# EXTENSIONS OF THE QUEUEING RELATIONS $L = \lambda W$ AND $H = \lambda G$

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(Received July 1986; revision received September 1987; accepted March 1988)

This paper extends the fundamental queueing relations $L = \lambda W$ and $H = \lambda G$ that relate customer averages (the customer-average waiting time $W$ or cost $G$) to associated time averages (the time-average queue length $L$ or cost $H$) given an arrival process with arrival rate $\lambda$. These relations can be established by focusing on a two-dimensional cumulative input process that has the two one-dimensional cumulative input processes of interest as marginals. Relations between the marginal averages are established for cumulative input processes that may not be representable as integrals or sums. The general framework includes the continuous versions of $L = \lambda W$ and $H = \lambda G$ due to T. Rolski and S. Stidham as well as the standard version of $H = \lambda G$, and can be extended to higher dimensions. Inequalities are also established when some of the conditions for equality do not hold. Moreover, central limit theorem versions of $H = \lambda G$ are established, extending our recent results for $L = \lambda W$.

The fundamental queueing formula $L = \lambda W$ (Little's law) states that the time-average queue length (number in the system) $L$ is equal to the product of the arrival rate $\lambda$ and the customer-average waiting time (time spent in the system) $W$. This formula is valid in great generality, as was shown by Little (1961), Stidham (1974) and others. Stidham (1972), Brumelle (1971, 1972), Maxwell (1970), and Heyman and Stidham (1980) showed that similar relations exist between more general customer-averages and time-averages, which is represented by the formula $H = \lambda G$. Rolski and Stidham (1983) also established continuous analogs of $L = \lambda W$ and $H = \lambda G$ for input-output models with general, nondecreasing cumulative input, such as occur in reservoirs and other storage systems.

The purpose of this paper is to present a more general version of $H = \lambda G$, which includes the continuous version of Rolski and Stidham as well as the standard version of Heyman and Stidham as special cases. We are motivated to consider further abstraction by examples not covered by any of the previous versions and by the desire to better understand what is essential. To see what we have in mind, consider the following example: Salmon migrate up river, jumping through a salmon ladder. The river narrows, creating a queue. The amount of food consumed by each fish while in the salmon ladder is modeled as a stochastic process with nondecreasing sample paths. (This single fish consumption process may be quite complicated, depending upon the current position and past consumption of all fish.) We wish to relate the average amount of food consumed *per fish* in the ladder among the first $n$ fish (throughout all time) to the average amount of food consumed *per time* in the ladder by time $t$ (by all fish). For further motivation, see Sections 1.4–1.7.

The analysis of $L = \lambda W$ and $H = \lambda G$ leads to the consideration of two-dimensional cumulative input processes (on the positive quadrant of the plane). In all previous versions of $L = \lambda W$ and $H = \lambda G$, the cumulative processes considered involve integrals of nonnegative functions. A key idea here is to consider general, nondecreasing cumulative processes, which need not be absolutely continuous with respect to Lebesgue or counting measures (expressible as an integral or a sum). This new representation is symmetric in the arguments, showing that applications need not be limited to time and a customer index. Essentially, there is just a two-dimensional cumulative input process which, loosely speaking, has its region of primary increase along some ray. We develop a version of $H = \lambda G$ in this general framework in Section 2. It leads to an easily proved statement, with conditions that in turn are verified easily in the previous settings.

Even for the standard version of $L = \lambda W$, we obtain some useful new results. All previous versions show

*Subject classifications* Queues: conservation laws, sample-path analysis; Little's Law.


0030-364X/89/3704-0634 $01 25

that the existence of limits for $\lambda$ and $W$ imply the existence of a limit for $L$. We show how to go the other way (from $\lambda$ and $L$ to $W$), with appropriate conditions. This reverse implication is also established in Theorem 2 of Glynn and Whitt (1986), but the proof here is different and adds additional insight. We also establish inequalities in situations where equality does not hold.

We follow Stidham (1974, 1982); Heyman and Stidham; Rolski and Stidham; and Glynn and Whitt (1986) by exploiting sample path methods. Our analysis in the first four sections is deterministic, so that in the usual stochastic model context, the functions here correspond to sample paths of stochastic processes, and the results hold with probability one; e.g., see p. 988 of Heyman and Stidham.

In the usual stochastic settings, the limits for the averages considered here correspond to strong laws of large numbers. In Glynn and Whitt (1986), we established corresponding relations among other classical limit theorems for $L = \lambda W$, such as central limit theorems, weak laws of large numbers and laws of the iterated logarithm. The central limit theorems are particularly useful for statistical estimation of queueing parameters; see Glynn and Whitt (1989). Similar results hold for $H = \lambda G$, as will be shown here in Section 5.

The rest of this paper is organized as follows. We introduce the general framework and show how it incorporates interesting special cases in Section 1. We establish the main theorem in Section 2. We treat the special case in which one variable is discrete and introduce new conditions that are verified more readily in applications in Section 3; i.e., we show that these new conditions imply the conditions in Section 2. In Section 4, we treat several special cases of Section 3, including the standard versions of $L = \lambda W$ in Stidham (1974) and $H = \lambda G$ in Heyman and Stidham. In Section 4, we also treat the continuous analog covering the recent results of Rolski and Stidham. Finally, in Section 5 we present central limit theorem versions of $H = \lambda G$, which extend Theorems 3 and 4 of Glynn and Whitt (1986).

## 1. A GENERAL FRAMEWORK FOR $H = \lambda G$

We begin with a *cumulative input function* $F(s, t)$, which is defined to be a real-valued function on $[0, \infty) \times [0, \infty)$ that is nondecreasing in both $s$ and $t$, and has finite limits $F(s, \infty)$ as $t \to \infty$ and $F(\infty, t)$ as $s \to \infty$. Typically, $F(s, t)$ is a cumulative distribution function associated with a positive measure on $[0, \infty) \times [0, \infty)$, i.e., $F(s, t)$ is the measure of the rectangle $[0, s] \times [0, t]$, so that

$$F(s_2, t_2) - F(s_2, t_1) - F(s_1, t_2) + F(s_1, t_1) \geq 0$$

for all $0 \leq s_1 \leq s_2$ and $0 \leq t_1 \leq t_2$, but we do not assume that $F$ has this property. For example, in the classical $L = \lambda W$ setting we let $F(s, t)$ represent the total time spent in the system in the time interval $[0, t]$ by the first $[s]$ arriving customers, where $[s]$ is the greatest integer less than or equal to $s$; see (5) and (7). In applications, $F(s, t)$ is *one sample path* of a bivariate *cumulative input stochastic process* associated with *a random measure*.

