WINNING THE HAND OF THE PRINCESS SARALINDA

by

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ABSTRACT

Suitors come to the castle in order to try to win the hand of the Princess Saralinda. The first suitor to perform n amazing feats will succeed. If the number n of feats is large and the times to perform the feats are all i.i.d., then how long is it before a suitor will win the hand of the Princess? We develop asymptotics describing the distribution of this random time as the number of feats gets large; e.g., we show that this time is of order $nm - \sigma\sqrt{n \log n}$, where m and σ^2 are the mean and variance of the time to perform a single feat, provided that the time to perform a single feat has a finite moment generating function and the suitor arrival process obeys a strong law of large numbers. The $-\sqrt{\log n}$ term reflects the fact that there is an arrival process of suitors instead of just one. The asymptotic effect of the arrival process is equivalent to having $\alpha\sqrt{n}$ suitors present at time 0 with no additional arrivals. We also obtain large deviations results that depend on the full interarrival-time distribution.

Keywords: princesses, dukes, asymptotics, tandem queues, infinite-server queues in series, parallel processing, large deviations, extreme values.

1. Introduction

"Once upon a time, in a gloomy castle on a lonely hill, where there were thirteen clocks that wouldn't go, there lived a cold, aggressive Duke, and his niece, the Princess Saralinda. She was warm in every wind and weather, but he was always cold. His hands were as cold as his heart. He wore gloves when he was asleep, and he wore gloves when he was awake, which made it difficult for him to pick up pins or coins or the kernels of nuts, or to tear the wings from nightingales... Wickedly scheming, he would limp and cackle through the cold corridors of the castle, planning new impossible feats for the suitors of Saralinda to perform. He did not wish to give her hand in marriage, since her hand was the only warm hand in the castle;" from *The Thirteen Clocks* by James Thurber [17].

Suitors come to the castle in order to win the hand of the Princess Saralinda. Suppose that the suitors arrive according to a stochastic point process and that the Duke requires each suitor to perform n feats, where the times required to perform the feats are all i.i.d. and independent of the arrival process. Then what can be said about the time T_n that one of the suitors finally succeeds as the number n of feats gets large? And what is the probability that the first suitor wins within time t of his arrival?

Expressed somewhat more prosaically, we have a series of initially empty, identical, infinite-server queues with all service times mutually independent, fed by an arrival process. We ask about the time T_n the first customer completes service from the n^{th} queue as $n \to \infty$. This model can be thought of as an infinite-server analog of the single-server queues in series considered in Glynn and Whitt [9] and Greenberg et al. [11]; see also Anantharam [1], Bambos and Walrand [2], Coffman and Whitt [4], Cox et al. [5] and Srinivasan [16]. The model here has potential application in the study of parallel processing, stochastic scheduling and transport in random media. We mention that Vere-Jones [18] already showed that the stationary departure process from the n^{th} queue is asymptotically Poisson as $n \to \infty$.

The model here also can be regarded as a single infinite-server queue. In particular, T_n is the time of the first departure from a single, initially empty $G/G/\infty$ queue with i.i.d. service times, each of which is the sum of n i.i.d. random variables. This first departure time considered here should be distinguished from the busy period or depletion time, which is the time until the system is first empty; see Browne and Steele [3].

Let A_i be the arrival epoch of the i^{th} suitor and let V_{ij} be the time required for the i^{th}

suitor to perform his j^{th} feat. Then clearly

$$T_n = \inf\{A_i + \sum_{j=1}^{n} V_{ij} : i \geq 1\} \ . \tag{1.1}$$

Let V be a generic time to perform a feat. Throughout this paper we assume that V_{ij} are i.i.d. and that $\mathrm{Var}(V) = \sigma^2$, $0 < \sigma^2 < \infty$. Let $m = EV$, F be the cdf of V and F_n its n-fold convolution. We will make additional assumptions in our theorems below.

We now put our problem in perspective. Let $\Rightarrow$ denote convergence in distribution as in the central limit theorem (CLT) and let $\rightsquigarrow$ denote almost sure convergence to a set for a subsubsequence for all subsequences, as in the law of the iterated logarithm (LIL). Let $N(a, b)$ be a normally distributed random variable with mean a and variance b. *If there were only one suitor (as is the case for single-server queues in series),* then we would have

$$(T_n - nm)/\sigma\sqrt{n} \Rightarrow N(0, 1) \ as \ n \to \infty \tag{1.2}$$

by the CLT and

$$(T_n - nm)/\sigma\sqrt{2n \log\log n} \rightsquigarrow [-1, 1] \ as \ n \to \infty \tag{1.3}$$

by the LIL, so that

$$\liminf_{n\to\infty}\{(T_n - nm)/\sigma\sqrt{2n \log\log n}\} = -1 \ . \tag{1.4}$$

In contrast, if $\delta = \inf\{t : P(V \leq t) > 0\}$ and *if all suitors were present at time* 0, then $T_n = n\delta$ w.p.1. Instead, *if there were a total of αn^β suitors, all present at time 0,* and if the times required for the feats were normally distributed, then

$$(T_n - nm)/\sigma\sqrt{2\beta n \log n} \Rightarrow -1 \ as \ n \to \infty \tag{1.5}$$

by the standard extreme-value limit for normally distributed random variables, Theorem 1.5.3 of Leadbetter et al. [12]. As a first result, we show that this result remains true for general distributions under regularity conditions. We give all proofs later.

