A Stochastic Model to Capture Space and Time Dynamics in Wireless Communication Systems

William A. Massey and Ward Whitt
AT&T Bell Laboratories
Murray Hill, NJ 07974–0636

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Abstract

We construct a version of the recently developed *Poisson-Arrival-Location Model* (PALM) to study communicating mobiles on a highway, giving the distribution of calls in progress and handoffs as a function of time and space. In a PALM, arrivals generated by a nonhomogeneous Poisson process move independently through a general state space according to a location stochastic process. If, as an approximation, we ignore capacity constraints, then we can use this model to describe the performance of wireless communication systems. Our basic model here is for traffic on a one-way single-lane semi-infinite highway, with movement specified by a deterministic location function. For the highway PALM considered here, key quantities are the call density, the handoff rate, the call-origination-rate density and the call-termination-rate density, which themselves are simply related by two fundamental conservation equations. We show that the basic highway PALM can be applied, together with independent superposition, to treat more complicated models. Our analysis provides connections between teletraffic theory and highway traffic theory.

Keywords: Wireless Communications, Mobile Cellular Communications, Personal Communication Systems, Highway Traffic, Nonhomogeneous Poisson Processes, Nonstationary Queues, Infinite-Server Queues, Queueing Networks.

1 Introduction

In a previous paper [11], we introduced a *Poisson-arrival-location model* (PALM), in which a nonhomogeneous Poisson process generates arrivals that move independently through a general location state space according to some location stochastic process. A PALM generalizes the notion of a finite network of infinite-server queues with nonhomogeneous Poisson arrival processes. Such a network is a PALM with a location space that is a finite set. In fact, the $M_t/G/\infty$ queue is a PALM with a location space containing a single point.

Our motivation for creating the PALM was to develop queueing models for wireless communication systems (see Lee [8]). The key features a PALM enables us to model are:

- Nonstationary behavior
- Non-Markovian behavior
- Service dependence on space and time.

The PALM is able to capture these important features because it does not represent interactions between customers. Our idea is that the PALM should be able to serve as a practical model for mobile phones, just as the infinite-server $M_t/G/\infty$ queue can serve as a practical model for telephone trunk groups. This idea for telephone trunk groups was advanced by Palm [15], which partly explains the name for our model.

Although the PALM ignores interactions between customers, it does capture many of the space-time dynamics of wireless communication systems. For a stationary telephone, only the time a call is in progress is important, so that service can be represented by the length of a telephone conversation. For mobile phones, both the location and time of a call in progress are important, so that service should be represented by a path through a location space as a function of time.

For moving automobiles, the two-dimensional space $\mathbb{R}^2$ is a natural location space. For personal communication systems associated with people or airplanes, three-dimensional space $\mathbb{R}^3$ is a natural location space. When these spaces are partitioned into a finite set of cells, the resulting PALM corresponds to a standard finite-node, infinite-server network. This PALM can then model the number of phone conversations in each cell.

In this paper, our basic model will be for traffic on a one-way, single-lane, semi-infinite highway. Thus we can let our location space be to the interval $[0,\infty)$; a point represents the distance along the highway from the origin. The boundary point 0 will be called the *spatial origin*; it will mark the entrance point to the highway. A nonhomogeneous Poisson arrival process $\{ A(t) \mid -\infty < t < \infty \}$ counts the number of cars that enter the highway. In particular, $A(t)$ counts the number of arrivals up to time t, which we assume is finite with probability one. This nonhomogeneous Poisson process A is characterized by its arrival rate function α. To have $A(t)$ finite with probability one, we assume that $\int_{-\infty}^{t} \alpha(s)ds < \infty$ for all t.

It is significant that many formulas here (e.g., for means) do *not* depend on the Poisson assumption; they hold if A is an arbitrary point process (without multiple points) with time-dependent arrival-rate function α. The Poisson assumption for A is important for distribution conclusions (e.g., the Poisson property and stochastic independence).

In a general PALM, the movement of each customer after arrival can be characterized in terms of a stochastic process. Here, as a further simplification, we assume that the movement of each car after arrival is described by a deterministic function $\chi(s,t)$ which gives the position at time t of a car entering the highway at time s. To specify the deterministic location function $\chi(s,t)$, we can draw on vehicular traffic theory. As in Gazis [3] and Haberman [4], we construct the function χ by a velocity field $v(x,t)$, which gives the velocity at position x at time t. Our analysis of the highway PALM thus brings together teletraffic theory and vehicular traffic theory. Other recent efforts to do this are the papers by Meier-Hellstern, Alonso and O'Neill [12], Montenegro, Sengoku, Yamaguchi and Abe [14], and Seskar, Maric, Holtzman and Wasserman [16]. We anticipate much more of this in the future.