Let $G(s)$ and $H(t)$ represent the associated *marginal averages*, defined by

$$G(s) = s^{-1}F(s, \infty) \quad \text{and} \quad H(t) = t^{-1}F(\infty, t). \tag{1}$$

Our object is to relate the limit of $G(s)$ as $s \to \infty$ to the limit of $H(t)$ as $t \to \infty$. However, we cannot do so without further assumptions. Loosely speaking, the essential idea is that the cumulative input function $F(s, t)$ should have its *primary region of increase* about a line $s = \lambda t$ in the positive quadrant where $\lambda > 0$. A simple example illustrating this imprecise notion is a cumulative input function $F(s, t)$ satisfying $F(s, \infty) = F(s, t)$ for $t \geq \lambda^{-1}s + x$ for some fixed positive $x$ and all positive $s$, and $F(\infty, t) = F(s, t)$ for $s \geq \lambda t + \lambda x$ for this same $x$ and all positive $t$. (Think of a positive measure on $[0, \infty) \times [0, \infty)$ with support on the set $\{(s, t)\colon \lambda(t - x) \leq s \leq \lambda(t + x)\}$.) For this example, it is easy to see that $G(s) \to G$ as $s \to \infty$ if and only if $H(t) \to H$ as $t \to \infty$, in which case $H = \lambda G$, for example

$$\begin{aligned} G(s) &= s^{-1}F(s, \infty) = s^{-1}F(s, \lambda^{-1}s + x) \\ &\leq s^{-1}F(s + 2\lambda x, \lambda^{-1}s + x) \\ &= s^{-1}F(\infty, \lambda^{-1}s + x) \\ &= (\lambda^{-1} + xs^{-1})H(\lambda^{-1}s + x) \end{aligned}$$

and similarly in the other direction.

To treat more interesting examples, we assume that $G(s)$ and $H(t)$ are related approximately by

$$G(s) \approx \frac{sH(T_1(s))}{T_1(s)} \quad \text{and} \quad H(t) \approx \frac{tG(S_1(t))}{S_1(t)} \tag{2}$$

with $T_1(s)$ being a nonnegative, nondecreasing, right-continuous real-valued function on $[0, \infty)$ such that $T_1(s) < \infty$ for all $s$ and $T_1(s) \to \infty$ as $s \to \infty$, and $S_1(t)$ being the right-continuous inverse of $T_1(s)$, defined by

$$S_1(t) = \inf\{s \geq 0\colon T_1(s) > t\}, \quad t \geq 0. \tag{3}$$

The functions $T_1(s)$ and $S_1(t)$ in (2) and (3) are *time-change functions* relating the growth of $F(s, t)$ in the two variables $s$ and $t$. When $F(s, t)$ is one sample path of a cumulative input stochastic process, $T_1(s)$ and $S_1(t)$ are single sample paths of associated random time-change stochastic processes. The idea is that $s^{-1}T_1(s) \to \lambda^{-1}$, $H(t) \to H$ and $G(s) \to G$, where $H = \lambda G$. Our goal is to show that some of these limits imply others and, when they hold, we indeed have the relation $H = \lambda G$. In Section 2, we give more precise expressions for (2) (see (14) and (15) there) and prove a theorem. We consider some examples next.

### 1.1. One Discrete Variable

In the usual setting, we start with a sequence of nonnegative, nondecreasing real-valued functions $\{[F_k(t): t \geq 0]: k = 1, 2, \ldots\}$ on $[0, \infty)$ and a nondecreasing sequence of numbers $\{T_k: k \geq 0\}$ with $T_0 = 0$ and $T_k < \infty$ for all $k$ and $T_k \to \infty$. We are interested in the limiting behavior of

$$G(n) = n^{-1} \sum_{k=1}^{n} F_k(\infty) \quad \text{and}$$

$$H(t) = t^{-1} \sum_{k=1}^{\infty} F_k(t). \tag{4}$$

We fit this into the general framework above by letting

$$F(s, t) = \sum_{k=1}^{[s]} F_k(t) \quad \text{and} \quad T_1(s) = T_{[s]}, \quad s \geq 0 \tag{5}$$

where $[s]$ is the greatest integer less than or equal to $s$. The inverse process $S_1(t)$ defined in (3) is thus $N(t) + 1$, where

$$N(t) = \max\{k \geq 0: T_k \leq t\}, \quad t \geq 0. \tag{6}$$

### 1.2. $L = \lambda W$

For the standard version of $L = \lambda W$ in Stidham (1974) and Section 2 of Glynn and Whitt (1986), let $T_n$ be the arrival epoch and $D_n$ the departure epoch of the $n$th customer for $n \geq 1$, so that the waiting time (time spent in the system) is $W_n = D_n - T_n$. ($T_0 = 0$ without there being a 0th customer.) We assume that $T_n \leq D_n$, $0 = T_0 \leq T_n \leq T_{n+1} < \infty$ for all $n$ and $T_n \to \infty$ as $n \to \infty$. To obtain a special case of Section 1.1, set

$$F_k(t) = \int_0^t I_{[T_k, D_k]}(s)\, ds, \quad t \geq 0 \tag{7}$$

where $I_A(t)$ is the indicator function of set $A$, defined by $I_A(t) = 1$ if $t \in A$ and $I_A(t) = 0$ otherwise

Figure 1 shows a natural picture associated with $L = \lambda W$. The customer number appears on the horizontal axis and time appears on the vertical axis. A unit density $f(s, t)$ is defined over a subset of the positive orthant by assigning the value 1 to each rectangle $[k - 1, k] \times [T_k, D_k]$ for $k \geq 1$. This represents customer $k$ being in the system from $T_k$ until $D_k$. The cumulative process $F(s, t)$ is the integral of this density over $[0, s] \times [0, t]$. Also depicted in Figure 1 are the two marginal cumulative processes, the sum of the first six waiting times, $\sum_{k=1}^{6} W_k$, and the integral of the queue length process (number in the system) from 0 to $t_0$, $\int_0^{t_0} Q(s)\, ds$. From this picture, it is easy to see that the two marginal cumulative processes count essentially the same thing for large $s$ and $t$, so that we should be able to relate the marginal averages.

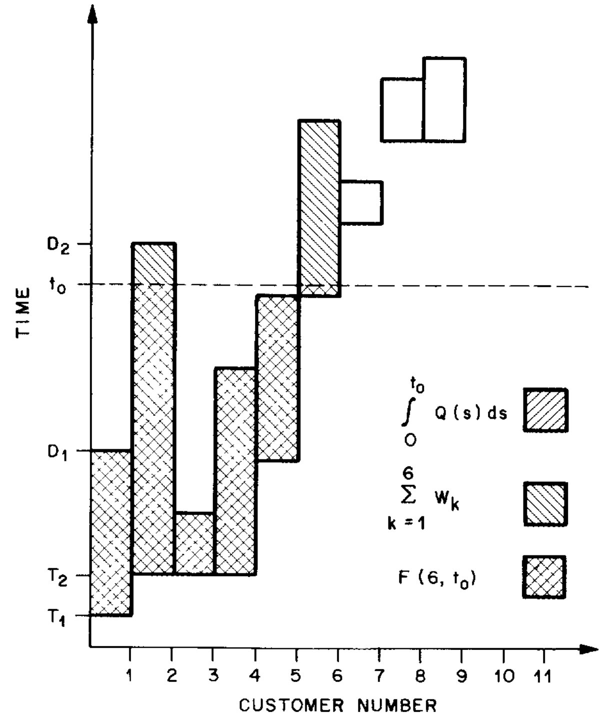


**Figure 1.** Cumulative processes associated with $Q(t)$ and $W_k$.

### 1.3. $H = \lambda G$

To obtain the version of $H = \lambda G$ in Heyman and Stidham, let $f_k(t)$ be a cost rate associated with the $k$th customer at time $t$. Assume that $f_k(t)$ is nonnegative and integrable. This becomes a special case of Section 1.1 by letting

