EXPONENTIAL APPROXIMATIONS FOR TAIL PROBABILITIES IN QUERIES II: SOJOURN TIME AND WORKLOAD

JOSEPH ABATE
Ridgewood, New Jersey

GAGAN L. CHOUDHURY
AT&T Laboratories, Holmdel, New Jersey

WARD WHITT
AT&T Laboratories, Murray Hill, New Jersey
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We continue to focus on simple exponential approximations for steady-state tail probabilities in queues based on asymptotics. For the $G/GI/1$ model with i.i.d. service times that are independent of an arbitrary stationary arrival process, we relate the asymptotics for the steady-state waiting time, sojourn time, and workload. We show that the three asymptotic decay rates coincide and that the three asymptotic constants are simply related. We evaluate the exponential approximations based on the exact asymptotic parameters and their approximations by making comparisons with exact numerical results for $BMAP/GI/1$ queues, which have batch Markovian arrival processes. Numerical examples show that the exponential approximations for the tail probabilities are remarkably accurate at the 90th percentile and beyond. Thus, these exponential approximations appear very promising for applications.

This paper is a sequel to Part I, Abate et al. (1995), in which we studied exponential approximations for steady-state waiting-time tail probabilities in infinite-capacity queues based on asymptotics. In Part I we presented numerical examples based on exact numerical solutions of the $BMAP/GI/1$ queue (having i.i.d. service times independent of a batch Markovian arrival process) and the $GI/GI/1$ queue to show that the exponential approximation is remarkably good, lending support to previous work in the same direction, notably by Tijms (1986) and Asmussen (1987, 1989). Moreover we developed simple effective approximations for the asymptotic parameters.

The primary purpose of this paper is to relate the asymptotic behavior of the steady-state waiting time $W$ to the asymptotic behavior of the steady-state sojourn time $T$ (response time, i.e., waiting time plus service time) and the steady-state workload $L$ (virtual waiting time). In particular, we show that corresponding asymptotics for $T$ and $L$ are valid in any $G/GI/1$ queue (having i.i.d. service times that are independent of a general stationary arrival process) whenever the exponential asymptotics for the waiting time are valid, and we show that the parameters are simply related. Moreover, we show by making comparisons with exact numerical values that the resulting approximations are remarkably good.

Even if we recognize that an exponential approximation is good for the waiting time, it may be surprising that a similar exponential approximation is also often good for the sojourn time without any special assumptions on the service-time distribution. This idea has been advanced by Fleming (1992), who proposes simple heavy-traffic approximations for sojourn-time percentiles as well as waiting-time percentiles in a class of $M/GI/1$ queues. (He focuses on two-point service-time distributions, which are realistic for computer systems.) We provide additional support for this idea, as well as develop new approximations for more general models. Having a corresponding exponential approximation for sojourn times is very important for applications, because the sojourn time (response time) is often the critical variable.

Here is how the rest of this paper is organized. In Section 1 we relate the asymptotic behavior of the waiting time, workload, and sojourn time in the $G/GI/1$ model. In Section 2 we discuss numerical examples for the workload and sojourn time, drawing on Choudhury et al. (1996), which in turn draws upon Lucantoni (1991).

Additional related work (done after this paper) appears in Abate et al. (1994), Glynn and Whitt (1994), and Choudhury et al. (1996). Abate et al. (1994) directly establish exponential asymptotics for the steady-state variables in the $BMAP/GI/1$ queue and give explicit expressions for the asymptotic parameters. Glynn and Whitt establish logarithmic limits for the steady-state waiting time and relate these limits to corresponding logarithmic limits for the steady-state workload and queue length (at an arbitrary time and at arrivals). These weaker logarithmic limits hold in greater generality than the limits considered in this paper. Finally, Choudhury et al. (1996) focus on the special

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case of superposition arrival processes, which is of special interest for communication networks, and show by numerical examples that the asymptotic approximations often do not perform as well in that setting. Due to developments in communication networks, this topic has recently received much attention. See the references (e.g., Part I) for additional references.