Theorem 1. *If there are αn^β suitors present at time 0 and no further arrivals, and if $E \exp \theta V < \infty$ for some positive θ, then (1.5) holds. If instead there are $\alpha\sqrt{n^\beta \log n}$ suitors present at time 0 with no additional arrivals, then*

$$P(T_n > nm - \sigma\sqrt{2\beta n \log n}) \to 1 - e^{-\alpha/2\sqrt{\pi\beta}} \text{ as } n \to \infty \ . \tag{1.6}$$

However, we are primarily interested in the case of an arrival process. We should anticipate that our situation with an arrival process should lead to a limit of the form (1.5) for appropriate β; we show that it does for $\beta = 1/2$ under the suitable regularity conditions. Let $A(t)$ count the number of arrivals in $[0, t]$. We say that $A(t)$ obeys a *strong law of large numbers* (SLLN) if $t^{-1}A(t)$ converges to a finite positive limit w.p.1 as $t \to \infty$, which here we take to be 1.

Theorem 2. *If $A(t)$ obeys a SLLN and $E \exp \theta V < \infty$ for some positive θ, then*

$$(T_n - nm)/\sigma\sqrt{n \log n} \Rightarrow -1 \quad as \quad n \ \to \ \infty \ . \tag{1.7}$$

Remark 1.1. More generally, by the same proof, if $A(t)/t^{\eta}$ converges to a finite positive limit w.p.1, for $0 < \eta < \infty$, then in (1.7) σ should be replaced by $\sigma\eta$. Having a SLLN for $A(t)$ with exponent η is equivalent to having αn^{β} suitors at time 0, with no future arrivals, when $\sqrt{2\beta} = \eta$ or $\beta = \eta^2/2$. □

Let $N(t)$ be the maximum number of feats completed by any suitor by time t. Note that T_n and $N(t)$ are inverse processes, i.e.,

$$N(t) \geq n \text{ if and only if } T_n \leq t \ , \tag{1.8}$$

so that we can apply Theorem 6 of Glynn and Whitt [8] (see also Theorem 4.1 of Massey and Whitt [15]) to deduce the following from Theorem 2.

Corollary 1. *Under the conditions of Theorem 2,*

$$(N(t) - t/m)/\sqrt{(\sigma^2/m^3)t \log t} \Rightarrow 1 \ as \ t \to \infty \ . \tag{1.9}$$

We also deduce simple weak law of large numbers (WLLN) statements from Theorem 2 and Corollary 1.

Corollary 2. *Under the conditions of Theorem 2, $T_n/n \Rightarrow m$ as $n \to \infty$ and $N(t)/t \Rightarrow m^{-1}$ as $t \to \infty$.*

From Theorem 2 we see that $T_n \approx nm - \sigma\sqrt{n \log n}$. We now describe the tail probabilities of $T_n - nm$ in more detail. Our next two results take the form of large deviations behavior, but they do not appear to be standard results from that theory. In our next result, we exploit the relation between the large deviations behavior of a counting process and its inverse, which was developed in Glynn and Whitt [10].

For our next result we assume that the arrival process $A(t)$ is a renewal process, but this property is not critical; see Remark 1.2 below. Let U_i be the i^{th} interarrival time, let U be a generic interarrival time and let $EU = 1$. Let ψ_U be the *logarithmic moment generating function* of U, i.e.,

$$\psi_U(\theta) = \log Ee^{\theta U} . \tag{1.10}$$

Since the interarrival times are i.i.d.,

$$n^{-1} \log Ee^{\theta(U_1+\ldots+U_n)} = \psi_U(\theta) \text{ for all } n , \tag{1.11}$$

so that trivially the *Gärtner-Ellis condition with decay rate function* ψ_U holds for the partial sums, i.e.,

$$n^{-1} \log Ee^{\theta(U_1+\ldots+U_n)} \to \psi_U(\theta) \quad \text{as } n \to \infty . \tag{1.12}$$

We now introduce *auxiliary large deviations conditions* for ψ_U, namely

$$\beta^u \equiv \inf\{\theta : \psi_U(\theta) = \psi_U(\infty)\} > 0 , \tag{1.13}$$

$$\psi_U \quad \text{is differentiable everywhere in } (-\infty, \beta^u) , \tag{1.14}$$

$$\lim_{\theta \uparrow \beta^u} \psi_U'(\theta) = +\infty \text{ if } \psi_U(\beta^u) < \infty \; (\psi_U \quad \text{is steep}) \text{ , and} \tag{1.15}$$

$$\lim_{\theta \uparrow \beta^u} \psi_U(\theta) = \psi_U(\beta^u) . \tag{1.16}$$

In [10] we prove that if (1.12)–(1.16) hold (without the i.i.d. conditions), then

$$t^{-1} \log Ee^{\theta A(t)} \to \psi_A(\theta) \quad \text{as } t \to \infty , \tag{1.17}$$

where the decay rate function ψ_A of $A(t)$ is

$$\psi_A(\theta) = -\psi_U^{-1}(-\theta) \tag{1.18}$$

and ψ_A satisfies (1.13)–(1.16), except that (1.17) might not hold for $\theta = \beta_A^u$. We exploit (1.17) in our next theorem. (We only use (1.17) for negative θ.)