$$F_k(t) = \int_0^t f_k(s)\, ds, \quad t \geq 0. \tag{8}$$

Then $G(n) = n^{-1} \sum_{k=1}^{n} F_k(\infty)$ is the average cost associated with the first $n$ customers and $H(t) = t^{-1} \sum_{k=1}^{\infty} F_k(t)$ is the average cost incurred over the interval $[0, t]$. As a special case, we might have cost rates in the $L = \lambda W$ setting of Section 1.2. We might start with the (integrable) cost rate $c_k(t)$ associated with the $k$th customer $t$ time units after his arrival, when he is in the system. Paralleling (7), we have

$$f_k(t) = c_k(t - T_k)I_{[T_k, D_k]}(t), \quad t \geq 0. \tag{9}$$

### 1.4. Nonintegral Formulation: Lump Costs Plus Cost Rate

In the setting of Section 1.3, suppose that a lump cost $C_k$ is incurred at the instant (if it occurs) that the waiting time of the $k$th customer exceeds $x$. (For $x = 0$, this represents a lump cost per customer upon arrival.) If

$$F_k(t) = \int_0^t f_k(s)\, ds + C_k I_{[x, \infty)}(W_k) I_{[T_k + x, \infty)}(t), \quad t \geq 0 \tag{10}$$

where $I_A(t)$ is the indicator function of $A$ as before, then $G_n = n^{-1} \sum_{k=1}^{n} F_k(\infty)$ is the average cost for the first $n$ customers and $H(t) = t^{-1} \sum_{k=1}^{\infty} F_k(t)$ is the average cost incurred in the interval $[0, t]$. This example is not covered by any of the previous versions of $H = \lambda G$ because $F_k(t)$ in (10) as a function of $t$ is not absolutely continuous with respect to the Lebesgue measure, i.e., (10) is not of the form (8).

**Remark 1.** We should not overemphasize this point because we can transform (10) into the integral form (8). To see this, recall that if $B$ is a cdf (cumulative distribution function) and $U$ is a random variable uniformly distributed on $[0, 1]$, then $B^{-1}(U)$ has cdf $B$ where $B^{-1}(s) = \inf\{u \geq 0: B(u) > s\}$ as in (3). Hence, if $B$ is any nondecreasing right-continuous function with $B(0) = 0$ and $f$ is integrable with respect to $B$, then

$$\int_{[0, t]} f(s)\, dB(s) = \int_0^{B(t)} f(B^{-1}(s))\, ds = \int_0^t f\left(B^{-1}\left[\frac{B(t)}{t} s\right]\right) \frac{B(t)}{t}\, ds$$

by a change of variables. However, the integrand is quite different when we make this transformation, and this can significantly complicate subsequent analysis.

### 1.5. Other Nonintegral Formulations: General Cost Processes

In the spirit of (9), there might be a general cumulative cost function $C_k(t)$ associated with the $k$th customer $t$ time units after arrival at $T_k$, when the customer is in the system. In the original stochastic setting, $\{[C_k(t): t \geq 0]: k \geq 1\}$ might be a sequence (indexed by $k$) of independent and identically distributed stochastic processes with nondecreasing sample paths. (Of course, we do not require these stochastic assumptions.) This is fit into the setting of Section 1.1 by setting

$$F_k(t) = C_k(t - T_k)I_{[T_k, D_k]}(t), \quad t \geq 0. \tag{11}$$

**Example 1.** Customers enter a store at time $\{T_k\}$ and leave at times $\{D_k\}$. The cumulative amount purchased by customer $k$ is modeled by a stochastic process $\{C_k(t): t \geq 0\}$ where $t$ represents the time after entering the store. The amount purchased by the $k$th customer by time $t$ is then $F_k(t)$ in (11). ($D_k$ and $\{C_k(t): t \geq 0\}$ might be highly dependent.) The average amount purchased per customer throughout all time by the first $n$ customers is $G(n)$; the average amount purchased per time by all customers up to time $t$ is $H(t)$.

**Example 2.** Shipments of a perishable commodity (e.g., blood) enter a storage facility at times $\{T_k\}$. The total amount lost from shipment $k$ by $t$ time units after arrival is modeled as a stochastic process $\{C_k(t): t \geq 0\}$. The amount lost from shipment $k$ by time $t$ is then $F_k(t)$ in (11), where $D_k$ represents a time after all of shipment $k$ perishes. The average amount eventually lost per shipment among the first $n$ shipments is $G(n)$; the average amount lost per time in all shipments by time $t$ is $H(t)$.

### 1.6. Continuous Analogs

A continuous analog of Section 1.1 is obtained by starting with a continuous family of nonnegative, nondecreasing real-valued functions $\{[F_u(t): t \geq 0]: u \geq 0\}$ and a nondecreasing right-continuous function $\{T_1(s): s \geq 0\}$ with $T_1(s) < \infty$ for all $s$ and $T_1(s) \to \infty$ as $s \to \infty$. We are interested in the limiting behavior of

$$G(s) = s^{-1} \int_0^s F_u(\infty)\, du$$

and (12)

$$H(t) = t^{-1} \int_0^\infty F_u(t)\, du.$$

For (12) to be well defined we assume that $F_u(t)$ is integrable in $u$ for each $t$. We fit this in the general framework by letting $F(s, t) = \int_0^s F_u(t)\, du$ for $s \geq 0$ and $t \geq 0$. We obtain the special cases considered by

Rolski and Stidham when $F_u(t)$ is an integral

$$F_u(t) = \int_0^t f(u, v)\, dv \tag{13}$$

where $f(u, v)$ is a nonnegative integrable function on $[0, \infty) \times [0, \infty)$.

### 1.7. Stochastic Integrals

Let $\{S_1(t): t \geq 0\}$ be a stochastic process with nondecreasing sample paths; let $\{B(s, t): s \geq 0, t \geq 0\}$ be a stochastic process with sample paths that are nondecreasing in $t$ for each $s$; and let $F(s, t)$ be the stochastic integral

$$F(s, t) = \int_0^s B(u, t - u) S_1(du)$$

which we assume is well defined.

**Example 3.** Water contaminated with radioactive elements pours into a river. Let $S_1(t)$ represent the total amount of water that pours into the river by time $t$, and let $B(s, t - s)S_1(ds)$ represent the amount of radioactivity emitted by time $t$ by water that arrives in the interval $[s, s + ds)$. Then $F(s, t)$ is the stochastic integral, $F(s, \infty)$ is the total radioactivity that will be emitted by water that arrives by time $s$, and $F(\infty, t) = F(t, t)$ is the total radioactivity emitted by time $t$.

To further develop the example, suppose that $B(s, t - s) = c(s)b(t - s)$ where $b(t) \to b(\infty)$ as $t \to \infty$. (We probably cannot assume that $B(s, t - s) = \alpha(1 - e^{-\lambda(t-s)})$ because, typically, there will be many different radioactive elements in the water, each with different decay rates.) If we let $\hat{S}_1(du) = c(u)S_1(du)$, then $F(s, t) = \int_0^s b(t - u)\hat{S}_1(du)$, $F(s, \infty) = b(\infty)\hat{S}_1(s)$ and $F(\infty, t) = \int_0^\infty b(t - u)\hat{S}_1(du)$. If $s^{-1}\hat{S}_1(s) \to \lambda$ as $s \to \infty$, then $G(s) = s^{-1}F(s, \infty) \to \lambda b(\infty)$ as $s \to \infty$. The results in this paper could be applied to establish convergence of $H(t) = t^{-1}F(\infty, t)$.

