=i,\, J(t)=k \mid X(0)=i_0,\, J(0)=j,\, \tau_0=0\,)$, and let $Y_{i_0 i}(t)$ have (j,k)-entry $Y_{i_0 i}^{jk}(t)$. Recall that $\tau_0 = 0$ means that there is a departure at time 0. Then clearly,

$$Y_{i_0 0}(t) = W_{i_0}(t,0) = \int_0^\infty dM_{i_0 0}(u)\, e^{D_0(t-u)},$$

by conditioning on the last departure before time t. Let $y_{i_0 i}(s)$ be the Laplace transform of $Y_{i_0 i}(t)$. Then $y_{i_0 0}(s) = G(s)^{i_0} p_{00}(s) = p_{i_0 0}(s)$. The probability generating function of the queue length at time t is defined by

$$\tilde{y}_{i_0}(z,s) \equiv \sum_{i=0}^{\infty} y_{i_0 i}(s) z^i.$$

Theorem 6: The matrix $\tilde{y}_{i_0}(z,s)$ is given by

$$\tilde{y}_{i_0}(z,s) = \left[z^{i_0+1}(I-\tilde{A}(z,s))(sI-D(z))^{-1} + (z-1)\hat{p}_{i_0 0}(s)\tilde{A}(z,s) \right] [zI-\tilde{A}(z,s)]^{-1}, \qquad (41)$$

for $\mathrm{Re}(s) > 0$ and $|z| < 1$, where $\hat{p}_{i_0 0}(s)$ is given in (17) and $\tilde{A}(z,s)$ is given in (10).

Equation (41) is the matrix analogue of Equation (77) on pg. 74 in [10]. Let the transform of the complementary queue length distribution be defined by

$$y_{i_0 i}^*(s) = \int_0^\infty e^{-st} \sum_{n=i+1}^{\infty} Y_{i_0 n}(t)\, dt,$$

with the corresponding generating function

$$\tilde{y}_{i_0}^*(z,s) \equiv \sum_{i=0}^{\infty} y_{i_0 i}^*(s) z^i.$$

Then since $\tilde{y}_{i_0}(1,s) = (sI-D)^{-1}$, we have the following corollary.

Corollary: The transform of the complementary queue length distribution, $\tilde{y}^*_{i_0}(z,s)$, is given by

$$\tilde{y}^*_{i_0}(z,s) = \frac{1}{1-z}\left[(sI-D)^{-1} - \tilde{y}_{i_0}(z,s)\right].$$

5.2 Simplifications for the $M^X/G/1$ Queue

For the $M^X/G/1$ queue, with the notation in §2.5, we have

$$\tilde{y}_{i_0}(z,s) = \frac{z^{i_0+1}(1-\tilde{A}(z,s))}{(s+\delta-\delta\gamma(z))(z-\tilde{A}(z,s))} + \frac{(z-1)\hat{p}_{i_0 0}(s)\tilde{A}(z,s)}{z - \tilde{A}(z,s)}, \tag{42}$$

where $\mathrm{Re}(s)>0$, $|z|<1$ and $\tilde{A}(z,s) = h(s+\delta-\delta\gamma(z))$, and $\hat{p}_{i_0 0}(s)$ is given in Equation (24).

6. An Algorithm for Inverting Two-Dimensional Laplace Transforms

Recently, we have developed effective algorithms for multi-dimensional transform inversions (Choudhury, Lucantoni, Whitt, [15]). The algorithms are based on the multi-dimensional Poisson summation formula (of continuous, discrete, and mixed variety, see e.g., (5.47) of [16] for the continuous variety), and are generalizations of the *EULER* and *Lattice – Poisson* algorithms presented in [16]. We briefly describe the algorithms used here and refer to [15] for further discussion.