Even though we directly consider only one-way traffic on $[0,\infty)$ originating at 0, our analysis applies to much more general models, by virtue of superposition. To include other cars on the highway going the same direction with different starting points, and other cars going in the opposite direction, we can simply superimpose independent versions of the highway PALM model considered here. Moreover, general movement in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be treated as superpositions of independent one-way traffic along paths in these spaces. Hence the one-way highway is not only a natural model of interest in its own right, but it is a key building block for more complicated models. We discuss such extensions further in Section 9.

It is important to keep in mind, though, that in all these PALM models we are assuming that there are *no interactions* between different cars. The locations of all the cars are mutually independent, conditional on their arrival times. None of this precludes, however, the cars being influenced by their common space-time environment. For example, we cannot *directly* model a car deciding to slow down based on the actions of another car, but we can model this behavior *indirectly* by having all cars slow down in a specific region of the highway at a specific time. In a sequel to this paper, Leung, Massey, and Whitt [9], we show that a variant of the highway PALM introduced here can indeed indirectly capture the effect of an accident.

Here is how this paper is organized. In Section 2, we construct the basic highway model and identify the four fundamental stochastic processes associated with it. They are $Q(x,t)$, the number of *calls in progress* in region $(0,x]$ before time t; $H(x,t)$, the number of *handoffs* at position x (the number of calls in progress passing x) before time t; $C^+(x,t)$, the number of *call originations* in region $(0,x]$ before time t; and $C^-(x,t)$, the number of *call terminations* in region $(0,x]$ before time t. Their means are characterized by a call density $q(x,t)$, a handoff rate $h(x,t)$, a call-origination-rate density $c^+(x,t)$, and a call-termination-rate density $c^-(x,t)$, which are all related by two fundamental conservation equations, see (2.6) and (2.7).

In Sections 3 and 4 we develop the PALM version of the highway model. As in [11], these four processes are Poisson when viewed properly; see Theorem 3.1. This additional stochastic structure allows us to analyze the interactions of cells and the aggregate flows of calls in and out of cells.

In Section 5 we review how infinite-capacity models can be used to approximate finite-capacity models. In particular, we review the modified-offered-load approximation (MOL) for the nonstationary Erlang loss model, i.e., the $M_t/G/c/0$ model, which has c servers and no extra waiting room. In Section 6 we describe how the highway PALM here can be applied to divide the highway into cells so that each cell can just handle its offered traffic. We contrast the case of fixed cell boundaries with dynamic cell boundaries (which are functions of time). Having dynamic cell boundaries is analogous to having dynamic (time-of-day) routing in a standard voice network. Fewer cells are required with dynamic cell boundaries. The highway PALM here provides a way to quantify the difference.

In Section 7 we indicate how we can describe the proportion of blocked originating calls in a cell and the proportion of blocked handoffs from one cell to another. Here we exploit the MOL approximations discussed in Section 5.

In Section 8 we analyze a special case of a highway PALM in which all cars have constant velocity. For this special case, we can easily express the call density $q(x,t)$ in terms of the external arrival rate function α and other model features. We consider a simple numerical example to investigate the influence of the distributions beyond their means.

Finally, in Section 9 we show how to exploit independent superposition in order to build more elaborate models. There we show that the two fundamental conservation equations still hold with appropriate definitions.

2 General Conservation Equations

Before we specialize to the Highway PALM model, we will describe the conservation equations that hold for general systems. For every space time pair (x,t), we can associate the following four random variables:

$$
\begin{aligned}
Q(x,t) &= \text{ number of active calls in the interval } (0,x] \text{ at time } t \\
H(x,t) &= \text{ number of calls in progress moving past position } x \text{ before time } t \\
C^{+}(x,t) &= \text{ number of call initiations in interval } (0,x] \text{ before time } t \\
C^{-}(x,t) &= \text{ number of call terminations in interval } (0,x] \text{ before time } t.
\end{aligned}
$$

We also call $H(x,t)$ the number of call *handoffs* at x before t, because in many mobile communication systems if a boundary of a cell were at x, then the call in progress would have to be handed off from a base station in one cell to a base station in the next cell when the mobile passes the cell boundary.