### 1.8. Multivariate Extensions

The general framework can be generalized to more than two dimensions. Here is the idea: Consider the nondecreasing function $F(t_1, \ldots, t_k)$ defined on $[0, \infty)^k$, the $k$-fold product of $[0, \infty)$ with itself. Let $F_i(t_i) = F(t_1, \ldots, t_k)$ with $t_j = \infty$ for all $j \neq i$. Under appropriate conditions, the limit $G_i$ of $G_i(t_i) = t_i^{-1}F_i(t_i)$ as $t_i \to \infty$ for one $i$ determines the limit for all $i$ and $G_i = (\alpha_i/\alpha_j)G_j$. The idea is that $F(t_1, \ldots, t_n)$ has its primary region of increase along some ray. For example, the domain could be made five-dimensional by adding three spatial variables. (Imagine a queue inside a spaceship heading toward outer space in a straight line at constant velocity, and ask for the average number in queue per distance along a particular spatial coordinate, over all customers, all time and all values of the other spatial coordinates.) The symmetry in our general framework makes this extension apparent. It is not difficult to extend Sections 2 and 5 to this setting.

## 2. THE MAIN THEOREM

Recall the general framework for $H = \lambda G$ introduced at the beginning of Section 1 with the cumulative input function $F(s, t)$ and the time-change function $T_1(s)$. We actually consider two nonnegative, nondecreasing, right-continuous real-valued functions on $[0, \infty)$, $T_1(s)$ and $T_2(s)$, with inverses $S_1(t)$ and $S_2(t)$ defined by (3). We could, of course, have $T_1(s) = T_2(s)$ for all $s$, so that there really is only one function, but we usually do not. We will make assumptions equivalent to requiring that $T_1(s)/T_2(s) \to 1$ as $s \to \infty$. (In the standard $L = \lambda W$ framework, $T_1(s) = T_{[s]}$ where $T_k$ is the arrival epoch of the $k$th customer and we could have $T_2(s) = T_{[s]} + U_{[s]}$ where $U_k$ is the maximum waiting time among the first $k$ customers; see (19) and Section 4.1.)

We use the following familiar elementary lemma relating the limits of the averages (the strong laws of large numbers in the stochastic setting) for $T_i(s)$ and $S_i(t)$, e.g., see Theorem 2(a) of Glynn and Whitt (1986).

**Lemma 1.** *$t^{-1}S_i(t) \to \lambda$, $0 < \lambda < \infty$, as $t \to \infty$ if and only if $s^{-1}T_i(s) \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, as $s \to \infty$.*

We use the following elementary composition property. Let $f(t-)$ be the left limit of $f(s)$ as $s$ approaches $t$ from below.

**Lemma 2.** *For all $t \geq 0$, $T_i(S_i(t)) \geq t \geq T_i(S_i(t)-)$.*

To establish our result, we use two approximation conditions:

$$\lim_{s\to\infty} s^{-1}[F(s, T_1(s-)) - F(\infty, T_1(s-))] = 0 \tag{14}$$

and

$$\lim_{s\to\infty} s^{-1}[F(s, \infty) - F(s, T_2(s))] = 0. \tag{15}$$

We rely heavily on (14) and (15) for our results. Roughly speaking, together they imply that

$$G(s) - H(T_1(s-)) = o(s) \quad \text{as } s \to \infty \tag{16}$$

provided that $T_2(s) \approx T_1(s-)$ because $G(s) = F(s, \infty)$ and $H(T_1(s-)) = F(\infty, T_1(s-))$.

Our main result gives inequalities for *lim infs* ($\underline{\lim}$) and *lim sups* ($\overline{\lim}$) as well as statements closer to the previous versions of $H = \lambda G$.

**Theorem 1.** (a) *If* $s^{-1}T_1(s) \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, *and* (14) *holds, then*

$$\underline{\lim}_{t\to\infty} G[S_1(t)] \leq \underline{\lim}_{t\to\infty} \lambda^{-1}H(t)$$

*and*

$$\overline{\lim}_{s\to\infty} G(s) \leq \overline{\lim}_{t\to\infty} G[S_1(t)] \leq \overline{\lim}_{t\to\infty} \lambda^{-1}H(t).$$

(b) *If* $s^{-1}T_2(s) \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, *and* (15) *holds, then*

$$\underline{\lim}_{t\to\infty} H(t) \leq \underline{\lim}_{t\to\infty} \lambda G[S_2(t)] \leq \underline{\lim}_{s\to\infty} \lambda G(s)$$

*and*

$$\overline{\lim}_{t\to\infty} H(t) \leq \overline{\lim}_{t\to\infty} \lambda G[S_2(t)].$$

(c) *If the conditions of both* (a) *and* (b) *hold, then*

$$\underline{\lim}_{t\to\infty} H(t) = \underline{\lim}_{s\to\infty} \lambda G(s) \quad \textit{and} \quad \overline{\lim}_{t\to\infty} H(t) = \overline{\lim}_{s\to\infty} \lambda G(s).$$

(d) *Under the conditions of* (a), *if either* $H(t) \to H$ *as* $t \to \infty$ *or* $G(s) \to G$ *as* $s \to \infty$, *then*

$$\underline{\lim}_{t\to\infty} H(t) \geq \overline{\lim}_{s\to\infty} \lambda G(s).$$

(e) *Under the conditions of* (b), *if either* $H(t) \to H$ *as* $t \to \infty$ *or* $G(s) \to G$ *as* $s \to \infty$, *then*

$$\overline{\lim}_{t\to\infty} H(t) \leq \underline{\lim}_{s\to\infty} \lambda G(s).$$

(f) *If the conditions of both* (a) *and* (b) *hold, then* $H(t) \to H$ *as* $t \to \infty$ *if and only if* $G(s) \to G$ *as* $s \to \infty$, *in which case* $H = \lambda G$.

**Proof.** We need to provide details for only (a) and (b). (a) Apply Lemma 1 to get $t^{-1}S_1(t) \to \lambda$. Since $t \geq T_1[S_1(t)-]$ by Lemma 2

$$\frac{tH(t)}{S_1(t)} \geq \frac{F(S_1(t), T_1(S_1(t)-))}{S_1(t)} \geq G(S_1(t)) + \frac{F(S_1(t), T_1(S_1(t)-) - F(\infty, T_1(S_1(t)-)}{S_1(t)}$$

so that under (14)

$$\underline{\lim}_{t\to\infty} H(t) \geq \underline{\lim}_{t\to\infty} \lambda G(S_1(t))$$

and

$$\overline{\lim}_{t\to\infty} H(t) \geq \overline{\lim}_{t\to\infty} \lambda G(S_1(t)).$$

Since $S_1[T_1(s)] \geq s$ by Lemma 2

$$G(S_1[T_1(s)]) \geq G(s)(s/S_1[T_1(s)]).$$

By Lemma 2, $s^{-1}S_1[T_1(s)] \to 1$ as $s \to \infty$. Hence, $\overline{\lim}_{t\to\infty} G(S_1(t)) \geq \overline{\lim}_{s\to\infty} G(s)$.