Let $F(t_1,t_2)$ represent a function of two non-negative real variables with Laplace transform

$$f(s_1,s_2) = \int_0^\infty\int_0^\infty F(t_1,t_2)e^{-(s_1t_1+s_2t_2)}\,dt_1\,dt_2. \tag{43}$$

The values of F may be obtained by inverting the transform via

$$F(t_1,t_2) = \frac{e^{(A_1+A_2)/2}}{2t_1t_2}\left[\mathrm{Re}\left[f(\delta_1,\delta_2)\right] + \sum_{j=0}^{\infty}(-1)^j\sum_{k=1}^{\infty}(-1)^k\mathrm{Re}\left[f(\delta_1-j\gamma_1,\delta_2-k\gamma_2)\right]\right.$$

$$\left. + \sum_{j=1}^{\infty}(-1)^j\sum_{k=-\infty}^{0}(-1)^k\mathrm{Re}\left[f(\delta_1-j\gamma_1,\delta_2-k\gamma_2)\right]\right] - e_d, \qquad (44)$$

where e_d is the discretization error, $i\equiv\sqrt{-1}$, and for j=1 and 2, A_j is a constant (discussed below), $\delta_j=A_j/2t_j$, and $\gamma_j=\pi i/t_j$. When $|F(t_1,t_2)| \le 1$, for all t_1,t_2, as when $F(t_1,t_2)$ represents a probability, the discretization error term e_d is bounded as follows:

$$|e_d| \le \frac{e^{-A_1} + e^{-A_2} - e^{-(A_1+A_2)}}{(1-e^{-A_1})(1-e^{-A_2})} \approx e^{-A_1} + e^{-A_2}. \qquad (45)$$

The constants A_1 and A_2 are chosen appropriately to control the error term. For example, choosing $A_1=A_2=19.1$ ensures that $|e_d|\le 10^{-8}$. With double precision arithmetic we do not aim much lower than this in order to avoid introducing significant roundoff errors. With higher precision arithmetic, we can aim for much lower discretization error.

Now let $F(t,n)$ be a function of two nonnegative variables where *t* is continuous and *n* is an integer. Its two-dimensional Laplace-*z* transform is defined by

$$f(s,z) = \sum_{n=0}^{\infty}\left[\int_0^{\infty}F(t,n)\,e^{-st}dt\right]z^n.$$

The inversion formula is given by

$$F(t,n) = \frac{e^{A/2}}{tmr^n}\left[\mathrm{Re}\left[f(\delta,r)\right] + (-1)^n\mathrm{Re}\left[f(\delta,-r)\right] + \sum_{j=0}^{\infty}(-1)^j\sum_{k=1}^{m/2-1}\mathrm{Re}\left[f(\delta-j\gamma,r\omega^k)\,\omega^{-kn}\right]\right.$$

$$\left. + \sum_{j=1}^{\infty}(-1)^j\sum_{k=-m/2}^{0}\mathrm{Re}\left[f(\delta-j\gamma,r\omega^k)\,\omega^{-kn}\right]\right] - \tilde{e}_d, \qquad (46)$$

where *m* is any even number bigger than or equal to *n* (we typically choose $m=2n$ or $m=4n$), *A* and *r* are chosen to control the discretization error term $\tilde{e}_d$, $\delta=A/2t$, $\gamma=\pi i/t$, and $\omega=e^{2\pi i/m}$. If $|F(t,n)| \le 1$, for

all t,n, as when $F(t,n)|$represents a probability, then we have

$$|\tilde{e}_d| \le \frac{e^{-A}+r^m-e^{-A}r^m}{(1-e^{-A})(1-r^m)} \approx e^{-A} + r^m. \tag{47}$$

For example, choosing $A = 19.1$ and $r = (10^{-8}/2)^{1/m}$ ensures that $|\tilde{e}_d| \le 10^{-8}$.