Assuming that all traffic moves only from left to right down the positive real line, these four random variables satisfy the following conservation relation:

$$C^{+}(x,t) = Q(x,t) + H(x,t) + C^{-}(x,t). \tag{2.1}$$

Now assume that their expectations are finite and differentiable in both x and t. Let $q(x,t)$ and $h(x,t)$ be respectively the *active call density* and the *call handoff rate*, defined by

$$q(x,t) \equiv \frac{\partial}{\partial x}\mathsf{E}[Q(x,t)] \quad \text{and} \quad h(x,t) \equiv \frac{\partial}{\partial t}\mathsf{E}[H(x,t)]. \tag{2.2}$$

Similarly, let $c^{+}(x,t)$ and $c^{-}(x,t)$ be respectively the *call initiation rate-density* and the *call termination rate-density*, defined by

$$c^{+}(x,t) \equiv \frac{\partial^2}{\partial x \partial t}\mathsf{E}[C^{+}(x,t)] \quad \text{and} \quad c^{-}(x,t) \equiv \frac{\partial^2}{\partial x \partial t}\mathsf{E}[C^{-}(x,t)]. \tag{2.3}$$

Applying the operator $\frac{\partial^2}{\partial x \partial t}\mathsf{E}[\cdot]$ to the conservation relation, we get the differential equation

$$\frac{\partial}{\partial t}q(x,t) + \frac{\partial}{\partial x}h(x,t) = c^+(x,t) - c^-(x,t). \tag{2.4}$$

Assuming that the calling density $q(x,t)$ is never zero unless the call handoff rate $h(x,t)$ is also, let

$$v(x,t) \equiv \frac{h(x,t)}{q(x,t)}, \tag{2.5}$$

when the ratio is well defined and set $v(x,t)$ equal to zero otherwise. We can give $v(x,t)$ the physical interpretation of the *aggregate mean velocity* for the active calls. We have now created by definition the relation

$$h(x,t) = v(x,t)q(x,t). \tag{2.6}$$

This parallels the fundamental conservation equations of highway traffic, see p. 274 of Haberman [4]. The difference here is that we are talking about calls instead of cars. Substituting this into the conservation differential equation above, we get

$$\frac{\partial}{\partial t}q(x,t) + \frac{\partial}{\partial x}v(x,t)q(x,t) = c^+(x,t) - c^-(x,t). \tag{2.7}$$

The resulting differential equation (2.7) is a one-dimensional version of the generalized conservation law for fluid motion, see p. 323 of Symon [17]. This equation governs the mean behavior of *any* stochastic highway traffic model. The standard continuity equation for charges in semiconductor statistics is a concrete application of this law, see p. 51 of Sze [18]. In the special case in which $c^+(x,t) = c^-(x,t)$, equation (2.7) reduces to the standard mass conservation equation in fluid dynamics, see p. 317 of Symon [17].

3 Constructing the Highway PALM Model

The conservation equations given above hold for any stochastic highway model. These equations govern the mean behavior for Q, H, C^+, and C^-, but do not give any insight into their distributions or behavior as stochastic processes indexed by both space and time. By specializing to the PALM version of the highway model, we can handle both issues.

For simplicity, we assume that each car goes through three successive phases: think mode, calling mode, and completion mode. Given that a car arrives to the highway at time s, the car (or its driver) is in *think mode* until some random *call origination time* T_s^+ when a call is placed. After the call is placed, the call is in progress and the car is in calling mode until some random *call termination time* T_s^-. After T_s^-, the call is finished and the car is in completion mode. Observing that we will always have $s \le T_s^+ \le T_s^-$, we will refer to $T_s^+ - s$ as the *think time* and $T_s^- - T_s^+$ as the *call holding time*. In Section 9 and in Leung, Massey, and Whitt [9] we discuss extensions to the highway PALM in which cars can make multiple calls.

Assuming that cars do not pass each other, we can let the aggregate mean velocity field $v(x,t)$ in (2.5) be the velocity field for the cars. We then construct the trajectory field $\chi(s,t)$ as the unique solution to the differential equation

$$\frac{\partial}{\partial t}\chi(s,t) = v(\chi(s,t),t) \tag{3.1}$$

for $t > s$, where $\chi(s,s) = 0$. From the trajectory field, we can define $\tau(s,x)$ and $\sigma(x,t)$ where:

$$\begin{aligned} \tau(s,x) &= \text{time at position } x \text{ for a car entering the highway at time } s, \\ \sigma(x,t) &= \text{highway entrance time for a car to be in position } x \text{ at time } t. \end{aligned}$$

Figure 3.1 is a space-time diagram depicting a car in think and call modes. The solid-line portion of the car's trajectory represents when the call is in progress. In the dashed-line portions before and after, the call is not in progress.

To have $\sigma(x,t)$ and $v(x,t)$ well defined, we need to assume that two cars cannot be in the same place at the same time or, equivalently, that *cars cannot pass each other.* (In Section 9, we will treat cases where cars can pass by using the principle of independent superposition.)

If $Q(x,t)$ is the number of cars with calls in progress at time t within the interval of space $(0,x]$. We can express it in terms of Poisson integration as:

$$Q(x,t) \equiv \int_{\sigma(x,t)}^{t} 1_{\{T_s^+ \le t < T_s^-\}} dA(s). \tag{3.2}$$

where 1_B is the indicator function of the set B; i.e., $1_B(x) = 1$ if $x \in B$ and 0 otherwise. By "Poisson integration" we mean stochastic integration with respect to the nonhomogeneous Poisson process A. This integration can be simply defined as a finite sum over the jumps of the Poisson process A for each sample path; see the Appendix for more details.