(b) Apply Lemma 1 to get $t^{-1}S_2(t) \to \lambda$ as $t \to \infty$. Since $t \leq T_2[S_2(t)]$ by Lemma 2

$$\frac{tH(t)}{S_2(t)} = \frac{F(S_2(t), \infty) + F(S_2(t), t) - F(S_2(t), \infty)}{S_2(t)}$$

$$\leq G(S_2(t)) + \frac{F(S_2(t), T_2[S_2(t)] - F(S_2(t), \infty)}{S_2(t)}$$

so that by (15)

$$\underline{\lim}_{t\to\infty} H(t) \leq \underline{\lim}_{t\to\infty} \lambda G(S_2(t))$$

and

$$\overline{\lim}_{t\to\infty} H(t) \leq \overline{\lim}_{t\to\infty} \lambda G(S_2(t)).$$

Since $S_2(T_2(s-)) \leq s$ by Lemma 2, $G(S_2[T_2(s-)]) \leq G(s)(s/S_2[T_2(s-)])$. By Lemma 1, $s^{-1}S_2[T_2(s-)] \to 1$ as $s \to \infty$. Hence, $\underline{\lim}_{t\to\infty} G(S_2(t)) \leq \underline{\lim}_{s\to\infty} G(s)$.

**Remark 2.** Parts a and b of Theorem 1 are important as separate statements because sometimes only one of (14) and (15) holds; see Example 1 of Glynn and Whitt (1986).

**Remark 3.** The standard versions of $H = \lambda G$ in the literature establish the limit for $H(t)$ given the limit for $G(s)$. Our Theorem 1 goes both ways. In fact, to a large extent, the formulation in this section is symmetric in the two variables $s$ and $t$. We thus can replace conditions (14) and (15) with

$$\lim_{t\to\infty} t^{-1}[F(S_1(t-), t) - F(S_1(t-), \infty)] = 0$$

and

$$\lim_{t\to\infty} t^{-1}[F(\infty, t) - F(S_2(t), t)] = 0$$

and obtain corresponding results. However, note that (14) and (15) are not symmetric.

**Remark 4.** Theorem 1 remains valid with weaker conditions than (14) and (15). As easily seen from the proof, it suffices to have, for each $\epsilon > 0$, functions $T_1(s)$ and $T_2(s)$, such that $s^{-1}T_i(s) \to \lambda^{-1}$,

$0 < \lambda^{-1} < \infty$, and

$$\overline{\lim_{s\to\infty}}\, s^{-1}[F(\infty, T_1(s-) - F(s, T_1(s-))] < \epsilon \tag{14'}$$

and

$$\overline{\lim_{s\to\infty}}\, s^{-1}[F(s, \infty) - F(s, T_2(s))] < \epsilon. \tag{15'}$$

Equivalently, we can work with sequences $\{[T_{2n}(s): s \geq 0]: n \geq 1\}$, e.g., (15′) is equivalent to a sequence $\{T_{2n}(s)\}$ such that

$$\lim_{n\to\infty} \overline{\lim_{s\to\infty}}\, s^{-1}[F(s, \infty) - F(s, T_{2n}(s))] = 0. \tag{15''}$$

This extension of (15), but not (14), is used in Section 4.5 to cover conditions III and III′ of Rolski and Stidham for the continuous versions of $L = \lambda W$ and $H = \lambda G$.

## 3. RESULTS WITH ONE DISCRETE VARIABLE

In the setting of Section 1.1, we use the following assumptions.

$$\text{(i)} \quad n^{-1} \sum_{k=n}^{\infty} F_k(T_n) \to 0 \quad \text{as } n \to \infty. \tag{17}$$

(ii) There exists a nonnegative finite sequence $\{S_k: k \geq 0\}$ such that

$$\frac{S_n}{T_n} \to 0 \quad \text{and} \quad n^{-1} \sum_{k=1}^{n} [F_k(\infty) - F_k(T_k + S_k)] \to 0 \quad \text{as } n \to \infty. \tag{18}$$

**Remark 5.** Condition ii in (18) is a slight generalization of the assumption (i and ii) on p. 985 of Heyman and Stidham.

We define $T_2(s)$ in terms of $T_k + S_k$ by $T_2(s) = T_{[s]} + U_{[s]}$, $s \geq 0$, where

$$U_n = \max\{S_k: 1 \leq k \leq n\}. \tag{19}$$

Thus $S_2(t) = M(t) + 1$, where

$$M(t) = \max\{k \geq 0: T_k + U_k \leq t\}, \quad t \geq 0. \tag{20}$$

The next two theorems show that Theorem 1 applies here. We simply relate conditions (17) and (18) to (14) and (15).

**Theorem 2.** *In the setting of Section* 1.1, (17) *is equivalent to* (14).

**Proof.** For $n < s < n + 1$,

$$\begin{aligned} n^{-1} \sum_{k=n+1}^{\infty} F_k(T_n) &= n^{-1}\left[\sum_{k=1}^{\infty} F_k(T_n) - \sum_{k=1}^{n} F_k(T_n)\right] \\ &= ([s])^{-1}[F(\infty, T_1(s-)) - F(s, T_1(s-))] \end{aligned}$$

while, for $s = n$, since $T_1(s-)$ is left-continuous

$$\begin{aligned} n^{-1} \sum_{k=n}^{\infty} F_k(T_{n-1}) &= n^{-1}\left[\sum_{k=1}^{\infty} F_k(T_{n-1}) - \sum_{k=1}^{n} F_k(T_{n-1})\right] \\ &= s^{-1}[F(\infty, T_1(s-)) - F(s, T_1(s-))]. \end{aligned}$$

Hence, (17) and (14) are equivalent.

We need the following lemma for relating (18) to (15).

**Lemma 3.** *If* $S_n/T_n \to 0$, *then* $U_n/T_n \to 0$ *for* $U_n$ *in* (19).

**Proof.** For arbitrary $\epsilon > 0$, let $n(\epsilon)$ be such that $S_n/T_n < \epsilon$ for $n > n(\epsilon)$. Then, for $n > n(\epsilon)$

$$\begin{aligned} U_n/T_n &= U_{n(\epsilon)}/T_n + \max\{S_k/T_n: n(\epsilon) < k \leq n\} \\ &\leq U_{n(\epsilon)}/T_n + \max\{S_k/T_k: n(\epsilon) < k \leq n\} \\ &\leq U_{n(\epsilon)}/T_n + \epsilon \to \epsilon \quad \text{as } n \to \infty. \end{aligned}$$

Since $\epsilon$ was arbitrary, the proof is complete.

**Theorem 3.** *In the setting of Section* 1.1, (18) *implies* (15) *with* $\lim_{s\to\infty} s^{-1}T_2(s) = \lim_{n\to\infty} n^{-1}T_n$ *whenever the latter limit exists.*

**Proof.** Using $U_n$ in (15), we have for $n \leq s < n + 1$

$$\begin{aligned} n^{-1} \sum_{k=1}^{n} [F_k(\infty) - F_k(T_k + S_k)] &\geq n^{-1} \sum_{k=1}^{n} [F_k(\infty) - F_k(T_k + U_k)] \\ &\geq n^{-1} \sum_{k=1}^{n} [F_k(\infty) - F_k(T_n + U_n)] \\ &\geq s^{-1}[F(s, \infty) - F(s, T_2(s))] \end{aligned}$$

with $T_2(n) = T_n + U_n$. By Lemma 3, $U_n/T_n \to 0$ as $n \to \infty$, so that $s^{-1}T_2(s)$ has the same limiting behavior as $s \to \infty$ as $n^{-1}T_n$ as $n \to \infty$.