In the next theorem we use a condition on the conditional residual lifetime cdf's; i.e., we assume that

$$G^c(t) = \sup\{H^c(t+x)/H^c(x) : x \geq 0\} \tag{1.19}$$

is a proper cdf, where H is the cdf of U. Simple sufficient conditions are for U to be bounded or to have a new-better-than-used (NBU) cdf (then $G^c(t) = H^c(t)$ for all t). Let Φ be the standard (mean 0, variance 1) normal cdf.

Theorem 3. *If the arrival process is a renewal process with (1.13)–(1.16) holding for $\psi_U(\theta)$ in (1.10), if G in (1.19) above is a proper cdf and if $EV^4 < \infty$, then*

$$n^{-1/2}\log P(T_n - nm > -x\sigma\sqrt{n}) \to \sigma \int_{-\infty}^{-x} \psi_A(\log(1-\Phi(z)))dz \text{ as } n \to \infty \tag{1.20}$$

for ψ_A in (1.18).

Remark 1.2. In our proof of Theorem 3 we exploit the renewal property of the arrival process via Lemmas 2 and 3 below, but from the proof it is evident that (1.20) holds for much more general arrival processes; then ψ_A should be defined by (1.17). For the case in which $A(t)$ is a cumulative process with respect to a sequence of regeneration times, we can apply Theorem 7 of Glynn and Whitt [10] to establish (1.17). Extensions of Lemmas 2 and 3 below are easily developed in terms of the regeneration cycles as well. For example, this extension covers the case in which $A(t)$ is a batch Markovian arrival process as in Lucantoni [14]. Thus, (1.20) should be regarded as a general relation, requiring regularity conditions not depending critically on the renewal property of the arrival process. □

We now obtain a stronger result in the case of a Poisson arrival process. Recall that $f(x) \sim g(x)$ as $x \to \infty$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. Let ϕ be the standard normal density, i.e., the density of Φ. The last assumption below is basically a density assumption, allowing us to apply a refined CLT on p. 541 of Feller [7].

Theorem 4. *If the arrival process is a Poisson process, $EV^4 < \infty$ and $\overline{\lim}_{t\to\infty}|E\exp(itV)| < 1$, then*

$$P(T_n - nm > -x\sigma\sqrt{n}) \sim \alpha(x)\exp(-\eta(x)\sqrt{n}) \text{ as } n \to \infty\ , \tag{1.21}$$

where

$$\alpha(x) = \exp(-(E(V-m)^3/6\sigma^3) \int_{-\infty}^{-x} (1-z^2)\phi(z)dz \quad (1.22)$$

and

$$\eta(x) = \sigma \int_{-\infty}^{-x} \Phi(z)dz \sim \ \sigma\phi(x)/x^2 \ \textit{as} \ x \to \infty \ . \quad (1.23)$$

Remark 1.3. Note that if U has an exponential distribution, then $\psi_A(\theta) = e^\theta - 1$, so that (1.20) becomes consistent with the stronger result (1.21).

Remark 1.4. Note that, given $EV^4 < \infty$, the asymptotic behavior in Theorem 4 only depends on the first three moments of V, while the asymptotic behavior in Theorem 3 only depends on the first two moments of V. In contrast, the asymptotic behavior in Theorem 3 depends on the entire distribution of U via its logarithmic moment generating function ψ_U and its inverse ψ_A.

Remark 1.5. Note that $\log \Phi^c(x) \sim -\Phi(x)$ as $x \to -\infty$, so that

$$\psi_A(\log \Phi^c(x)) \sim -\psi_A'(0)\Phi(x) \text{ as } x \to \infty \ ,$$

where $\psi_A'(0) = 1/EU = 1$, so that

$$\sigma \int_{-\infty}^{-x} \psi_A(\log \Phi^c(y))dy \sim -\sigma \int_{-\infty}^{-x} \Phi(z)dz \text{ as } \ -x \to -\infty \ ,$$

which is the exponent in the Poisson case. In other words, the limit in Theorem 3 ceases to depend on the interarrival time distribution as $-x \to -\infty$. □

The special case of Poisson arrivals is much easier to analyze, because the departure process is a nonhomogeneous Poisson process. This is true even when the arrival process itself is nonhomogeneous; e.g., see Theorem 1 of Eick et al. [6]. Suppose that the arrival rate function is $\lambda(t)$. Then the departure rate function as a function of n is

$$\delta_n(t) = E[\lambda(t - \sum_{j=1}^{n} V_{ij})] = \int_0^t \lambda(t-y)dF_n(y) \ , \ t \geq 0 \ , \quad (1.24)$$

where F is the cdf of V and F_n is its n-fold convolution. In the special case of $\lambda(t) = 1$, $t \geq 0$, Eq. (1.24) becomes $\delta_n(t) = F_n(t)$ and

$$P(T_n > x) = \exp(-\int_0^x \delta_n(y)dy) = \exp(-\int_0^x F_n(y)dy) \ . \quad (1.25)$$

To make our paper self-contained, we give a direct proof of (1.25) as well. (See Lemma 1 below.)

The Poisson departure property enables us to easily deduce many other properties. For example, let $D_n(t)$ be the number of departures from the n^{th} queue in the time interval $[0,t]$.