The process Q can also be constructed by applying Poisson integration to a *location process* $L_s(t)$, as we did in [11]. We define the *location space* to be the set of ordered pairs $(x,0)$ and $(x,1)$ for $x \in \mathbb{R}_+$, i.e., where x is a nonnegative real number. The pair $(x,0)$ corresponds to the car being at position x on the highway and in the think mode. The pair $(x,1)$ corresponds to the car being at position x in in calling mode. We can then let $L_s(t)$ be defined by

$$L_s(t) = \begin{cases} \Delta_* & s < t, \\ (\chi(s,t),0) & t \le s < T_s^+, \\ (\chi(s,t),1) & T_s^+ \le s < T_s^-, \\ \Delta^* & T_s^- \le s, \end{cases} \tag{3.3}$$

where Δ_* and Δ^* are the "pre-arrival" and "completion-mode" states as defined in [11]. This gives us an alternate construction of $Q(x,t)$ as

$$Q(x,t) = \int_{\sigma(x,t)}^{t} 1_{\{L_s(t)\in(0,x]\times\{1\}\}} dA(s). \tag{3.4}$$

We now assume in addition that σ is twice continuously differentiable, α is once differentiable, and for almost all s (with respect to the measure $\alpha(s)ds$), the distributions of both T_s^+ and T_s^- have densities. With these assumptions, the cell density q in (2.2) is well defined.

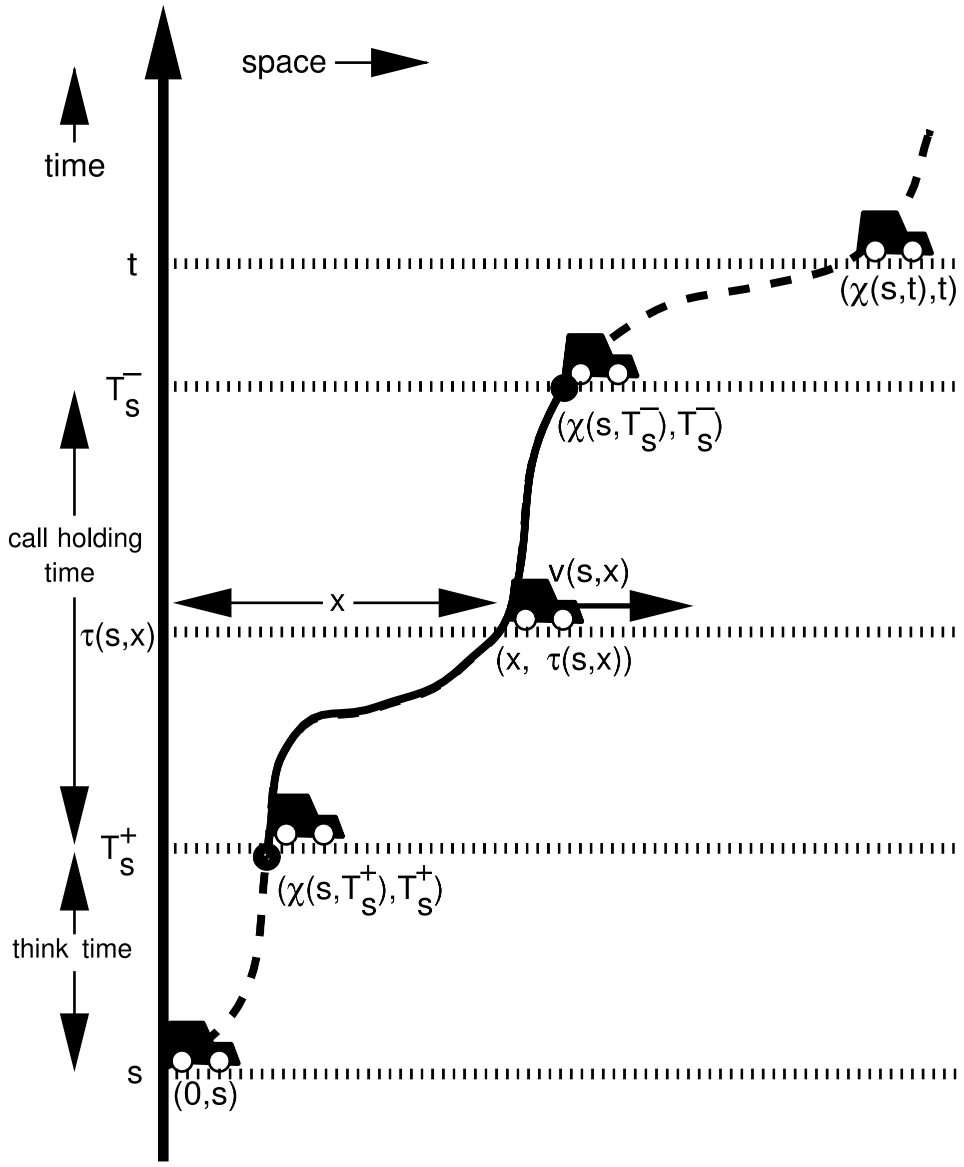

Figure 3.1: Space-time diagram for a car in think and call modes.