The next result is an immediate consequence of Theorems 1 through 3.

**Theorem 4.** *Consider the setting of Section* 1.1 *with one discrete variable.*

(a) *Suppose that* $n^{-1}T_n \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, *and* $G(n) \to G$ *as* $n \to \infty$.
  (i) *If* (17) *holds, then* $\overline{\lim}_{t\to\infty} H(t) \geq \lambda G$.
  (ii) *If* (18) *holds, then* $\underline{\lim}_{t\to\infty} H(t) \leq \lambda G$.
  (iii) *If* (17) *and* (18) *both hold, then* $\lim_{t\to\infty} H(t) = H = \lambda G$.

(b) *Suppose that* $t^{-1}N(t) \to \lambda$, $0 < \lambda < \infty$, *and* $H(t) \to H$ *as* $t \to \infty$.
  (i) *If* (17) *holds, then* $\overline{\lim}_{n\to\infty} G(n) \leq \lambda^{-1}H$.
  (ii) *If* (18) *holds, then* $\underline{\lim}_{n\to\infty} G(n) \geq \lambda^{-1}H$.
  (iii) *If* (17) *and* (18) *both hold, then*

$$\lim_{n\to\infty} G(n) = G = \lambda^{-1}H.$$

## 4. SPECIAL CASES

In this section, we discuss the special cases in Sections 1.2–1.5.

### 4.1. $L = \lambda W$

In the setting of Section 1.2, $G(n) = n^{-1}\sum_{k=1}^{n} F_k(\infty)$ is the customer-average waiting time and $H(t) = t^{-1}\sum_{k=1}^{\infty} F_k(t)$ is the time-average queue length (number in the system). This gives us a special case of Section 3 with $L$ and $W$ playing the roles of $H$ and $G$, respectively.

To establish conditions for $L = \lambda W$, for (18) let $S_n = W_n$, $n \geq 1$. Note that

$$F_k(T_n) = 0 \quad \text{for all } k \geq n \tag{21}$$

and

$$F_k(\infty) - F_k(T_k + S_k) = 0 \quad \text{for all } k \tag{22}$$

so that (17) always holds and (18) holds if and only if $W_n/T_n \to 0$. When $n^{-1}T_n \to \lambda^{-1}$, $0 < \lambda^{-1} < \infty$, (18) reduces to $n^{-1}W_n \to 0$. Just as in Theorem 2 of Glynn and Whitt (1986), if $n^{-1}\sum_{k=1}^{n} W_k \to W$ as $n \to \infty$, we automatically get $n^{-1}W_n \to 0$; otherwise it needs to be assumed; see Example 1 of Glynn and Whitt (1986). Here we obtain a new result from Theorem 4b (ii). We summarize all implications starting from the limits for the time averages $t^{-1}S_1(t)$ and $H(t)$ to $\lambda$ and $L$. Let $Q(t)$ be the queue length (number in the system) at time $t$, defined by $Q(t) = \sum_{k=1}^{\infty} I_{[T_k, D_k]}(t)$.

**Theorem 5.** (a) *In the* $L = \lambda W$ *framework of Section* 1.2, *if the time-averages* $t^{-1}S_1(t)$ *and* $H(t)$ *converge to* $\lambda$ *and* $L$ *as* $t \to \infty$ *where* $0 < \lambda < \infty$, *then*

$$\underline{\lim}_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k \geq \lambda^{-1}L.$$

(b) *If, in addition, there exists an increasing sequence* $\{t_k: k \geq 1\}$ *with* $t_k \to \infty$ *such that* $Q(t_k) = 0$ *for all k, then*

$$\underline{\lim}_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k = \lambda^{-1}L.$$

(c) *If* $n^{-1}W_n \to 0$ *as* $n \to \infty$ *in addition to the other assumptions of* (a), *then*

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k = \lambda^{-1}L.$$

**Proof.** Part a follows directly from Theorem 4b (ii). Part b follows from Corollary 1.1 of Glynn and Whitt (1986). Given the time-average limit for $\lambda$ and Lemma 1, the assumption in c is just what is needed to have both (17) and (18), so that we can apply Theorem 4b (iii).

**Remark 6.** Theorem 5 is illustrated by Example 1 of Glynn and Whitt (1986). The conditions of Theorem 5a and b hold, but not c. There

$$\overline{\lim}_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k = 2 > 1 = \underline{\lim}_{n\to\infty} n^{-1}\sum_{k=1}^{n} W_k = \lambda^{-1}H$$

as can be shown directly.

### 4.2. $H = \lambda G$

Consider the setting of Section 1.3. Usually, $f_k(t) = 0$ for $t \notin [T_k, D_k]$. Then we set $S_n = W_n$, as in Section 4.1. Otherwise, conditions (17) and (18) can be applied directly. For example, they are implied by assumptions i and ii on p. 985 of Heyman and Stidham. In contrast to Theorem 1 of Heyman and Stidham, we do not require that $G < \infty$ or $\int_0^\infty f_k(t)\,dt < \infty$.

### 4.3. A Nonintegral Formulation

Consider the new examples in Sections 1.4 and 1.5. As in Sections 4.1 and 4.2, if $f_k(t) = 0$ for $t \notin [T_k, D_k]$ as in (9), then we can set $S_n = W_n$, so that conditions (17) and (18) are satisfied for $F_k(t)$ in (10) if $W_n/T_n \to 0$ as $n \to \infty$. Similarly, (17) and (18) are satisfied for $F_k(t)$ in (11) if $W_n/T_n \to 0$ as $n \to \infty$. Otherwise apply conditions (17) and (18) directly.

### 4.4 Continuous Analogs

Now consider the setting of Section 1.6. Paralleling Section 3, we use two assumptions:

$$\text{(i)}\quad s^{-1}\int_s^{\infty} F_u(T_1(s))\,du \to 0 \quad \text{as } s\to\infty \tag{23}$$

and

(ii) there exists a nondecreasing right-continuous function $T_2(s)$ such that

$T_2(s)/T_1(s) \to 1$

and

$$s^{-1}\int_0^s [F_u(\infty) - F_u(T_2(u))]\,du \to 0 \quad \text{as } s\to\infty. \tag{24}$$

The results of Section 3 and the examples in Section 4 easily carry over to this setting. In particular, the analog of Theorem 4 is immediate.

With the additional integral representation (13), condition (23) holds by the assumption that $f(u, v) = 0$ for $v < T_1(u)$. As in Sections 4.1 and 4.2, the second part of (24) holds by assuming that $f(u, v) = 0$ for $v > T_2(u)$, but satisfying (24) can be nontrivial. We first discuss the standard elementary case and then a generalization.