1. SOJOURN TIME AND WORKLOAD

We consider the G/GI/1 queueing model, which has one server, unlimited waiting space, the first-come first-served discipline, and i.i.d. service times that are independent of a general stationary arrival process. We assume that the mean service time is 1 and that the arrival rate is \( \rho < 1 \). We assume that the various steady-state distributions discussed below exist as proper probability distributions.

Let \( W \) be the steady-state waiting time (before beginning service). As discussed in Part I, in great generality,

\[
P(W > x) \sim \alpha_W e^{-\alpha_W x} \quad \text{as } x \to \infty,
\]

i.e., \( e^{\alpha_W} P(W > x) \to \alpha_W \) as \( x \to \infty \), where \( \eta \) and \( \alpha_W \) are positive constants (independent of \( x \)) called the asymptotic decay rate and the asymptotic constant, respectively. Let \( V \) be a generic service time random variable. Theorem 10 and Example 5 of Part I show that in order for (1) to be valid in the G/GI/1 queue, it is necessary, but not sufficient, to have \( E e^{\eta V} < \infty \) for some \( s > 0 \).

In this section we show that the steady-state sojourn time \( T \) and the steady-state workload \( L \) satisfy analogs of (1) when (1) holds with the same asymptotic decay rate \( \eta \) and asymptotic constants \( \alpha_L \) and \( \alpha_T \) that are easily related to \( \alpha_W \).

For G/I\(/PH/1 \) queues, the asymptotic behavior of \( W, L, \) and \( T \) was described in detail by Neuts (1981). These relationships are also a consequence of interesting phase-type results in Asmussen (1992); see Corollary 2.2. In particular, Asmussen shows that if the service-time distribution is phase-type characterized by the pair \((\pi, \hat{Q})\) where \( \pi \) is a \( d \)-dimensional vector and \( \hat{Q} \) is a \( d \times d \) generator matrix, then \( W, L, \) and \( T \) have distributions, which except for a probability mass at the origin, are also phase-type with representations \((\pi_W, Q), (\pi_L, Q)\) and \((\pi_T, Q)\) where \( Q \) is a common \( d \times d \) generator matrix and \( \pi_W, \pi_L \), and \( \pi_T \) are in general different \( d \)-dimensional vectors. Since the asymptotic decay rate \( \eta \) is the Perron-Frobenius eigenvalue of \( \hat{Q} \), it is identical for all three random variables. The asymptotic constants involve the eigenvectors associated with the dominant eigenvalue and the vectors \( \pi_W, \pi_L, \) and \( \pi_T \).

We conjecture that this structural solidity result extends to GI/PH/s models with \( s > 1 \), but without having the number of phases in \( \hat{Q} \) be equal to the number \( d \) of service-time phases. Indeed, we conjecture that the number of phases in \( \hat{Q} \) is

\[
(d + s - 1) = (d + s - 1)!/(d - 1)!s!.
\]

This is based on the structural solidity result for GI/H\(/s \) queues established by de Smit (1983); the waiting-time distribution is again hypexponential (plus a mass at the origin) with this larger number of exponential terms.

We extend Neuts (1981) and Asmussen (1992) for the sojourn time by replacing the GI and PH in GI/PH/1 by \( G \) and GI, respectively, but we consider only the asymptotic parameters. This next result extends easily to \( s \) servers.

**Theorem 1.** In the G/GI/1 model, if \( e^{\alpha_W} P(W > x) \to \alpha_W \) as \( x \to \infty \), then \( E e^{\eta V} < \infty \) and

\[
e^{\alpha_W} P(T > x) \to \alpha_T e^{\alpha_W} E e^{\eta V} > \alpha_W \text{ as } x \to \infty.
\]

**Proof.** By Theorem 10 of Part I, \( E e^{\eta V} < \infty \). Since \( T = W + V \) where \( W \) and \( V \) are independent,

\[
e^{\alpha_W} P(T > x) = \int_0^x e^{\eta(x-u)} P(W > x - u) e^{\alpha_W} dP(V \leq u)
\]

\[
+ e^{\alpha_W} P(V > x)
\]

Since \( E e^{\eta V} < \infty \), \( e^{\alpha_W} P(V > x) \to 0 \) as \( x \to \infty \). Then the assumed convergence for \( W \) plus the bounded convergence theorem implies the desired conclusion.