Theorem 5. *If the arrival process is a Poisson process, then* $D(nm+\sigma x_2\sqrt{n}) - D(nm+\sigma x_1\sqrt{n})$ *has a Poisson distribution with mean*

$$\sigma\sqrt{n}\int_{x_1}^{x_2} F_n(nm+y\sigma\sqrt{n})dy \tag{1.26}$$

provided that $x_1 < x_2$ *and* $nm+\sigma x_1\sqrt{n} > 0$. *Consequently,*

$$\frac{D_n(nm+\sigma x_2\sqrt{n}) - D_n(nm+\sigma x_1\sqrt{n})}{\sigma\sqrt{n}} \Rightarrow N(m,m) \text{ as } n\to\infty\ , \tag{1.27}$$

where

$$m = \int_{x_1}^{x_2}\Phi(y)dy\ . \tag{1.28}$$

Note that the asymptotic mean m in (1.28) is approximately (x_2-x_1) for x_1 suitably large. The asymptotic relation in (1.23) describes the interesting case of x_2 small or negative.

Now let us consider which suitor wins the hand of the Princess.

Remark 1.6. If the arrival process is a Poisson process, and if F is absolutely continuous with density f, then the probability that the first suitor wins and does so within time t of his arrival is

$$\int_0^t f_n(y)e^{-\int_0^y F_n(z)dz}dy \tag{1.29}$$

by (1.25).

Remark 1.7. Theorem 2 enables us to estimate the probability that the various suitors win. Under the conditions of Theorem 1, the probability that the first suitor completes his n feats by time $nm-\sigma(1+\epsilon)\sqrt{n\log n}$ is

$$\begin{aligned} F_n(nm-\sigma(1+\epsilon)\sqrt{n\log n}) &\sim \Phi(-(1+\epsilon)\sqrt{\log n}) \\ &\sim (2\pi n^{(1+\epsilon)}\log n)^{-1/2} \text{ as } n\to\infty \end{aligned} \tag{1.30}$$

by Theorem 1 on p. 549 of Feller [7].

Suitor $\lfloor \alpha\sqrt{n} \rfloor$ has a comparable chance to win, because he arrives at time $\alpha\sqrt{n}+o(\sqrt{n})$. Hence the probability that suitor $\lfloor \alpha\sqrt{n} \rfloor$ arrives before time $nm - \sigma(1+\epsilon)\sqrt{n\log n}$ is also given by (1.30). Roughly speaking, the first $\lfloor \sqrt{n} \rfloor$ suitors each have probability slightly less than $1/\sqrt{n}$ of winning.

On the other hand, suitor $\lfloor \sigma\sqrt{n\log n} \rfloor$ has a significantly smaller chance to win, because he arrives at time $\sigma\sqrt{n\log n}+o(\sqrt{n\log n})$. So the probability that suitor $\lfloor \sigma\sqrt{n\log n} \rfloor$ completes his n feats by time $nm - \sigma(1+\epsilon)\sqrt{n}\log n$ is approximately

$$
\begin{aligned}
F_n(nm - \sigma(2+\epsilon)\sqrt{n\log n}) \quad &\sim \quad (-(2+\epsilon)\sqrt{\log n}) \\
&\sim \quad (2\pi n^{2+\epsilon}\log n)^{-1/2} \text{ as } n \to \infty \ . \qquad (1.31)
\end{aligned}
$$

Remark 1.8. We have indicated that our model corresponds to a single G/G/∞ queue with service times that are the sum of n i.i.d. random variables. In contrast, suppose that instead the service times are distributed as nV (corresponding to one amazing feat for each suitor, which really is more in the spirit of [17]), so that $F_n(y) = F(y/n)$, where F is the cdf of V. In the case of Poisson arrivals, we can apply (1.25) to deduce that

$$
P(T_n > nt) = \exp(-n\int_0^t F(y)dy) \ , \qquad (1.32)
$$

so that

$$
P(T_n > nt)^{1/n} = \exp(-\int_0^t F(y)dy) \ , \qquad (1.33)
$$

$$
n^{-1}\log(P(T_n > nt)) = -\int_0^t F(y)dy \ , \qquad (1.34)
$$

and

$$
n^{-1}\log P(T_n - nm > -xn^{\alpha}) \to -\int_0^m F(y)dy \text{ as } n \to \infty \qquad (1.35)
$$

for $0 < \alpha < 1$. The expression (1.34) and the limit (1.35) should be compared to (1.20).

Theorem 3 describes the large deviations behavior of the right tail probabilities of T_n. Our final result describes the large deviations behavior of the left tail probabilities of T_n. Theorem 3 concerns deviations of order $\sqrt{n\log n}$ from the "asymptotic mean" $nm - \sigma\sqrt{n\log n}$. Our next result concerns deviations of order n. Notice that there is an asymmetry in the conditions. Theorem 3 depends on the large deviations behavior of $A(t)$, whereas the next result depends on the large deviations behavior of the V_{ij} random variables. Here we do not need to directly assume that V_{ij} are i.i.d.

Theorem 6. *If U_i are i.i.d. with $P(0 < U_1 < \infty) > 0$, $\{V_{ij} : j \geq 1\}$, $i \geq 1$, are identically distributed, and for all x in a neighborhood of $\epsilon > 0$,*

$$n^{-1} \log P\left(\sum_{j=1}^{n} V_{ij} < n(m - x)\right) \to \gamma(-x) \quad \text{as} \quad n \to \infty \ , \tag{1.36}$$

where γ is continuous in a neighborhood of ϵ, then

$$n^{-1} \log P(T_n \leq n(m - \epsilon)) \to \gamma(-\epsilon) \quad \text{as} \quad n \to \infty \ . \tag{1.37}$$

2. Proof of Theorems 4 and 5

For any cdf H, let H^c be its complementary cdf, i.e., $H^c(x) = 1 - H(x)$. Let $A(t)$ count the number of arrivals in $[0, t]$. We start with a direct proof of (1.15).

