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WEAK CONVERGENCE OF FIRST PASSAGE TIME PROCESSES

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1. Summary

Let $D \equiv D[0,\infty)$ be the space of all real-valued right-continuous functions on $[0,\infty)$ with limits from the left. For any stochastic process X in D, let the associated *supremum process* be $S(X)$, where

$$S(x)(t) = \sup_{0 \leqq s \leqq t} x(s), \quad t \geqq 0, \tag{1.1}$$

for any $x \in D$. It is easy to verify that $S: D \to D$ is continuous in any of Skorohod's (1956) topologies extended from $D[0,1]$ to $D[0,\infty)$ (cf. Stone (1963) and Whitt (1970a, c)). Hence, weak convergence $X_n \Rightarrow X$ in D implies weak convergence $S(X_n) \Rightarrow S(X)$ in D by virtue of the continuous mapping theorem (cf. Theorem 5.1 of Billingsley (1968)).

Let $E \equiv E[0,\infty)$ be the subset of D containing those $x \in D$ for which $\lim_{t\to\infty} S(x)(t) = +\infty$. Let E be endowed with the relative topology. For any stochastic process X in E, let the associated *first passage time process* in E be $T(X)$, where

$$T(x)(t) = \inf\{s \geqq 0 : x(s) > t\}, \qquad t \geqq 0, \tag{1.2}$$

for any $x \in E$. (The infimum is always attained.) We show that $T: E \to E$ is continuous in Skorohod's (1956) M_1 topology on E. Hence, weak convergence $X_n \Rightarrow X$ in $E(M_1)$ implies weak convergence $T(X_n) \Rightarrow T(X)$ in $E(M_1)$. Moreover, $S: E \to E$ is also continuous in the M_1 topology, $T(S) = T$ and $T(T) = S$. Hence, $S(X_n) \Rightarrow S(X)$ if and only if $T(X_n) \Rightarrow T(X)$ in $E(M_1)$ where $S(X) = T(T(X))$ and $T(X) = T(S(X))$. For any stochastic process X in $E(M_1)$, we therefore call the associated stochastic processes $S(X)$ and $T(X)$ *dual processes*. For example, the Wiener process or Brownian motion W is in E with probability one. The associated dual processes are $S(W)$, the reflecting Brownian motion or one-dimensional Bessel process, and $T(W)$, the one-sided stable process with exponent $\frac{1}{2}$. It is interesting that $T(T(W)) = S(W)$. The corresponding distributions at a single time point are displayed in Feller ((1949), p. 109) where the duality relationship is also discussed to some extent.

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By relating the weak convergence of $\{S(X_n)\}$ and $\{T(X_n)\}$, we are providing a supplement to Iglehart and Whitt (1969) in which the equivalence of functional central limit theorems for point processes or counting processes and associated partial sums was demonstrated. The first passage time processes here can be regarded as counting processes in the special case in which $T(X_n)$ and $T(X)$ are integer-valued; then $S(X_n) = T(T(X_n))$ and $S(X) = T(T(X))$ are the associated partial sum processes. In [9] the random functions in D had positive translation terms; here the random functions have no translation terms. The results in [9] can obviously be extended to the slightly more general setting of this paper.

In Section 2 we prove the results described above; in Section 3 we discuss two examples; and in Section 4 we mention possible applications.

2. The results

Let $D \equiv D[0, \infty)$ be the space of all real-valued right-continuous functions on $[0, \infty)$ with limits from the left. We shall consider two different topologies on D. Stone (1963) extended Skorohod's (1956) J_1 topology to D by defining convergence of a sequence of functions $\{x_n\}$ to a function x in D by the existence of a sequence of continuous, one-to-one functions $\{\lambda_n\}$ of $[0, \infty)$ onto itself such that for each $m > 0$

$$\sup_{0 \leqq t \leqq m} \left| \lambda_n(t) - t \right| \to 0 \tag{2.1}$$

and

$$\sup_{0 \leqq t \leqq m} \left| x_n(t) - x(\lambda_n(t)) \right| \to 0 \tag{2.2}$$