Suppose that the highway is partitioned into subintervals called *cells*. When a car is in a cell, its calls are handled by a designated *base station*. To describe the handoffs of calls from one cell to another at the cell boundary point x within the time interval $(-\infty, t]$, we use the function $\tau(s,x)$. Let one radio base station cover the cell or highway region $(w, x]$ and another cover $(x, y]$, where $0 \le w < x < y \le \infty$. We can express $H(x,t)$ in terms of Poisson integration as:

$$H(x,t) \equiv \int_{-\infty}^{\sigma(x,t)} 1_{\{T_s^+ \le \tau(s,x) < T_s^-\}} dA(s). \tag{3.5}$$

Figure 3.2 gives a space-time diagram for the processes Q and H for a possible location function χ and a possible realization of the nonhomogeneous Poisson arrival process. This particular realization has seven car arrivals. As in Figure 3.1, the solid-line portion of each trajectory represents when the call is in progress. For the specified x and t, $Q(x,t) = 2$, while $H(x,t) = 3$. The same conditions that allow us to define $q(x,t)$ also enable us to define a *handoff rate* $h(x,t)$ in (2.2).

Finally, the Poisson integral representations of $C^+(x,t)$ and $C^-(x,t)$ are:

$$C^+(x,t) \equiv \int_{-\infty}^{t} 1_{\{T_s^+ \le \min(\tau(s,x),t)\}} dA(s) \tag{3.6}$$

and

$$C^-(x,t) \equiv \int_{-\infty}^{t} 1_{\{T_s^- \le \min(\tau(s,x),t)\}} dA(s). \tag{3.7}$$

Using the same conditions as for $q(x,t)$ and $h(x,t)$ (the second derivatives of σ are needed here), we see that the *call-origination-rate density* $c^+(x,t)$ and the *call-termination-rate density* $c^-(x,t)$ in (2.3) are well defined.

We now exploit the PALM structure to deduce distributional properties for the basic processes Q, H, C^+, and C^-. We will prove the next two results in the Appendix.

Theorem 3.1 *The following results hold:*

(a) For all real t, $\{ Q(x,t) \mid x \ge 0 \}$ is a Poisson process with

$$\mathsf{E}[Q(x,t)] = \int_{\sigma(x,t)}^{t} \mathsf{P}\left(T_s^+ \le t < T_s^-\right) \alpha(s) ds. \tag{3.8}$$

(b) For all non-negative x, $\{ H(x,t) \mid -\infty < t < \infty \}$ is a Poisson process with

$$\mathsf{E}[H(x,t)] = \int_{-\infty}^{\sigma(x,t)} \mathsf{P}\left(T_s^+ \le \tau(s,x) < T_s^-\right) \alpha(s) ds. \tag{3.9}$$

(c) Both $\{ C^+(x,t) \mid x \ge 0, -\infty < t < \infty \}$ and $\{ C^-(x,t) \mid x \ge 0, -\infty < t < \infty \}$ are two dimensional Poisson processes with

$$\mathsf{E}[C^+(x,t)] = \int_{-\infty}^{\sigma(x,t)} \mathsf{P}\left(T_s^+ \le \tau(s,x)\right) \alpha(s) ds + \int_{\sigma(x,t)}^{t} \mathsf{P}\left(T_s^+ \le t\right) \alpha(s) ds. \tag{3.10}$$

and

$$\mathsf{E}[C^-(x,t)] = \int_{-\infty}^{\sigma(x,t)} \mathsf{P}\left(T_s^- \le \tau(s,x)\right) \alpha(s) ds + \int_{\sigma(x,t)}^{t} \mathsf{P}\left(T_s^- \le t\right) \alpha(s) ds. \tag{3.11}$$