Lemma 1. *With a Poisson arrival process, (1.25) holds.*

Proof. We condition on the arrival times and then exploit the fact that the conditional distribution of each arrival time given $A(x)$ arrivals in $[0, x]$ is uniform in $[0, x]$ (in the fifth line below) to obtain

$$\begin{aligned}
P(T_n > x) &= E[P(T_n > x | A_k, \ k \geq 1)] \\
&= E[\Pi_{k=1}^{\infty} F_n^c(x - A_k)] \\
&= E[\Pi_{k=1}^{A(x)} F_n^c(x - A_k)] \\
&= E[E[\Pi_{k=1}^{A(x)} F_n^c(x - A_k) | A(x)]] \\
&= E[(\int_0^x F_n^c(x - y) dy / x)^{A(x)}] \\
&= \exp(\lambda x (\int_0^x F_n^c(x - y) dy / x - 1)) \\
&= \exp(-\lambda \int_0^x F_n(y) dy) \ . \quad \square
\end{aligned}$$

Under the assumptions, we can exploit a refined CLT, Theorem 3 on p. 541 of Feller [7], to obtain

$$F_n(nm + \sigma z\sqrt{n}) \quad = \quad \Phi(z) + \frac{E(V-m)^3}{6\sigma^3\sqrt{n}}(1-z^2)\phi(z)$$

$$+ \quad \frac{R_4(z)\phi(z)}{n} \; + \; o(n^{-1}) \; as \; n \to \infty \tag{2.1}$$

uniformly in z, where R_4 is a polynomial.

From (1.15) and (2.1), we obtain

$$P(T_n - nm > x\sigma\sqrt{n}) \;=\; \exp(-\int_0^{nm - x\sigma\sqrt{n}} F_n(y)dy)$$

$$= \; \exp(-\sigma\sqrt{n}\int_{-(m/\sigma)\sqrt{n}}^{-x} F_n(nm + z\sigma\sqrt{n})dz) \; , \tag{2.2}$$

where

$$\sqrt{n}\int_{-(m/\sigma)\sqrt{n}}^{-x} F_n(nm + \sigma z\sqrt{n})dz = \sqrt{n}\int_{-(m/\sigma)\sqrt{n}}^{-x} \Phi(z)dz$$

$$+ \; \frac{E[(V-m)^3]}{6\sigma^3}\int_{-(m/\sigma)\sqrt{n}}^{-x}(1-z)^2\phi(z)dz$$

$$+ \; n^{-1/2}\int_{-(m/\sigma)\sqrt{n}}^{-x} R_4(z)\phi(z)dz + o(n^{-1/2})(m\sqrt{n} - x) \text{ as } n \to \infty \; . \tag{2.3}$$

Since

$$\int_{-(m/\sigma)\sqrt{n}}^{-x} |R_4(z)|\phi(z)dz = O(1) \text{ as } n \to \infty \; , \tag{2.4}$$

we obtain the desired conclusion.

To establish the asymptotic relation in (1.23), note that

$$\int_{-\infty}^{-x} \Phi(z)dz \;=\; \int_x^{\infty} \Phi^c(z)dz$$

$$= \; \int_x^{\infty} (y-x)\phi(y)dy = \phi(x) - x\Phi^c(x) \sim \phi(x)/x^2 \text{ as } x \to \infty \; .$$

Finally, Theorem 5 follows by first a change of variables given the mean, i.e.,

$$\int_{nm+\sigma x_1\sqrt{n}}^{nm+\sigma x_2\sqrt{n}} F_n(y)dy = \sigma\sqrt{n}\int_{x_1}^{x_2} F_n(nm + y\sigma\sqrt{n})dy \; ,$$

and then the CLT, i.e.,

$$F_n(nm + x\sigma\sqrt{n}) \to \Phi(x) \ as \ n \to \infty \ .$$

3. Proof of Theorem 3

We use stochastic bounds on the renewal counting function. Let $\le_{st}$ denote stochastic order on $\mathbb{R}^k$ or $\mathbb{R}^\infty$ with the usual order: $x \le y$ if $x_k \le y_k$ for all k. Stochastic order $X \le_{st} Y$ holds if $Eh(X) \le Eh(Y)$ for all nondecreasing real-valued h for which the expectations are well defined; e.g., see Chapter IV of Lindvall [13].

Lemma 2. *For $c > 0$,*

$$\{A(ck) - A(c(k-1)) : k \ge 1\} \le_{st} \{1 + A_k(c) : k \ge 1\}$$

where A_k, $k \ge 1$, are i.i.d. copies of A.

Proof. We apply Theorem 5.8 of Lindvall [13]. For each k, $A(ck) - A(c(k-1))$ is less than or equal to 1 plus the number of points after the first point in the interval $(c(k-1),\ ck]$, which in turn is stochastically less than $1 + A_k(c)$, regardless of the history of A before the left endpoint $c(k-1)$. Inductively, we can choose A_k independent of $A_1, \ldots, A_{k-1}$. Hence, the conditions of Theorem 5.8 of [13] are satisfied. $\square$

Let $x \wedge y = \min\{x, y\}$.