Equations (44) and (46) contain double infinite sums and single infinite sums, respectively. Straightforward computation of those sums by truncation may in general require the computation of a large number of terms. However, since each infinite sum is nearly an alternating series, the sums are efficiently computed via the Euler summation technique using finite differences; see §6 of [16]. In particular, (see pg. 230 of Davis and Rabinowitz, [34]), we have

$$\sum_{i=0}^{\infty}(-1)^i u_i = \sum_{i=0}^{n-1}(-1)^i u_i + (-1)^n\left[\tfrac{1}{2}u_n - \tfrac{1}{4}\Delta u_n + \tfrac{1}{8}\Delta^2 u_n - \cdots\right], \tag{48}$$

where $\Delta u_n = u_{n+1}-u_n$, $\Delta^2 = \Delta(\Delta u_n) = u_{n+2}-2u_{n+1}+u_n$, etc. In many cases, the series on the right hand side of (48) converges much more rapidly than the series on the left. Our experience shows that each infinite sum may be computed accurately by evaluating only about 50 terms for most cases of interest.

7. Numerical Results

In this section, we demonstrate the computability of our results. We consider a *BMAP* which is a superposition of four independent and identical *MMPP*s. Each *MMPP* alternates between a high-rate and a low-rate state where the ratio of the arrival rates in the two states is 4:1. The durations of each state are such that there is an average of four arrivals during the sojourns in each state. The individual arrival rates are scaled appropriately to achieve the desired traffic intensity, ρ. The auxiliary phase in the overall *BMAP* can be characterized by the number of individual *MMPP*'s that are in the high-rate state. Let j_0 be the initial number. The service time distribution is assumed to be Erlang of order 16, E_{16}, with unit mean so that the time units are in mean service times. The squared coefficient of variation of this service-time distribution is 1/16.

Figures 1-5 show several transient workload and queue length distributions on log scales. In each case the stationary distribution is shown by a solid line and the transient distributions are shown by dashed or dotted lines. In all figures except Figure 3, we assume that $j_0 = 2$, i.e., at time 0, two sources are in the high state and two are in the low state.

Figure 1 shows the transient workload tail probabilities (i.e., the transient complementary cdf of the workload) at time $t=10$ with different initial queue lengths and $\rho=0.7$. We have summed over all auxiliary phase states at time $t=10$, so that we obtain a one-dimensional distribution. In particular, we display $P(\ V(10)>x \mid X(0)=i_0,\ J(0)=j_0\ )$ for the designated initial phase state $j_0=2$ corresponding to two of the four *MMPP*'s starting in the high-rate state. We consider four different initial queue lengths: i_0 = 0, 2, 8 and 32. It is interesting to note that for small x the transient complementary cdf may be higher or lower than the corresponding stationary values, depending on the initial conditions, but for larger x, the transient results alway decay faster then the stationary distribution. We elaborate on this point in [15].

Figure 2 shows how the transient workload distribution approaches the stationary distribution as t increases. In particular, Figure 2 displays the workload tail probabilities $P(\ V(10)>x \mid X(0)=i_0,\ J(0)=j_0\ )$ as a function of t for two values of i_0, i_0 = 0 and i_0 = 32, with j_0 = 2. Note that the convergence to steady-state clearly depends on the initial queue length.

The transient behavior also depends on the *BMAP* as is shown in Figure 3. Here we show the transient distribution for a fixed time $t=10$ and a fixed initial queue length of 2. We vary the number of sources in the high-rate state, considering the cases of 0, 2 and 4.

The transient distributions are proper for $\rho\geq 1$, as well. This is demonstrated in Figure 4 where the workload tail probabilities are displayed for several values of t when $\rho=2.0$. For each case in this example, i_0 = j_0 = 2, i.e., the initial queue length is two and two *MMPP*'s start out in the high-rate state. As $t\to\infty$, $V(t)\to\infty$ w.p. 1, so that $V(t)\to V(\infty)$, where $V(\infty)$ has the degenerate distribution $P(V(\infty)>x)$ = 1 for all x, as is shown by the solid line. As expected, the transient distributions approach the steady-state behavior as t increases, but note however, that if the overload is limited in duration, the system performance might well be acceptable. In particular, we believe that transient solutions can shed light on the problem of overload controls.