as $n \to \infty$. We shall also extend Skorohod's (1956) M_1 topology to D. We define the graph $G(x)$ of $x \in D$ as the subset of R^2 which contains all pairs (x, t) such that for all t the point x belongs to the segment joining $x(t-)$ and $x(t)$. The graph $G(x)$ is a continuous curve in R^2. The pair of functions $(x(s), t(s))$ gives a parametric representation of the graph $G(x)$ if those and only those pairs (x, t) belong to it for which an s can be found such that $x = x(s)$ and $t = t(s)$, where $t(s)$ is continuous and monotonically increasing and $x(s)$ is continuous. We say the sequence $\{x_n\}$ is M_1-convergent to x in D if there exist parametric representations $(x(s), t(s))$ of $G(x)$ and $(x_n(s), t_n(s))$ of $G(x_n)$ such that for every $m > 0$

$$\sup_{\substack{s \\ 0 \leqq t(s) \leqq m}} \{\left| x_n(s) - x(s) \right| + \left| t_n(s) - t(s) \right|\} \to 0 \tag{2.3}$$

as $n \to \infty$. We remark that the M_1 topology is weaker than the J_1 topology. If the limit x is contained in $C \equiv C[0, \infty)$, then both the J_1 and M_1 topologies reduce to the topology of uniform convergence on compacta. For further discussion, see [19], [20], [25], and [27].

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Let E have the relative topology from D. Let $S: E \to E$ be the supremum function and let $T: E \to E$ be the first passage time function defined in (1.1) and (1.2). The function T is in a sense an inverse mapping for S because $T(S) = T$ and $T(T(S)) = S$. For weak convergence we need

Lemma. The supremum function is continuous on E in both the J_1 and M_1 topologies. The first passage time function is continuous on E in the M_1 topology, but not in the J_1 topology.

Proof. Since

$$\sup_{0 \leqq t \leqq m} \left| \sup_{0 \leqq s \leqq t} x_n(s) - \sup_{0 \leqq s \leqq t} x(\lambda_n(s)) \right| \leqq \sup_{0 \leqq t \leqq m} \left| x_n(t) - x(\lambda_n(t)) \right|$$

for all m and n, S is continuous in the J_1 topology, and hence also in the M_1 topology.

Note that $[S(x)(s), t(s)]$ serves as a parametric representation for $T(S(x))$ as well as $S(x)$ when the roles of $S(x)$ and t are switched because $S(x)$ is non-decreasing. Hence, $T(S(x_n)) \to T(S(x))$ (M_1) if $S(x_n) \to S(x)$ (M_1). Since S is continuous (M_1), $T(S(x_n)) \to T(S(x))$ (M_1) if $x_n \to x$ (M_1). Finally, since $T(S) = T$, T itself is continuous (M_1). To show that T is not continuous in the J_1 topology, define x and x_n by

$$x(t) = \begin{cases} t, & 0 \leqq t \leqq 1 \\ 1, & 1 \leqq t \leqq 2 \\ t-1, & 2 \leqq t \end{cases}$$

and

$$x_n(t) = \begin{cases} t, & 0 \leqq t \leqq 1 - 1/n \\ 1 - 1/n, & 1 - 1/n \leqq t < 3/2 \\ 1 + 1/n, & 3/2 \leqq t \leqq 2 + 1/n \\ t-1, & 2 + 1/n \leqq t. \end{cases}$$

Note that $\sup_{t \geqq 0} \left| x_n(t) - x(t) \right| = 1/n \to 0$, but $\left| T(x_n)(s) - T(x)(\lambda_n(s)) \right| \geqq \frac{1}{2}$ for all n and λ_n with $s = 1$.

Let X_n $(n \geqq 1)$ and X be random functions in E, and write $X_n \Rightarrow X$ for weak convergence. By the lemma and the continuous mapping theorem ([1], p. 29), we have

Theorem. (i) If $X_n \Rightarrow X$ in E (J_1 or M_1), then $S(X_n) \Rightarrow S(X)$ and $T(X_n) \Rightarrow T(X)$ in $E(M_1)$.

(ii) Let $\{X_n\}$ be a sequence of random functions in E. There exists a random function $A \in E$ such that $S(X_n) \Rightarrow A$ (M_1) if and only if there exists a random function $B \in E$ such that $T(X_n) \Rightarrow B$ (M_1). Moreover, $T(A) = B$ and $T(B) = A$.