Lemma 3. *If G defined by (1.19) is a proper cdf, i.e., if $G^c(t) \to 0$ as $t \to \infty$, then*

$$\{A(ck) - A(c(k-1) : k \ge 1\} \ge_{st} \{A_k(c - (X_k \wedge c)) : k \ge 1\} \ ,$$

where $\{A_k\}$ and $\{X_k\}$ are independent i.i.d. sequences with A_k distributed as A and X_k having cdf G in (1.19).

Proof. Again we apply Theorem 5.8 of Lindvall [13]. Proceeding inductively, consider the interval $(c(k-1),\ ck]$. Conditioned on A before $c(k-1)$, the first point to the right of $c(k-1)$ is stochastically less than X_k, by virtue of (1.19). Hence

$$A(ck) - A(c(k-1)) \geq_{st} A_k(c - (X_k \wedge c))$$

where X_k and A_k are chosen independently of $X_1, \ldots, X_{k-1}$ and $A_1, \ldots, A_{k-1}$. Hence, the conditions of Theorem 5.8 of [13] are satisfied. □

We exploit a basic relation for the tail probability for any arrival process.

Lemma 4. *For any arrival process,*

$$P(T_n > x) = E \exp\left(\int_0^x \log(F_n^c(x-y))A(dy)\right) . \tag{3.1}$$

Proof. Note that

$$\begin{aligned} P(T_n > x) &= E[\Pi_{k=1}^{\infty} F_n^c(x - A_k)] \\ &= E \exp\left(\sum_{k=1}^{\infty} \log(F_n^c(x - A_k))\right) \\ &= E \exp\left(\int_0^x \log(F_n^c(x-y))A(dy)\right) . \quad \square \end{aligned}$$

We exploit the moment condition EV^4 to obtain an inequality on the tail probability of F_n.

Lemma 5. *If $EV^4 < \infty$, then*

$$F_n(nm - x\sigma n^{1/2} - \sigma n^{3/4}) \leq M/n$$

for some constant M.

Proof. Note that

$$\begin{aligned} F_n(nm - x\sigma n^{1/2} - \sigma n^{3/4}) &= P\left(\sum_{j=1}^{n} V_{1j} \leq nm - x\sigma^{1/2} - \sigma n^{3/4}\right) \\ &= P\left(\frac{\sum_{j=1}^n V_{1j} - nm}{\sigma\sqrt{n}} \leq -x - n^{1/4}\right) \\ &\leq E\left(\frac{\sum_{j=1}^n V_{1j} - nm}{\sigma\sqrt{n}}\right)^4 \frac{1}{(x+n^{1/4})^4} \leq \frac{M}{n} \end{aligned}$$

for some M. $\square$

Let $O(n^{-1})$ be a deterministic quantity of order n^{-1}. We now apply Lemmas 4 and 5 to obtain

$$
\begin{aligned}
P(T_n > nm - x\sigma\sqrt{n}) &= E\exp(\int_0^{nm} \log(F_n^c(nm - x\sigma\sqrt{n} - y))A(dy)) \\
&= E\exp(\int_0^{\sigma n^{3/4}} \log(F_n^c(nm - x\sigma\sqrt{n} - y))A(dy) \\
&\quad + \int_{\sigma n^{3/4}}^{nm} \log(1 - O(n^{-1}))A(dy)) \\
&= E\exp(\int_0^{\sigma n^{3/4}} \log(F_n^c(nm - x\sigma\sqrt{n}n - y))A(dy)) \\
&\quad + O(n^{-1})(A(nm) - A(\sigma n^{3/4}))) \ . \qquad (3.2)
\end{aligned}
$$

We now construct upper and lower bounds for (3.2). Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. First, we apply (2.1) and Lemma 2 to construct a lower bound. In particular,

$$
\begin{aligned}
P(T_n &> nm - x\sigma\sqrt{n}) \geq \\
&E\exp(\sum_{k=0}^{\lfloor n^{1/4}h \rfloor} \log(F_n^c(nm - x\sigma\sqrt{n} - (1+k)h\sigma\sqrt{n}))(A((k+1)h\sigma\sqrt{n}) \\
&\quad - A(kh\sigma\sqrt{n}))) + \ O(n^{-1})(A(nm) - A(\sigma n^{3/4}))) \\
&= E\exp(\sum_{k=0}^{\lfloor n^{1/4}h \rfloor} \log(\Phi^c(-x - (1+k)h) + O(n^{-1/2})))(A(k+1)h\sigma\sqrt{n}) \\
&\quad - A(kh\sigma\sqrt{n})) + O(n^{-1})(A(nm) - A(\sigma n^{3/4}))) \\
&\geq E\exp(\sum_{k=0}^{\lfloor n^{1/4} \rfloor} \log(\Phi^c(-x - (k+1)h) + O(n^{-1/2}))(1 + A_k(h\sigma\sqrt{n}))
\end{aligned}
$$

$$+ O(n^{-1})(1 + A_0(nm)) \ , \tag{3.3}$$

where $A_k,\ k \geq 0$, are i.i.d. versions of the renewal process A.