Finally, in Figure 5, we plot the transient queue-length probability mass function with an initial queue length of 32. We note that, as expected, as t increases, the initial distribution (concentrated at a point mass at 32) gradually spreads out to approach the stationary distribution. Note the striking qualitative differences between the stationary distribution and the transient results for moderate values of t. This is further indication that predictions of system performance based on stationary analysis could be very far from what is observed during the short run.

8. Proofs

In several of the following proofs, multiple interchanges of integrals are required. In all cases the integrands are either probabilities, generating functions or Laplace transforms so that the interchanges are justified by the Bounded Convergence Theorem (see, e.g., p.81 of (Royden, [35])).

Proof of Theorem 1

We know from Lemma 2 in (Lucantoni and Neuts, [36]) that the the Laplace transform of the time required for the system to empty given an initial workload of *x*, and keeping track of the phase change, is given by $e^{-(sI-D[G(s)])x}$. Therefore,

$$p_{x0}(s) = e^{-(sI-D[G(s)])x} p_{00}(s).$$

Now, if we condition on the first arrival before *t* (if any), we get

$$P_{00}(t) = e^{D_0 t} + \int_0^t e^{D_0 u} \sum_{k=1}^{\infty} D_k \, du \int_0^{t-u} dG^{(k)}(v) P_{00}(t-u-v). \tag{49}$$

The first term corresponds to the case where there are no arrivals before *t*. The second term corresponds to the case where there is a batch arrival of size *k* at time *u* and the system next empties out *v* time units later (at time $u+v$). Taking Laplace transforms, exploiting the convolution in (49) and letting $y=t-u$, we obtain

$$p_{00}(s) = (sI-D_0)^{-1} + \int_0^{\infty} e^{-su} e^{D_0 u} du \int_0^{\infty} e^{-sy} dy \sum_{k=1}^{\infty} D_k \int_0^{y} dG^{(k)}(v) P_{00}(y-v)$$

$$= (sI-D_0)^{-1} + (sI-D_0)^{-1} \left[D[G(s)] - D_0 \right] p_{00}(s).$$

Rearranging the terms gives $p_{00}(s) = (sI-D[G(s)])^{-1}$ which combined with (49) gives (16).

Proof of Theorem 2

We first prove the following lemma.

Lemma 1: The following integral is explicitly evaluated as

$$\int_0^\infty e^{-(\xi I - D(h(s)))y} dy \int_0^\infty e^{-sw} d_w H(y+w)[\xi I - sI - D(h(s))] \;=\; h(s)I \;-\; \tilde{A}(h(s),\xi). \tag{50}$$

Proof of Lemma 1: Using the change of variable, $v = y + w$, we have

$$\int_0^\infty e^{-(\xi I - D(h(s)))y} dy \int_0^\infty e^{-sw} d_w H(y+w)[\xi I - sI - D(h(s))]$$

$$= \int_0^\infty e^{-vs} dH(v) \int_0^v e^{-(\xi I - sI - D(h(s)))y} dy [\xi I - sI - D(h(s))]$$

$$= [h(s)I - h(\xi I - D(h(s)))]$$

which, with (10), proves the result. ■

We now prove Theorem 2. First note that the mass at the origin, $W_{i_0}(t,0) = \hat{P}_{i_0 0}(t)$, with Laplace transform $m_{i_0 0}(\xi)(\xi I - D_0)^{-1}$ (from (18)). Now, by conditioning on the last departure before time t, we can write

$$W_{i_0}(t,x) \;-\; W_{i_0}(t,0)$$

$$= \sum_{i=0}^\infty \int_0^t dM_{i_0 0}(u) \int_0^{t-u} e^{D_0 v} \sum_{k=1}^\infty D_k dv \int_0^x P(i, t-u-v)\, d_w H(t+w-u-v)\, H^{(i+k-1)}(x-w)$$