From Proposition 9 of Glynn and Whitt (1994), we know that the correction term \( E e^{\eta V} \) in Theorem 1 is \( \sigma^{-1} \), where \( \sigma \) is the queue-length asymptotic decay rate. Glynn and Whitt relate the logarithmic asymptotics for the waiting time and queue length. It remains to relate (1) to analogs of (1) for the queue length. For special cases, connections are established in Neuts (1981, 1986) and Abate et al. (1994). The BMAP/G/1 model in the last paper is very close to the G/GI/1 model considered here.

Theorem 1 is especially easy to understand when the service-time distribution is deterministic; then \( P(T > x) = P(W > x - 1) \sim a e^{-\eta(x-1)} \) as \( x \to \infty \), so that \( \alpha_T = a \eta \).

The case of a service-time distribution with finite support is a minor modification. Theorem 1 is the natural generalization.

We now consider the workload in the G/GI/1 model. For this, we use a relation between a distribution and its associated stationary-excess distribution. If \( X \) is a nonnegative random variable with finite mean, then \( X_e \) is a random variable with the associated stationary-excess distribution, i.e.,

\[
P(X_e > x) = \frac{1}{EX} \int_x^\infty P(X > y)dy, \; x \geq 0.
\]

Applying integration by parts, we see that if \( E e^{aX} < \infty \), then

\[
E e^{aX} e^{aX_e} = \frac{E(e^{aX} - 1)}{sEX}.
\]
Theorem 2. In the $GI/GI/1$ model, if $e^{\eta W}(W > x) \to \alpha_W$ as $x \to \infty$, then
\[ e^{\eta}P(L > x) \to \alpha_L = \frac{\alpha_W \rho}{\eta} (e^{\eta} - 1) \]

Proof. By the generalized Takács formula in Franken et al. (1981 (4.5.9) on p. 129),
\[ P(L > x) = \rho P(W + V_x > x) \]
for all $x$, where $V_x$ is independent of $W$ and has the stationary-excess distribution of the service-time distribution. The rest of the argument is as in Theorem 1. We use (3) (and the fact that $EV = 1$) to obtain
\[ \alpha_P e^{\eta V} = \frac{\alpha_P}{\eta} (e^{\eta} - 1) \]

Given that $E e^{\eta V} = \sigma^{-1}$, we can express the correction term for $L$ in Theorem 2 directly in terms of the asymptotic decay rates $\eta$ and $\sigma$, i.e., $\rho e^{\eta V} - 1)/\eta = \rho(1 - \sigma)/\eta$. Notice that Theorem 2 is consistent with the well-known property that $L$ has the same distribution as $W$ in the $M/G/1$ queue, because then $\rho e^{\eta V} - 1)/\eta = 1$ by the defining property of $\eta$. Similarly, for $GI/M/1$,
\[ \alpha_L = \alpha \cdot \frac{(E e^{\eta V} - 1)}{\eta} = \frac{\alpha_P}{\eta} = \rho \]

because $\alpha = \sigma = 1 - \eta$. Finally, for the $GI/PH/1$ queue, Theorems 1 and 2 agree with Section 2 of Neuts (1981).

When $\eta$ is sufficiently small, we can use the approximation
\[ E e^{\eta V} \approx 1 + \eta V + \frac{\eta^2 V^2}{2} + \frac{\eta^3 V^3}{6} \]

Inserting (5) into the formula for $\alpha_L$, we get
\[ \alpha_L = \alpha_W \rho \left( \frac{E e^{\eta V}}{\eta} \right) \]

Given formula (33) in Part I for $GI/GI/1$, we see that for $GI/GI/1$ as $\rho \to 1$
\[ \alpha_L = \alpha_W \left( 1 - \frac{c_2 - 1}{c_2 + c_3} \right)(1 - \rho) - \left( \frac{c_2^2 + 2c_3}{c_2^2 + c_3} \right)(\eta^* + 1) \]

where
\[ \eta^* \approx \frac{2\eta_3 - 3c_2^2 + 2c_3}{3(c_2^2 + c_3^2)} \]

Theorems 1 and 2 show how to compute $\alpha_F$ and $\alpha_L$ given $\eta$ and $\alpha_W$ or approximations for them. When it is not convenient to calculate $E e^{\eta V}$, (5)–(8) show how to approximate $\alpha_F$ and $\alpha_L$ given $\eta$ and $\alpha_W$ or approximations for them. Paralleling Section 6 of Part I, we also suggest the approximations $\alpha \approx \eta E T$ and $\alpha \approx \eta E L$.