Turning to the upper bound, we apply (2.1) and Lemma 3 to get, for some integer M,

$$P(T_n \ > \ nm - x\sigma\sqrt{n}) \leq$$

$$E \exp(\sum_{k=0}^{\lfloor M/h \rfloor} \log(F_n^c(nm - x\sigma\sqrt{n} - kh\sigma\sqrt{n})(A((k+1)h\sigma\sqrt{n}) - A(kh\sigma\sqrt{n})) \\ + O(n^{-1})(A(nm) - A(\sigma n^{3/4}))$$

$$\leq \ E \exp(\sum_{k=0}^{\lfloor M/h \rfloor} (\log(\Phi^c(-x - kh)) + O(N^{-1/2}))A_k(h\sigma\sqrt{n} - X_k \wedge h\sigma\sqrt{n}) \\ + O(N^{-1})(A_0(nm) \ . \tag{3.4}$$

Next, we take logarithms and divide by $\sqrt{n}$ in (3.3) and (3.4), and then let $n \to \infty$ using (1.17), to get

$$\sum_{k=0}^{\infty} h\sigma\psi_A(\log(\Phi^c(-x - (k+1)h))) \leq \underline{\lim}_{n\to\infty} n^{-1/2} \log P(T_n > nm - x\sigma\sqrt{n})$$

$$\leq \overline{\lim}_{n\to\infty} n^{-1/2} \log P(T_n > nm - x\sigma\sqrt{n}) \leq \sum_{k=0}^{\lfloor M/h \rfloor} h\sigma\psi_A(\log \Phi^c(-x - kh))) \ . \tag{3.5}$$

Finally, we let $h \downarrow 0$ in (3.5) to obtain (1.20). The direct Riemann integrability follows partly from (1.14) for ψ_A. In addition to (1.14), integrability of $\psi_A(\log(\Phi^c(-x-y)))$ holds, because

$$\log(\Phi^c(-y)) \sim \log(1 - \phi(y)/y) \sim -\phi(y)/y \text{ as } y \to \infty \ ,$$

so that

$$\psi_A(\log(\Phi^c(-y))) \sim -\psi_A'(0)\phi(y)/y \text{ as } y \to \infty \ .$$

4. Proof of Theorem 2

Using the basic identity in Lemma 4 and Theorem 1 on p. 549 of Feller [7] (for the third line below), we have for any $\epsilon > 0$

$$\begin{aligned}
P(T_n &> nm - \sigma(1-\epsilon)\sqrt{n\log n}) \\
&= E\exp(\int_0^{nm} \log(F_n^c(nm - \sigma(1-\epsilon)\sqrt{n\log n} - y))A(dy)) \\
&\leq E\exp(\int_0^{\sigma\sqrt{n}} \log(F_n^c(nm - \sigma(1-\epsilon)\sqrt{n}\log n - \sigma\sqrt{n}))A(dy)) \\
&= E\exp(\int_0^{\sigma\sqrt{n}} \log(\Phi^c(-(1-\epsilon)\sqrt{\log n} - 1)(1+o(1))A(dy)) \\
&= E\exp(-\Phi(-(1-\epsilon)\sqrt{\log n} - 1)(1+o(1))A(\sigma\sqrt{n})) ,
\end{aligned}$$

but

$$\begin{aligned}
\sqrt{n}\Phi(-(1-\epsilon)\sqrt{\log n} - 1) &\sim \sqrt{n}\exp[(-(1-\epsilon)\sqrt{\log n} + 1)^2/2]/\sqrt{2\pi(1-\epsilon)\log n} \\
&\approx \frac{n^{(1/2-(1-\epsilon)^2/2)}}{\sqrt{2\pi(1-\epsilon)\log n}} \to \infty \text{ as } n \to \infty .
\end{aligned}$$

Also, by the SLLN assumption, $A(\sigma\sqrt{n})/\sqrt{n} \to 1/EU_1 \equiv 1$ as $n \to \infty$. Hence, by the bounded convergence theorem,

$$P(T_n > nm - \sigma(1-\epsilon)\sqrt{n\log n}) \to 0 \text{ as } n \to \infty \tag{4.1}$$

for each $\epsilon > 0$.

On the other hand, for $\epsilon > 0$, we have

$$\begin{aligned}
P(T_n &> nm - \sigma(1+\epsilon)\sqrt{n\log n}) \\
&= E\exp(\int_0^{nm} \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n\log n} - y)A(dy)) \\
&= E[\exp(\int_0^{\sigma n^{(1+\epsilon)/2}} \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n\log n} - y)A(dy)
\end{aligned}$$

$$\exp(\int_{\sigma n^{(1+\epsilon)/2}}^{nm} \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n \log n} - y)A(dy))] \ .$$

First,

$$\int_0^{\sigma n^{(1+\epsilon)/2}} \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n \log n} - y)A(dy)$$

$$\geq \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n \log n})A(\sigma n^{(1+\epsilon)/2})$$

$$= -(1+o(1))\Phi(-(1+\epsilon)\sqrt{\log n})A(\sigma n^{(1+\epsilon)/2}) \to 0 \text{ as } n \to \infty \ ,$$

because $\sigma^{-1} n^{-(1+\epsilon)/2} A(\sigma n^{(1+\epsilon)/2}) \to 1$ as $n \to \infty$ by the SLLN assumption, and

$$n^{(1+\epsilon)/2}\Phi(-(1+\epsilon)\sqrt{\log n}) \sim n^{(1+\epsilon)/2} \frac{\exp(-(1+\epsilon)^2 \log n)}{(1+\epsilon)\sqrt{2\pi \log n}}$$

$$\sim \frac{n^{-\epsilon/2-\epsilon^2/2}}{(1+\epsilon)\sqrt{2\pi \log n}} \to 0 \text{ as } n \to \infty \ .$$