$$+ \sum_{i=1}^\infty \sum_{k=1}^i \int_0^t dM_{i_0 k}(u) \int_0^x P(i-k, t-u)\, d_w H(t+w-u)\, H^{(i-1)}(x-w), \tag{51}$$

where $H^{(i)}$ is the i-fold convolution of H with itself. The first term corresponds to the case where the last departure occurs at time u and leaves the system empty; there is a batch arrival of size k at time $u+v$; the service time of the first customer lasts until time $t+w$; there are i additional arrivals between $u+v$ and t and the total service time of all customers present at time t is less than or equal to x. The second term corresponds to the case where the last departure left the system at time u with $k \geq 1$ customers remaining, and there are $i-k$ additional arrivals by time t. Also, we have

$$w_{i_0}(t,s) = W_{i_0}(t,0) + \int_0^\infty se^{-sx}(W_{i_0}(t,x) - W_{i_0}(t,0))\,dx. \tag{52}$$

Multiplying the first term in (51) by se^{-sx} and integrating with respect to x from 0 to ∞, gives

$$\int_0^\infty se^{-sx}dx \sum_{i=0}^\infty \int_0^t dM_{i_0 0}(u) \int_0^{t-u} e^{D_0 v} \sum_{k=1}^\infty D_k dv \int_0^x P(i,t-u-v)\,d_w H(t+w-u-v) H^{(i+k-1)}(x-w).$$

Changing the order of integration with respect to x and w and making the change of variables $y = x - w$ leads to

$$\int_0^t dM_{i_0 0}(u) \int_0^{t-u} e^{D_0 v} dv [D(h(s)) - D_0] \int_0^\infty e^{-sw} e^{D(h(s))(t-u-v)}\, d_w H(t+w-u-v)\, h(s)^{-1},$$

by using (5). Forming the Laplace transform of this by multiplying by $e^{-\xi t}$ and integrating followed by several change of variables gives

$$h(s)^{-1} m_{i_0 0}(\xi)(\xi I - D_0)^{-1}[D(h(s)) - D_0] \int_0^\infty e^{-(\xi I - D(h(s)))y} dy \int_0^\infty e^{-sw} d_w H(y+w). \tag{53}$$

Now, multiplying the second term in (51) by se^{-sx}, integrating with respect to x from 0 to ∞, and performing similar manipulations leads to

$$\sum_{k=1}^\infty \int_0^t dM_{i_0 k}(u) \int_0^\infty e^{-sw} e^{D(h(s))(t-u)}\, d_w H(t+w-u)\, h(s)^{k-1}.$$

Forming the Laplace transform of this by multiplying by $e^{-\xi t}$ and integrating leads to

$$h(s)^{-1}[\tilde{m}_{i_0}(h(s),\xi) - m_{i_0 0}(\xi)] \int_0^\infty e^{-(\xi I - D(h(s)))y} dy \int_0^\infty e^{-sw} d_w H(y+w). \tag{54}$$

Next, adding (53) and (54), post-multiplying by $[\xi I - sI - D(h(s))]$ and using Lemma 1 and (12), we have

$$(\tilde{w}_{i_0}(\xi,s) - m_{i_0 0}(\xi))[\xi I - sI - D(h(s))]$$

$$= \left[m_{i_0 0}(\xi)(\xi I - D_0)^{-1}[D(h(s)) - D_0] + \tilde{m}_{i_0}(h(s),\xi) - m_{i_0 0}(\xi)\right][h(s)I - \tilde{A}(h(s),\xi)]h(s)^{-1},$$

so that

$$\tilde{w}_{i_0}(\xi,s)[\xi I - sI - D(h(s))] = h(s)^{i_0} I - s\hat{p}_{i_0 0}(\xi). \tag{55}$$

This yields (25). Taking the Laplace transform of $w_{i_0}(t,s)$ in (26) readily leads to (25).