In the $L = \lambda W$ setting in Section 2 of Rolski and Stidham

$$f(u, v) = I_{[T_1(u),\,T_1(u)+w(u)]}(v) \tag{25}$$

where $w(u)$ is the waiting time at the $u$-arrival epoch $T_1(u)$, which we assume is right-continuous. Then, paralleling Section 3, the second, nondecreasing right-continuous function $T_2(s)$ can be defined by

$$T_2(s) = T_1(s) + \sup_{0\le u\le s} w(u). \tag{26}$$

The function $T_2(s)$ corresponds to the sequence $\{T_n + U_n\colon n \ge 0\}$ in Section 3.

As in Sections 4.1 and 4.2, conditions (23) and (24) reduce to the requirement that $T_2(s)/T_1(s) \to 1$ as $s\to\infty$ for $T_2(u)$ such that $F_u(\infty) - F_u(T_2(u)) = 0$ for all $u$. Instead of the rather complex condition (III) on p. 212 of Rolski and Stidham, we simply use $\lim_{s\to\infty} s^{-1}w(s) = 0$. We summarize our results for the continuous $L = \lambda W$ case in the following theorem.

**Theorem 6.** *In the continuous $L = \lambda W$ setting, i.e., under* (13) *and* (25), *assume that $s^{-1}T_1(s) \to \lambda^{-1}$ and $s^{-1}w(s) \to 0$ as $s\to\infty$. Then $\lim_{n\to\infty} G(n) = G \equiv W$ if and only if $\lim_{t\to\infty} H(t) = H \equiv L$, in which case $L = \lambda W$.*

**Proof.** The conditions imply, first, that

$s^{-1}\sup_{0\le u\le s} w(u) \to$

and, second, that $T_2(s)/T_1(s) \to 1$ as $s\to\infty$ with $T_2(s)$ defined by (26). Hence, both conditions (23) and (24) hold. Then apply the analog of Theorem 4a (iii) and b (iii).

**Remark 7.** Unlike the discrete case in Section 4.1, the convergence $t^{-1}\int_0^t w(u)\,du \to w$ as $t\to\infty$ with $w < \infty$ does not imply that $t^{-1}w(t) \to 0$ as $t\to\infty$; see Example 2 of Rolski and Stidham.

**Remark 8.** Theorem 6 does not cover the rather pathological Example 1 of Rolski and Stidham in which $T_1(s) = s$ and $w(s) = nI_{[(2^n-1)/2^n,\,n^{-1})}(s)$, but it is easy to verify (23) and (24) directly using $T_2(s) = T_1(s) = s$, $s \ge 0$. The condition $t^{-1}w(t) \to 0$ in Theorem 6 seems more natural than III in Rolski and Stidham. More generally, note that condition (24), extended as indicated in Remark 4, actually coincides with condition III in Rolski and Stidham in the special case of (26). To see this, observe that

$$\begin{aligned} &F_u(T_2(u))) \\ &= \int_0^{T_2(u)} 1_{[T_1(u),\,T_1(u)+w(u)]}(v)\,dv \\ &= \min\{w(u), [T_2(u) - T_1(u)]^+\} \\ &\le \min\{w(u), d(u)\} \end{aligned}$$

where $d(u) = \sup_{0\le v\le u}\{[T_2(v) - T_1(v)]^+\}$. It is easy to see that $d(u)$ is nondecreasing and $u^{-1}d(u) \to 0$ because $T_2(u)/T_1(u) \to 1$. As indicated by Rolski and Stidham, their more general condition III has appeal because it is satisfied w.p.1 whenever $\{w(t)\colon t \ge 0\}$ is the sample path of a nonnegative ergodic stationary stochastic process with finite mean.

## 5. CENTRAL LIMIT THEOREM VERSIONS OF $H = \lambda G$

Glynn and Whitt (1986) show that in the $L = \lambda W$ framework, relations exist between other classical limit theorems for the averages besides the standard strong laws of large numbers. This is also true for the more general framework here, as we illustrate with the central limit theorem (CLT). Our CLT version of $H = \lambda G$ is a generalization of Theorem 3 of Glynn and Whitt (1986). Since the methods are similar, we will not state analogs of the other theorems there and we will omit much of the proof here. These extensions are important for identifying efficient statistical estimators, as shown in Glynn and Whitt (1989).

We actually establish a relation among functional central limit theorems (FCLTs). As in Section 3 of Glynn and Whitt (1986), we consider random elements of the function space $D \equiv D[0, \infty)$, the space of all real-valued right-continuous functions on $[0, \infty)$ with left limits, endowed with the usual Skorohod ($J_1$) topology; see Whitt (1980). Let $C \equiv C[0, \infty)$ be the subset of continuous functions. Let $\Rightarrow$ denote weak convergence.

In this section, we abandon the purely deterministic treatment employed so far. We now regard $\{F(s, t): s \geq 0,\ t \geq 0\}$ as a stochastic process with a two-dimensional time parameter; e.g., Straf (1972). However, as in Whitt, and Glynn and Whitt (1986), the proof here also can be done deterministically by sample-path methods.

We define the following random elements of $D$.

$$\begin{aligned}
\mathbf{G}_n(t) &= n^{-1/2}[G(nt) - gnt]\\
\mathbf{H}_n(t) &= n^{-1/2}[H(nt) - hnt]\\
\mathbf{T}^i_n(t) &= n^{-1/2}[T_i(nt) - \lambda^{-1}nt] \qquad (27)\\
\mathbf{S}^i_n(t) &= n^{-1/2}[S_i(nt) - \lambda nt]\\
\theta(t) &= 0 \quad \text{and} \quad \mathbf{e}(t) = t, \quad t \geq 0
\end{aligned}$$

where $\lambda$, $h$ and $g$ are positive real numbers. We assume that $\mathbf{G}_n(t)$ and $\mathbf{H}_n(t)$ are legitimate random elements of $D$. Let $\mathbf{R}^1_n$ and $\mathbf{R}^2_n$ be remainder processes in $D$, based on (14) and (15), that is

$$\begin{aligned}
\mathbf{R}^i_n(t) &= \lim_{\epsilon \to 0} \sup_{0 \leq s \leq t+\epsilon} n^{-1/2} R_i(ns)\\
R_1(t) &= |F(\infty, T_1(t-)) - F(t, T_1(t-))| \qquad (28)\\
R_2(t) &= |F(t, \infty), - F(t, T_2(t))|
\end{aligned}$$

for $t \geq 0$.

**Remark 9.** The supremum is included in $\mathbf{R}^i_n$ above to ensure that $\mathbf{R}^i_n$ is an element of $D$. Otherwise, it entails no extra conditions, because if $\mathbf{X}_n \Rightarrow \mathbf{X}$ in $D$ with $P(\mathbf{X} \in C) = 1$, then $f(\mathbf{X}_n) \Rightarrow f(\mathbf{X})$ by the continuous mapping theorem where $f: D \to D$ is the mapping $f(x)(t) = \sup_{0 \leq s \leq t+\epsilon} x(s)$ for $x \in D$.