2. NUMERICAL EXAMPLES FOR THE SOJOURN TIME AND THE WORKLOAD

We noted that it is easy to see that the asymptotic behavior of $T$ and $W$ are closely related when the service-time distribution is deterministic. We now consider what happens with service-time distributions that are substantially more variable than an exponential distribution.

As in Part I, we obtain the exact tail probabilities from the algorithm described in Choudhury et al. (1996), which draws upon Lucantoni. We obtain the exact values of the asymptotic parameters from the moment-based generating-function-inversion algorithm in Choudhury and Lucantoni (1996). We also estimate the asymptotic parameters by linear regression applied to the numerically calculated tail probabilities (after taking logarithms) as described in Part I.

Example 1. Consider the $M/H^2_1$ queue with a hyperexponential-service-time distribution with balanced means. Let the arrival rate be $\rho = 0.7$ and, as always, let the service-time distribution have mean $1$. The $H^2_1$ distribution is defined in Example 1 of Part I. Consider the case of service-time squared coefficient of variation (variance divided by the square of the mean) $c_2 = 4.0$. Then the parameters of the density are $p_1 = 0.8872983$, $\lambda_1 = 1.7744966$ and $\lambda_2 = 0.2254034$.

Since the arrival process is Poisson (M), the distributions of $W$ and $L$ coincide. We apply the Pollaczek-Khintchine formula to obtain $EL = 5.833$ and $ET = 6.833$. The exact asymptotic parameters for $L$ and $T$ obtained from Choudhury and Lucantoni (1994) and the linear regression are $\eta = 0.1000040$, $\alpha_L = \alpha_W = 0.5727238$ and $\alpha_T = 0.6545448$, so that $\alpha = \alpha_W/\alpha_T = 0.87500$.

The approximations from Section 4 and Section 6 of Part I are $\eta_{HT} = 0.1200$, $\eta_{ap} = 0.0984$, $\alpha_{HT} = \eta_{EL} = 0.5833$, $\alpha_{ap} = \eta_{ET} = 0.6833$, $\eta_{ap}EL = 0.5740$, and $\eta_{ap}ET = 0.6724$. As in Part I, the approximations for the asymptotic parameters are quite good.

Tables I and II display exact values of the tail probabilities $P(L > x)$ and $P(T > x)$ and the associated exponential approximations. The regression estimates are displayed as well to show the (in this case, spectacular) rate of convergence to the exponential limit. In this case, the linear regression easily produces the exact asymptotic parameters.

Example 2. To see what happens with a nonrenewal arrival process and a service-time distribution very unlike an exponential distribution, we now consider the MMPP$_2$/ $D_2/1$ model of Example 3 in Part I. As before, $\rho = 0.7$ and $c_3^2 = 2.0$. First, the asymptotic decay rates calculated for $W$, $T$, and $L$ by the algorithm in Choudhury and Lucantoni agreed to eight decimal places, yielding $\eta = 0.11159727$. For this model, it is easy to see that $\sigma^{-1} = E e^{\eta V} = \ldots$
Table I
A comparison of exponential approximations with exact values of the workload tail probabilities, 
\( P(L > x) \), in the \( M/H^2_2/1 \) queue with \( \rho = 0.7 \)
and \( c^2 \) = 4.0 in Example 1. Also included are the local linear regression estimates of the asymptotic parameters.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact</th>
<th>( \alpha_x e^{-\alpha x} )</th>
<th>( \hat{\alpha}_x(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.4278</td>
<td>0.4243</td>
<td>0.5931</td>
</tr>
<tr>
<td>6.0</td>
<td>0.31441</td>
<td>0.31431</td>
<td>0.5740</td>
</tr>
<tr>
<td>9.0</td>
<td>0.252846</td>
<td>0.23844</td>
<td>0.57279</td>
</tr>
<tr>
<td>12.0</td>
<td>0.17249290</td>
<td>0.17249293</td>
<td>0.57272727</td>
</tr>
<tr>
<td>18.0</td>
<td>0.094663802</td>
<td>0.094663802</td>
<td>0.572723848</td>
</tr>
<tr>
<td>24.0</td>
<td>0.051951350</td>
<td>0.051951350</td>
<td>0.572723841</td>
</tr>
</tbody>
</table>