Proof of Theorem 3

Conditioning on the amount of work at time $t = 0$, we can write

$$W(t,x) = \sum_{i=0}^{\infty} \int_0^t P(i,y)\, W_i(t-y,x)\, dF(y) + \sum_{i=0}^{\infty} \int_t^{t+x} P(i,t) H^{(i)}(t+x-y)\, dF(y), \tag{56}$$

where the first term corresponds to the case where the amount of work at time $t = 0$ is less than or equal to t and the second term corresponds to the case where the amount of work at $t = 0$ is greater than t. (Note that it must be less than or equal to $t + x$ for the work in the system at time t to be less than x.) The Laplace-Stieltjes transform with respect to x is

$$w(t,s) \equiv \int_0^{\infty} se^{-sx} W(t,x)\, dx$$

$$= \sum_{i=0}^{\infty} \int_0^t P(i,y)\, w_i(t-y,s)\, dF(y) + \sum_{i=0}^{\infty} \int_0^{\infty} se^{-sx} dx \int_t^{t+x} P(i,t) H^{(i)}(t+x-y)\, dF(y). \tag{57}$$

The second term becomes, after an interchange of integrals and a subsequent change of variables,

$$\sum_{i=0}^{\infty}\int_{t}^{\infty} dF(y)\int_{y-t}^{\infty} se^{-sx}P(i,t)H^{(i)}(t+x-y)\,dx \;=\; \sum_{i=0}^{\infty}\int_{t}^{\infty} dF(y)\int_{0}^{\infty} se^{-s(y+u-t)}\,P(i,t)H^{(i)}(u)\,du$$

$$=\; \sum_{i=0}^{\infty} h(s)^i e^{st}P(i,t)\int_{t}^{\infty} e^{-sy}dF(y) \;=\; e^{(sI+D(h(s)))t}\int_{t}^{\infty} e^{-sy}dF(y),$$

by using (5). The Laplace transform of $w(t,s)$ in (57) with respect to t is given by

$$\tilde{w}(\xi,s) \;=\; \sum_{i=0}^{\infty}\int_{0}^{\infty} dF(y)\int_{y}^{\infty} e^{-\xi t}P(i,y)\,w_i(t-y,s)\,dt \;+\; \int_{0}^{\infty} dF(y)\int_{0}^{y} e^{-(\xi I-sI-D[h(s)])t}e^{-sy}dt. \qquad (58)$$

Upon applying (5) and (25), we see that the first term in (58) becomes

$$\sum_{i=0}^{\infty}\int_{0}^{\infty} e^{-\xi y}P(i,y)\,\tilde{w}_i(\xi,s)\,dF(y)$$

$$=\; \left[f(\xi I-D(h(s))) \;-\; sf(\xi I-D[G(\xi)])\left[\xi I-D[G(\xi)]\right]^{-1}\right][\xi I-sI-D(h(s))]^{-1}, \qquad (59)$$

where we only consider pairs (ξ,s) for which the inverse in (59) exists. The second term in (58) is simplified as follows.

$$\int_{0}^{\infty} dF(y)\int_{0}^{y} e^{-(\xi I-sI-D(h(s)))t}e^{-sy}dt \;=\; \int_{0}^{\infty} e^{-sy}(I-e^{-[\xi I-sI-D(h(s))]y})[\xi I-sI-D(h(s))]^{-1}\,dF(y)$$

$$=\; [\,f(s)-f(\xi I-D(h(s)))][\xi I-sI-D(h(s))]^{-1}. \qquad (60)$$

Adding (59) and (60) yields (28). Finally, taking the Laplace transform of (29) yields (28).

Proof of Theorem 4

Multiplying both sides of (28) by $[\xi I-sI-D(h(s))]$, differentiating with respect to s and setting $s=0$ readily leads to

$$-\frac{\partial \tilde{w}(\xi,0)}{\partial s} = [\ \beta I + p_0(\xi) + \tilde{w}(\xi,0)(\alpha D^{(1)} - I)\ ](\xi I - D)^{-1}. \tag{61}$$

Inverting (61) by inspection, noting that $W(t,0) = e^{Dt}$, we obtain (30). Note that D and $D^{(1)}$ do not commute in general. Equations (31) and (32) follow routinely from (30).