Next, for the other term, we invoke the assumption that V has a finite moment generating function. For appropriate θ and for $y \leq \sigma n^{(1+\epsilon)/2}$,

$$F_n(nm - \sigma(1+\epsilon)\sqrt{n \log n} - y)$$

$$\leq P([\sum_{i=1}^{n} V_i - nm]/\sigma\sqrt{n} \leq -(1+\epsilon)\sqrt{\log n} - n^{\epsilon/2})$$

$$\leq E \exp(\theta \left| \sum_{i=1}^{n} V_i - nm \right| /\sigma\sqrt{n}) \exp(-\theta(n^{\epsilon/2} + (1+\epsilon)\sqrt{\log n}))$$

$$\sim E \exp(\theta |N(0,1)|) \exp(-\theta(n^{\epsilon/2} + (1+\epsilon)\sqrt{\log n})) \ .$$

Hence,

$$\int_{\sigma n^{(1+\epsilon)/2}}^{nm} \log(F_n^c(nm - \sigma(1+\epsilon)\sqrt{n \log n} - y))A(dy))$$

$$\geq (1+o(1))E \exp(\theta |N(0,1)|) \exp(-\theta n^{\epsilon/2})A(nm) \quad \to 0 \text{ as } n \to \infty \ .$$

Hence, by the bounded convergence theorem again,

$$P(T_n > nm - \sigma(1+\epsilon)\sqrt{n\log n}) \to 1 \text{ as} n \to \infty \ . \tag{4.2}$$

Since ϵ was arbitrary, combining (4.1) and (4.2) yields the desired result.

5. Proof of Theorem 1

Just as in the proof of Theorem 2, apply Theorem 1 on p. 549 of Feller [7]. For the first statement, note that

$$\begin{aligned} P(T_n > nm - (1+\epsilon)\sigma\sqrt{2\beta n\log n}) &= 1 - F_n^c(nm - (1+\epsilon)\sigma\sqrt{2\beta n\log n})^{\alpha n^\beta} \\ &= 1 - (1 - F_n(nm - (1+\epsilon)\sigma\sqrt{2\beta n\log n})^{\alpha n^\beta}, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} F_n(nm - (1+\epsilon)\sigma\sqrt{2\beta n\log n}) &\sim \Phi(-(1+\epsilon)\sqrt{2\beta\log n}) \\ &\sim n^{-\beta(1+\epsilon)^2}(2\pi(1+\epsilon)2\beta\log n)^{-1/2} \text{ as } n \to \infty \ . \end{aligned} \tag{5.2}$$

Since $(1+c_n/n)^n \to e^c$ as $n \to \infty$ if $c_n \to c$ as $n \to \infty$, the tail probability in (5.1) converges to 0 for any $\epsilon < 0$ and 1 for any $\epsilon > 0$. If there are instead $\alpha\sqrt{n^\beta\log n}$ suitors initially, then the same argument leads to the nondegenerate limit in (1.6).

6. Proof of Theorem 6

There exists $c_1 > 0$ and $c_2 > 0$ such that $P(c_1 < U_1 < c_2) > 0$. To establish a lower bound, it suffices to consider a single suitor. Note that

$$\begin{aligned} P(T_n \le n(m-\epsilon)) &\ge P\left(A_1 + \sum_{j=1}^n V_{1j} \le n(m-\epsilon)\right) \\ &\ge P\left(\sum_{j=1}^n V_{1j} \le n(m-\epsilon) - A_1, \quad A_1 < c_2\right) \\ &\ge P(U_1 < c_2)P\left(\sum_{j=1}^n V_{1j} \le n(m-\epsilon) - c_2\right) \\ &\ge P(U_1 \le c_2)P\left(\sum_{j=1}^n V_{1j} \le n(m-\eta)\right) \end{aligned}$$

for $\eta > \epsilon$ and n sufficiently large. Taking logarithms, dividing by n, sending n to ∞, and then letting η decrease to ϵ yields the desired lower bound.

Turning to the upper bound, it suffices to consider n^2 suitors. In particular, note that

$$\begin{aligned} P(T_n \le n(m-\epsilon) &= P\left(\min_{1\le i\le A(n(m-\epsilon))}\left\{A_i + \sum_{j=1}^n V_{ij}\right\} \le n(m-\epsilon)\right) \\ &\le P\left(\min_{1\le i\le A(n(m-\epsilon))}\left\{\sum_{j=1}^n V_{ij}\right\} \le n(m-\epsilon)\right) \\ &\le P\left(\min_{1\le i\le n^2}\left\{\sum_{j=1}^n V_{ij}\right\} \le n(m-\epsilon)\right) + P(A(n(m-\epsilon)) > n^2) \\ &\le n^2 P\left(\sum_{j=1}^n V_{1j} \le n(m-\epsilon)\right) + P(A(nm) > n^2)\ , \end{aligned}$$

but

$$\begin{aligned} P(A(nm) \ge n^2) &= P(A_{n^2} \le nm) \\ &\le P\left(\sum_{i=1}^{n^2} c_1 I(U_i > c_1) \le nm\right) = P(\text{Binomial }(n^2, p) \le rn)\ , \end{aligned}$$

where $p = P(U_i > c_1)$ and $r = m/c_1$. It is easily see that

$$n^{-1}\log P(\text{Binomial }(n^2,p) \le rn) \to -\infty \quad \text{as} \quad n \to \infty\ ,$$

from which the upper bound follows.

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