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Proof. Only (ii) needs comment. Recall that $T = T(S)$ so that $B = T(A)$. Also $T(T) = S$ so that $T(B) = A$.

Our principal concern here is the first passage time function T and the weak convergence $T(X_n) \Rightarrow T(X)$. We remark that even though the M_1 topology is weaker than the J_1 topology, weak convergence in the M_1 topology means convergence of all finite-dimensional distributions in an everywhere dense subset of R^1 plus a form of tightness (cf. Chapter 3 of [1] and Section 3.2 of [19]).

3. Examples

If $X_n \Rightarrow W$ in $E(M_1)$, where W is the standard Wiener process, then $S(X_n) \Rightarrow S(W)$, and $T(X_n) \Rightarrow T(W)$ in $E(M_1)$, where $S(W)$ is the reflecting Brownian motion or one-dimensional Bessel process (cf. [10], pp. 40 and 59) and $T(W)$ is the one-sided stable process with exponent $\frac{1}{2}$ and rate $2^{\frac{1}{2}}$, that is, $T(W)$ has increasing paths, may be regarded as in E, is differential, and is homogeneous with law

$$\begin{aligned} P\{T(W)(t) - T(W)(s) \leqq x\} &= P\{T(W)(t-s) \leqq x\} \\ &= \int_0^x (t-s)(2\pi v^3)^{-\frac{1}{2}} \exp\{-(t-s)^2/2v\}dv, \end{aligned} \tag{3.1}$$

for $0 \leqq s < t \leqq 1$ and $0 \leqq x < 1$, and characteristic function

$$\phi_{T(W)(t)}(\theta) = \exp\{-(-2it^2\theta)^{\frac{1}{2}}\} \tag{3.2}$$

for all real θ (cf. [10] p. 25). For further properties, see [15] and [16]. It is interesting to note that $T(T(W)) = S(W)$ and $T(S(W)) = T(W)$ too.

Let $C(t) = ct$ for $t \geqq 0$ and $c > 0$. Then $W + C$ is the Wiener process with a positive drift, and $W + C$ is in E with probability one. If $X_n \Rightarrow W + C$ in $E(M_1)$, then $T(X_n) \Rightarrow T(W + C)$, where $T(W + C)$ is the inverse Gaussian process, which may regarded as in E, has stationary and independent increments, and has the law

$$P\{T(W + C)(t) \leqq x\} = \int_0^x (c^3t/2\pi v^3)^{\frac{1}{2}} \exp\{-(c/2tv)(v-ct)^2\}dv, \tag{3.3}$$

$0 \leqq t \leqq 1$ and $0 \leqq x < 1$ (cf. [17]). For further properties of the inverse Gaussian process, see [17], [21], [22], [23], and [24].

Recall that the projections $\pi_{t_1,\cdots,t_k}: D \to R^k$, defined for any $x \in D$ by $\pi_{t_1,\ldots,t_k}(x) = [x(t_1), \cdots, x(t_k)]$, are not necessarily continuous in either the J_1 or the M_1 topology, but each such projection is continuous almost everywhere with respect to the two processes discussed above. Hence, we have convergence in R^1 to the laws described in (3.1) and (3.3) for $\pi_t(T(X_n))$ with weak convergence.

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We remark that the discussion here applies also to more general passage times and stopping times. For example, we could consider $P: E \to E$, defined for any $x \in E$ by

$$P(x)(t) = \inf\{s \geqq 0 : x(s) > g(t)\}, \tag{3.4}$$

where $g(t) = ct^{\alpha}$, $\alpha > 0$ (cf. [2], [3], and [18]). Of course, the problem with this greater generality is that it is hard to evaluate the weak convergence limits.

4. Applications

We do not intend to pursue any of the many possible applications of the weak convergence theorems for first passage times in this paper. Some of the application areas with related references are: first emptiness in dams and storage models [5], [6], [13], and [14]; the ruin problem in collective risk theory [8]; the busy period in queueing and the virtual waiting time for lower priority customers when there is a preemptive-resume discipline [7] and [28]; k-dimensional renewal theory [11] and [26]; and the superposition and thinning of point processes [12].

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