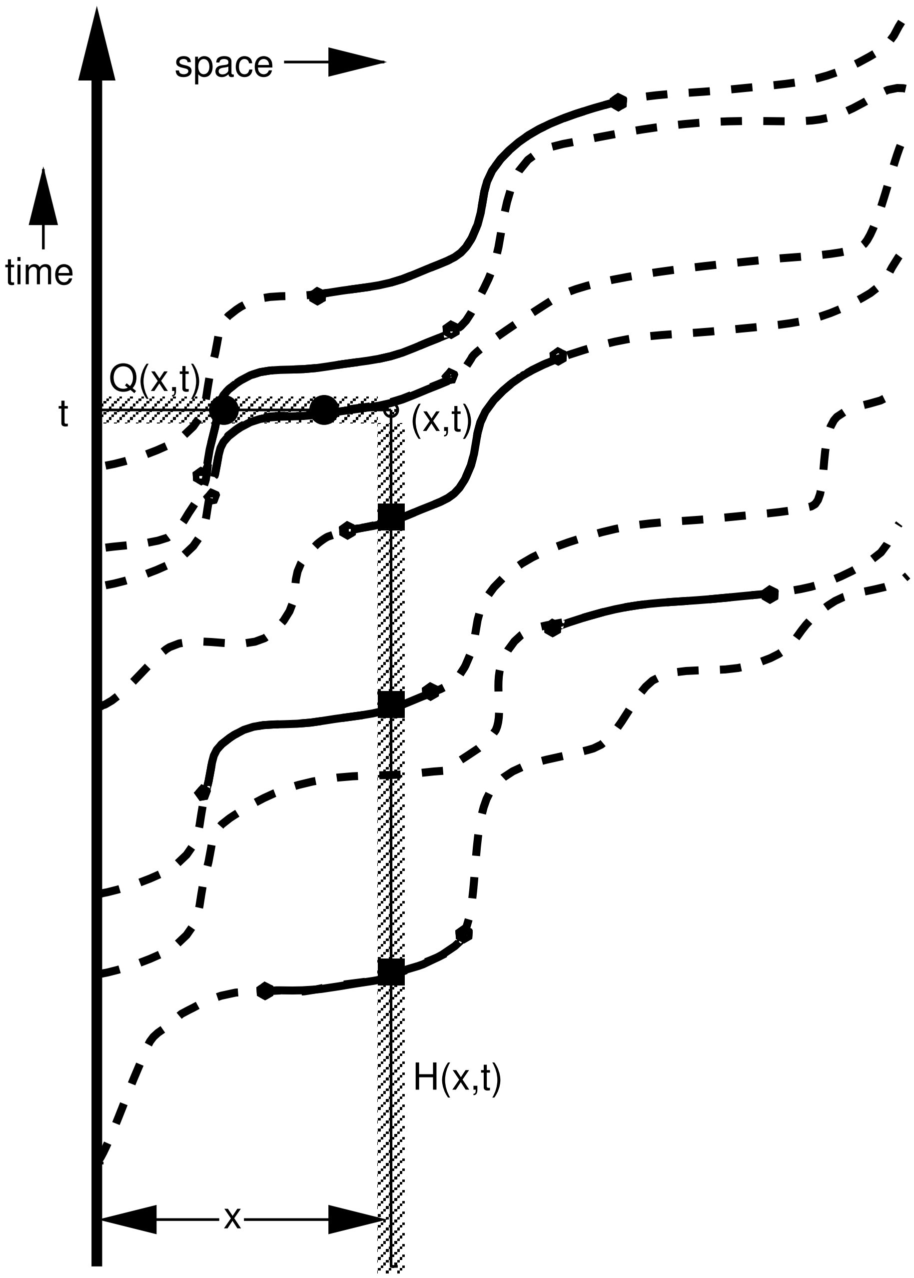

Figure 3.2: Space-time diagram for the processes $Q(x,t)$ and $H(x,t)$.

As in Remark 2.3 of [11], it is significant that the mean formulas in Theorem 3.1 do *not* depend on the arrival process A being a Poisson process. Of course, the Poisson process conclusions *do* depend on A being Poisson. As in [11], let $\oplus$ denote the sum of two random quantities, where the summands are stochastically independent.

Theorem 3.2 *The following independence relations hold for the random processes Q, H, C^+, and C^-:*

(a) For all space-time pairs (x,t) and (x',t') with $\sigma(x,t) \geq \sigma(x',t')$, $Q(x,t)$ is independent of $H(x',t')$.

(b) For all t, the processes $\{ C^-(x,t) \mid x \geq 0 \}$ and $\{ Q(x,t) \mid x \geq 0 \}$ are independent.

(c) Similarly, for all $x \geq 0$, the process $\{ C^-(x,t) \mid -\infty < t < \infty \}$ is independent of the process $\{ H(x,t) \mid -\infty < t < \infty \}$.

(d) Finally, for all space-time pairs (x,t),

$$C^+(x,t) = Q(x,t) \oplus H(x,t) \oplus C^-(x,t). \tag{3.12}$$

4 Cell Traffic Analysis

At this point, we want to think more in terms of cells, so let $(x,y]$ be a generic cell where of course $x < y$. Note that calls can arrive at a cell in two ways: (1) by a car already in the cell initiating a call, and (2) by a handoff of a call in progress for a car entering the cell. Similarly, calls can leave a cell in two ways: (1) by a car already in the cell terminating a call, and (2) by a handoff of a call in progress for a car leaving the cell. In this section we describe these four processes and their relationships.