1.13873. The successive approximations in (5) are: 1.0, 1.1115, 1.1301, and 1.1362. The relative error in the approximation for \( E e^{\eta V} \) is 0.8% and 0.2% using two and three moments.

The asymptotic constants are \( \alpha_w = 0.65738, \alpha_r = 0.74867, \) and \( \alpha_x = 0.57261. \) These provide empirical evidence supporting Theorems 1 and 2. For the asymptotic constant, \( (\alpha / \alpha_w) \) = 0.87104. The successive approximations in (6) are 0.7, 0.817, 0.8717. (Note that (7) does not apply because the arrival process is not renewal.)

Table III compares exponential approximations for the tail probabilities of the steady-state workload and sojourn time with exact values. Again the exponential approximations perform well. Our experience indicates that, consistent with intuition, the quality of the exponential approximations for the waiting time and workload is usually somewhat better than for the sojourn time. However, the difference is not perceptible in Table III.

Example 3. We conclude with an \( MMPP/\Gamma_1/1 \) example, which is used to evaluate heavy-traffic asymptotic expansions for the asymptotic decay rates of the waiting time in Section 7 of Choudhury and Whitt (1994). The service-time distribution is gamma with shape parameter 1/2, so that the transform is not rational and thus the distribution is not PH. It is moderately highly variable, with first three moments 1, 3, and 15.

Table II
A comparison of exponential approximations with exact values of the sojourn-time tail probabilities, 
\( P(T > x) \), in the \( M/H^2_2/1 \) queue with \( \rho = 0.7 \)
and \( c^2 \) = 4.0 in Example 1. Also included are the local linear regression estimates of the asymptotic parameters.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact</th>
<th>( \alpha_x e^{-\alpha x} )</th>
<th>( \hat{\alpha}_x(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>0.4943</td>
<td>0.4849</td>
<td>0.7107</td>
</tr>
<tr>
<td>6.0</td>
<td>0.35947</td>
<td>0.35921</td>
<td>0.6581</td>
</tr>
<tr>
<td>9.0</td>
<td>0.266115</td>
<td>0.266108</td>
<td>0.6547</td>
</tr>
<tr>
<td>12.0</td>
<td>0.1971358</td>
<td>0.1971356</td>
<td>0.6545</td>
</tr>
<tr>
<td>18.0</td>
<td>0.108187743</td>
<td>0.108187743</td>
<td>0.654544812</td>
</tr>
<tr>
<td>24.0</td>
<td>0.059373268</td>
<td>0.059373268</td>
<td>0.654544803</td>
</tr>
</tbody>
</table>

The arrival process is a two-phase \( MMPP \), which has four parameters (the arrival rate and mean holding time in each phase), one of which we determine by letting the arrival rate be \( \rho \). A second parameter is determined by assuming that the long-run arrival rate in each phase is \( \rho/2 \). A third parameter is determined by assuming that the expected number of arrivals during each visit to each phase is 5. Finally, the last parameter is determined by making the ratio of the arrival rates in the two phases 4.

Tables IV and V display approximations and exact values for higher percentiles of the steady-state workload and sojourn-time distributions, respectively. In each case, two values of \( \rho \) are considered: \( \rho = 0.8 \) and \( \rho = 0.5 \). Three approximations are considered. All approximations are exponential approximations \( ae^{-\alpha x} \) with the exact \( \eta \) converted to percentiles as in (2) of Part I. The first approximation has the exact asymptotic constant, \( \alpha_w \) and \( \alpha_r \), respectively; the second approximation approximates \( \alpha_x \) by \( \eta EL \) and \( \alpha_r \) by \( \eta ET \); and the third approximation approximates \( \alpha_w \) and \( \alpha_r \) by 1.