Proof of Theorem 5

By successively differentiating with respect to s in

$$\tilde{w}(\xi,s)[\xi I - sI - V(s)] = f(s)I + sp_0(\xi),$$

we obtain, for $k \geq 2$,

$$\frac{\partial^k}{\partial s^k}\tilde{w}(\xi,s)[\xi I - sI - V(s)] = f^{(k)}(s) + k\frac{\partial^{k-1}}{\partial s^{k-1}}\tilde{w}(\xi,s) + \sum_{j=0}^{k-1}\binom{k}{j}\frac{\partial^j}{\partial s^j}\tilde{w}(\xi,s)\frac{d^{k-j}}{ds^{k-j}}V(s).$$

Setting $s = 0$, multiplying by $(-1)^k$ and inverting the transform by inspection, we obtain (35). Differentiating (35) with respect to t yields (36).

Proof of Theorem 6

Once again, by conditioning on the last departure before time t we can write

$$Y_{i_0 i}(t) = \int_0^t dM_{i_0 0}(u) \int_0^{t-u} e^{D_0 v} \sum_{k=1}^{i} D_k\, dv P(i-k, t-u-v)[1 - H(t-u-v)] \tag{62}$$

$$+ \sum_{j=1}^{i} \int_0^t dM_{i_0 j}(u) P(i-j, t-u)[1 - H(t-u)].$$

The first term corresponds to the case where the last departure left the system empty and the second term corresponds to where the last departure left j customers in the system. Taking Laplace transforms leads successively to

$$y_{i_0 i}(s) = \int_0^\infty e^{-st} dt \int_0^t dM_{i_0 0}(u) \int_0^{t-u} e^{D_0 v} \sum_{k=1}^{i} D_k \, dv P(i-k, t-u-v)[1-H(t-u-v)]$$

$$+ \sum_{j=1}^{i} \int_0^\infty e^{-st} dt \int_0^t dM_{i_0 j}(u) P(i-j, t-u)[1-H(t-u)]$$

$$= m_{i_0 0}(s) \int_0^\infty dv \int_v^\infty e^{-sx} e^{D_0 v} \sum_{k=1}^{i} D_k P(i-k, x-v)[1-H(x-v)]\, dx$$

$$+ \sum_{j=1}^{i} m_{i_0 j}(s) \int_0^\infty e^{-sx} P(i-j, x)[1-H(x)]\, dx$$

$$= m_{i_0 0}(s)(sI - D_0)^{-1} \int_0^\infty e^{-sw} \sum_{k=1}^{i} D_k P(i-k, w)[1-H(w)]\, dw$$

$$+ \sum_{j=1}^{i} m_{i_0 j}(s) \int_0^\infty e^{-sx} P(i-j, x)[1-H(x)]\, dx.$$

Taking probability generating functions yields

$$\tilde{y}_{i_0}(z,s) \equiv \sum_{i=0}^{\infty} y_{i_0 i}(s) z^i = y_{i_0 0}(s) + m_{i_0 0}(s)(sI - D_0)^{-1}[D(z) - D_0] \int_0^\infty e^{-sw} P^*(z,w)[1-H(w)]\, dw$$

$$+ [m_{i_0}(z,s) - m_{i_0 0}(s)] \int_0^\infty e^{-sw} P^*(z,w)[1-H(w)]\, dw. \qquad (63)$$

Recall that $y_{i_0 0}(s) = G(s)^{i_0} p_{00}(s)$. The integral on the right side of each of the above terms is evaluated by applying (5), (10) and integration by parts to give

$$\int_0^\infty e^{-(sI - D(z))x}[1-H(x)]\, dx = (sI - D(z))^{-1}[I - \tilde{A}(z,s)]. \qquad (64)$$

Substituting (64) into (63) and simplifying leads to (41).

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