If we only focus on whether or not a call is in progress in a cell and suppress the position within a cell, then a partition of the highway into a finite number of cells allows us to represent the highway PALM as an $(M_t/G_t/\infty)^N/G_t$ network in the terminology of [11]. Then cells are nodes and N is the number of nodes. If we number the cells starting at the origin, then the network topology is as shown in Figure 4.1. Each cell has an external arrival process corresponding to call originations in the cell and a departure process corresponding to call terminations in the cell. There are also flows from node $i-1$ to i for $2 \leq i \leq N+1$ (the horizontal lines in Figure 4.1) corresponding to handoffs of calls in progress when a car passes a cell boundary.

We now focus on one cell within this $(M_t/G_t/\infty)^N/G_t$ network. From [11], we know that this cell itself can be regarded as an $M_t/G_t/\infty$ queue. We do a direct analysis here. The results below can be extended to more general location spaces. However, here we exploit the fact that no call can re-enter a cell after it has left; e.g., see Theorem 3.5 of [11].

We now define $C^+_{(x,y]}(t)$ and $C^-_{(x,y]}(t)$ to be respectively the *number of calls that originate and terminate in cell* $(x,y]$. We can express them in terms of C^+ and C^- as:

$$C^+_{(x,y]}(t) \equiv C^+(y,t) - C^+(x,t) \tag{4.1}$$

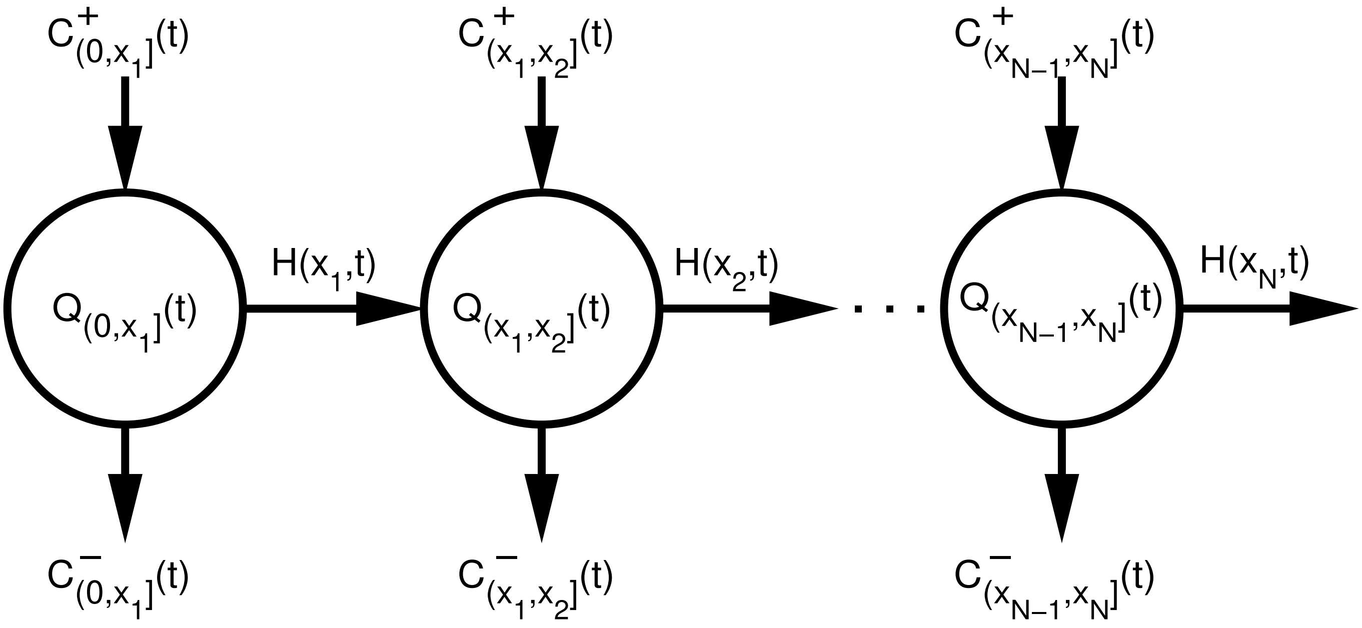

Figure 4.1: An Infinite Server Network Representation of the Highway Model.