From Tables IV and V, we see that the approximations for higher percentiles are very impressive. The accuracy improves as the percentile increases and as the traffic intensity increases. At \( \rho = 0.8 \), the relative error of the asymptotic approximation (with exact \( \alpha \)) is less than 0.1% even at the 80th percentile. The approximation based on \( \alpha \approx \eta \) mean performs remarkably well, substantially better than the approximation with \( \alpha \approx 1.0 \). However, for high percentiles such as 99.99, even \( \alpha \approx 1.0 \) yields a useful approximation.

However, as shown in Choudhury et al. (1996), the quality of these asymptotic approximations can deteriorate when the arrival process is the superposition of many independent component arrival processes. Nevertheless, since
Table IV
A comparison of approximations with exact values of high percentiles of the steady-state workload in the $MMPP/\Gamma_{1/2}/1$ queue in Example 3

<table>
<thead>
<tr>
<th>Percentile required</th>
<th>Exact</th>
<th>Approx., exact $\alpha_{\ell} = 0.73234$</th>
<th>Approx., $\alpha_{\ell} \approx 0.7439$</th>
<th>Approx., $\alpha_{\ell} \approx 1.0$</th>
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<tbody>
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<td>80</td>
<td>16.1555</td>
<td>16.1489</td>
<td>16.34</td>
<td>20.0</td>
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<tr>
<td>90</td>
<td>24.7714</td>
<td>24.7709</td>
<td>24.97</td>
<td>28.6</td>
</tr>
<tr>
<td>99</td>
<td>53.4126</td>
<td>53.4126</td>
<td>53.61</td>
<td>87.3</td>
</tr>
<tr>
<td>99.9</td>
<td>82.0542</td>
<td>82.0542</td>
<td>82.25</td>
<td>85.9</td>
</tr>
<tr>
<td>99.99</td>
<td>110.6959</td>
<td>110.6959</td>
<td>110.89</td>
<td>114.6</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Percentile required</th>
<th>Exact</th>
<th>Approx., exact $\alpha_{\ell} = 0.36257$</th>
<th>Approx., $\alpha_{\ell} \approx 0.4135$</th>
<th>Approx., $\alpha_{\ell} \approx 1.0$</th>
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<tbody>
<tr>
<td>80</td>
<td>3.6059</td>
<td>3.0228</td>
<td>3.70</td>
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</tr>
<tr>
<td>90</td>
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<tr>
<td>99</td>
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<td>41.6518</td>
<td>41.6517</td>
<td>42.32</td>
<td>46.8</td>
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Table V
A comparison of approximations with exact values of high percentiles of the steady-state sojourn time in the $MMPP/\Gamma_{1/2}/1$ queue in Example 3

<table>
<thead>
<tr>
<th>Percentile required</th>
<th>Exact</th>
<th>Approx., exact $\alpha_{\ell} = 0.87702$</th>
<th>Approx., $\alpha_{\ell} \approx 0.8899$</th>
<th>Approx., $\alpha_{\ell} \approx 1.0$</th>
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<tbody>
<tr>
<td>80</td>
<td>18.3940</td>
<td>18.3914</td>
<td>18.57</td>
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<tr>
<td>90</td>
<td>27.0136</td>
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<tr>
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<td>55.6551</td>
<td>55.83</td>
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<td>99.9</td>
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<td>84.48</td>
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<td>99.99</td>
<td>112.9384</td>
<td>112.9384</td>
<td>113.12</td>
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<table>
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<th>Percentile required</th>
<th>Exact</th>
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<th>Approx., $\alpha_{\ell} \approx 0.72288$</th>
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</tr>
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<td>21.1745</td>
<td>21.80</td>
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</tr>
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<td>33.50</td>
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<td>99.99</td>
<td>44.5786</td>
<td>44.5786</td>
<td>45.20</td>
<td>46.8</td>
</tr>
</tbody>
</table>

the desired probabilities are often extremely small, even in that demanding setting the asymptotic approximation is often good enough provided that we know (or have a good approximation for) the asymptotic constant as well as the asymptotic decay rate.

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