**Theorem 7.** *Suppose that* $h = \lambda g$ *and* $\mathbf{R}^i_n \Rightarrow 0$ *for* $i = 1, 2$. *Then* $(\mathbf{G}_n, \mathbf{T}^1_n, \mathbf{T}^2_n) \Rightarrow (\mathbf{G}, \mathbf{T}, \mathbf{T})$ *with* $P(\mathbf{T} \in C) = 1$ *and only if* $(\mathbf{H}_n, \mathbf{S}^1_n, \mathbf{S}^2_n) \Rightarrow (\mathbf{H}, \mathbf{S}, \mathbf{S})$ *with* $P(\mathbf{S} \in C) = 1$, *in which case*

$$\begin{aligned}
&(\mathbf{G}_n, \mathbf{T}^1_n, \mathbf{T}^2_n, \mathbf{H}_n, \mathbf{S}^1_n, \mathbf{S}^2_n)\\
&\quad \Rightarrow (\mathbf{G}, \mathbf{T}, \mathbf{T}, \mathbf{H}, \mathbf{S}, \mathbf{S}) \text{ in } D^6 \qquad (29)
\end{aligned}$$

*where*

$$\begin{aligned}
\mathbf{S}(t) &= -\lambda\mathbf{T}(\lambda t) \stackrel{d}{=} \lambda^{3/2}\mathbf{T}(t)\\
\mathbf{H}(t) &= \mathbf{G}(\lambda t) - h\mathbf{T}(\lambda t) \qquad (30)\\
&\stackrel{d}{=} \lambda^{1/2}(\mathbf{G}(t) - h\mathbf{T}(t)), \quad t \geq 0.
\end{aligned}$$

**Proof.** The idea is to apply the continuous mapping theorem and related arguments, as for Theorem 3 of Glynn and Whitt (1986). Paralleling Lemma 1 here, $\mathbf{T}^i_n \Rightarrow \mathbf{T}$ with $P(\mathbf{T} \in C)$ if and only if $\mathbf{S}^i_n \Rightarrow \mathbf{S}$ with $P(\mathbf{S} \in C)$, in which case $(\mathbf{T}^i_n, \mathbf{S}^i_n) \Rightarrow (\mathbf{T}, \mathbf{S})$; where $\mathbf{S}$ and $\mathbf{T}$ are related as in (30); see Theorem 7.3 plus the corollary to Lemma 7.6 of Whitt. Let $\Psi^i_n \equiv \Psi^i_n(t) = n^{-1}S_i(nt)$, $t \geq 0$. Suppose that $(\mathbf{G}_n, \mathbf{T}^1_n, \mathbf{T}^2_n) \Rightarrow (\mathbf{G}, \mathbf{T}, \mathbf{T})$. Then

$$\begin{aligned}
&(\mathbf{G}_n, \mathbf{T}^1_n, \mathbf{T}^2_n, \mathbf{S}^1_n, \mathbf{S}^2_n, \Psi^1_n, \Psi^2_n)\\
&\quad \Rightarrow (\mathbf{G}, \mathbf{T}, \mathbf{T}, \mathbf{S}, \mathbf{S}, \lambda\mathbf{e}, \lambda\mathbf{e}) \quad \text{in } D^7.
\end{aligned}$$

Following the proof of Theorem 1, we obtain

$$\begin{aligned}
&G(S_1(nt)) - R_1(S_1(nt))\\
&\quad \leq H(nt) \leq G(S_2(nt)) + R_2(S_2(nt)) \qquad (31)
\end{aligned}$$

for all $t \geq 0$. Hence, as in the proof of Theorem 3 of Glynn and Whitt (1986), $\mathbf{H}_n$ has the same weak convergence limit as the random function

$$(\mathbf{GS}_i)_n(t) = n^{1/2}[G(S_i(nt)) - \lambda gnt], \quad t \geq 0$$

which is $\mathbf{G}_n$ modified by a random time transformation, with a limit as described in (30).

A similar argument applies the other way, starting with convergence of $(\mathbf{H}_n, \mathbf{S}^1_n, \mathbf{S}^2_n)$. From the proof of Theorem 1

$$\begin{aligned}
&(s^{-1}S_2[T_2(s-)])[H(T_2(s-)) - R_2(S_2[T_2(s-)])\\
&\quad \leq (s^{-1}S_2[T_2(s-)])G(S_2[T_2(s-)])\\
&\quad \leq G(s) \leq (s^{-1}S_1[T_1(s)])G(S_1[T_1(s)])\\
&\quad \leq (s^{-1}S_1[T_1(s)])[H(T_1(s)) + R_1(S_1[T_1(s)])]. \qquad (32)
\end{aligned}$$

We complete the proof by substituting $nt$ for $s$ in (32) and reasoning as with (31), once again exploiting random time transformations. Since

$$\sup_{0 \leq t \leq t} \{|n^{-1}S_i[T_i(nt)] - t|\} \Rightarrow 0$$

$$\sup_{0 \leq t \leq t} \{|(nt)^{-1}S_i[T_i(nt)] - 1|\} \Rightarrow 0 \text{ too.}$$

**Remark 10.** More processes can be put in the final joint limit (30), as in Theorems 3 and 4 of Glynn and Whitt (1986). More description of the limit process is also given there; it is usually multivariate Brownian motion, which has multivariate normal marginal distributions. (The component Brownian motions are typically dependent; i.e., the covariance matrix typically has nonzero off-diagonal elements.)

As in Sections 4.1 and 4.2 here, in some settings the remainder terms drop out automatically, e.g., $\mathbf{R}_n^{\prime} = \theta$. In the standard $H = \lambda G$ setting of Section 1.3, suppose that $\{f_k(t): t \geq 0\}$ is a nonnegative integrable stochastic process for each $k$, with

$$f_k(t) = 0, \quad t \notin [T_k, T_k + S_k] \tag{33}$$

where $T_k$ is the random arrival epoch of the $k$th customer and

$$S_k/T_k \to 0 \quad \text{w.p.1} \quad \text{as } k \to \infty. \tag{34}$$

Let the random functions be as in (27) with $T_1(s) = T_{\lfloor s \rfloor}$, $s \geq 0$. The following result is an elementary consequence of Theorem 7.

**Theorem 8.** *Suppose that* $h = \lambda g$ *in the standard* $H = \lambda G$ *framework of Section* 1.3 *satisfying* (33) *and* (34). *Then* $(\mathbf{G}_n, \mathbf{T}_n^1) \Rightarrow (\mathbf{G}, \mathbf{T})$ *with* $P(\mathbf{T} \in C)$ *if and only if* $(\mathbf{H}_n, \mathbf{S}_n) \Rightarrow (\mathbf{H}, \mathbf{S})$ *with* $P(\mathbf{S} \in C)$, *in which case the joint convergence* (29) *holds with the limit processes related by* (30).

**Remark 11.** If (33) holds with $S_k \leq W_k$, then $(\mathbf{G}_n, \mathbf{T}_n^1) \Rightarrow (\mathbf{G}, \mathbf{T})$ implies (34) and thus also $(\mathbf{H}_n, \mathbf{S}_n) \Rightarrow (\mathbf{H}, \mathbf{S})$.

**Remark 12.** An application of the CLT version of $H = \lambda G$ to determine the asymptotic efficiency of statistical estimators of queueing parameters is illustrated in Section 11 of Glynn and Whitt (1989).

## ACKNOWLEDGMENT

This work was done while Peter Glynn was in the Department of Industrial Engineering at the University of Wisconsin-Madison, supported by the National Science Foundation under grant ECS-8404809 and by the U.S. Army under contract DAAG 29-80-C-0041.

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