and

$$C^-_{(x,y]}(t) \equiv C^-(y,t) - C^-(x,t). \tag{4.2}$$

Similarly, if we define $Q_{(x,y]}(t)$ to equal the number of calls in cell $(x,y]$ at time t, then

$$Q_{(x,y]}(t) \equiv Q(y,t) - Q(x,t). \tag{4.3}$$

By Theorem 3.1, $Q_{(x,y]}(t)$ has a Poisson distribution with mean $q_t(x,y)$ in (6.1). The fact that Q for fixed t is a spatial Poisson process can be reinterpreted for cellular traffic. If $\Gamma_1, \ldots, \Gamma_N$ are a pairwise disjoint collection of cells, we then have

$$\mathsf{P}\left(Q_{\Gamma_1}(t) = k_1, \ldots, Q_{\Gamma_N}(t) = k_N\right) = \prod_{i=1}^{N} \mathsf{P}\left(Q_{\Gamma_i}(t) = k_i\right) = e^{-q_t(\Gamma)} \prod_{i=1}^{N} \frac{q_t(\Gamma_i)^{k_i}}{k_i!}, \tag{4.4}$$

where Γ is the union of the Γ_i's. Here we see that like transient distribution for infinite server networks (appropriately initialized, see [11]) and the equilibrium distribution for the Jackson network [5], the PALM model has a product form structure. Although the PALM model assumes that the number of active calls in disjoint cells at time t are mutually independent, it does *not* assume that this is the case for disjoint cells observed at *different* times. In general, we can say that

$$\mathsf{Cov}\left[Q_{\Gamma_1}(t), Q_{\Gamma_2}(u)\right] = \int_{\sigma(\Gamma_1,t)\cap\sigma(\Gamma_2,u)} \mathsf{P}\left(T_s^+ \le t \le u < T_s^-\right) \alpha(s)ds. \tag{4.5}$$

where $\sigma(\Gamma,t) \equiv \{\, \sigma(x,t) \mid x \in \Gamma \,\}$.

As in Section 3, let $\oplus$ denote the sum of two random variables that are stochastically independent. We write $X = Y \ominus Z$ if $X \oplus Z = Y$. We will prove the following result in the Appendix.

Theorem 4.1 *For any given cell $(x, y]$, the following results hold:*

(a) The arrival process of calls to cell $(x, y]$ is Poisson and equals

$$\left\{ H(x,t) \oplus C^{+}_{(x,y]}(t) \mid -\infty < t < \infty \right\}. \tag{4.6}$$

(b) The departure process of calls from cell $(x, y]$ is Poisson and equals

$$\left\{ H(y,t) \oplus C^{-}_{(x,y]}(t) \mid -\infty < t < \infty \right\}. \tag{4.7}$$

(c) Finally, we have for all t,

$$Q_{(x,y]}(t) = [H(x,t) \oplus C^{+}_{(x,y]}(t)] \ominus [H(y,t) \oplus C^{-}_{(x,y]}(t)]. \tag{4.8}$$

5 Approximate Analysis of the $M_t/G/c/0$ Queue

In the next three sections, we will show how we can apply the highway PALM to the design and performance analysis of a wireless communication system. We will formulate an approximate analysis of the wireless communication system based upon an exact analysis of the highway PALM. Before doing so, we want to give the underlying motivation for this approximate analysis by reviewing how the exact analysis of the $M_t/G/\infty$ model plays the same role in the approximate analysis of the $M_t/G/c/0$ model.

First, the $M/M/c/0$ queue is the Markovian model for a telephone trunk group. We can encode the complete dynamics of this Markov model by the state space transition diagram in Figure 5.1. There λ is the call arrival rate, μ is the service rate, and c is the number of channels. Let Q be the steady-state number of busy trunklines. From standard birth-and-death process theory, we obtain a closed-form solution for the distribution of Q and derive the classical *Erlang blocking formula*

$$\mathsf{P}(Q = c) = \beta_c(\lambda/\mu) \equiv \frac{(\lambda/\mu)^c}{c!} \Bigg/ \sum_{i=0}^{c} \frac{(\lambda/\mu)^i}{i!}\,. \tag{5.1}$$

The Erlang blocking formula gives the probability that all trunklines are busy, which also equals the proportion of arriving customers that are blocked from entering the system. Moreover, if Q is the steady-state number of trunklines for the non-Markovian $M/G/c/0$ queue with i.i.d. service times distributed as S with a general distribution, then $\mathsf{P}(Q = c) = \beta_c(\lambda \mathsf{E}[S])$; i.e., the $M/G/s/0$ model has the insensitivity property: the distribution of Q depends on the distribution of S only through its mean.

A more thorough performance analysis of telephone trunk groups forces us to deal with a call arrival process that is nonstationary. When this process is nonstationary Poisson, we call it an $M_t/G/c/0$ system. Although this model is more realistic, it is far less tractable. In particular, we lose the simplicity of the Erlang blocking formula.

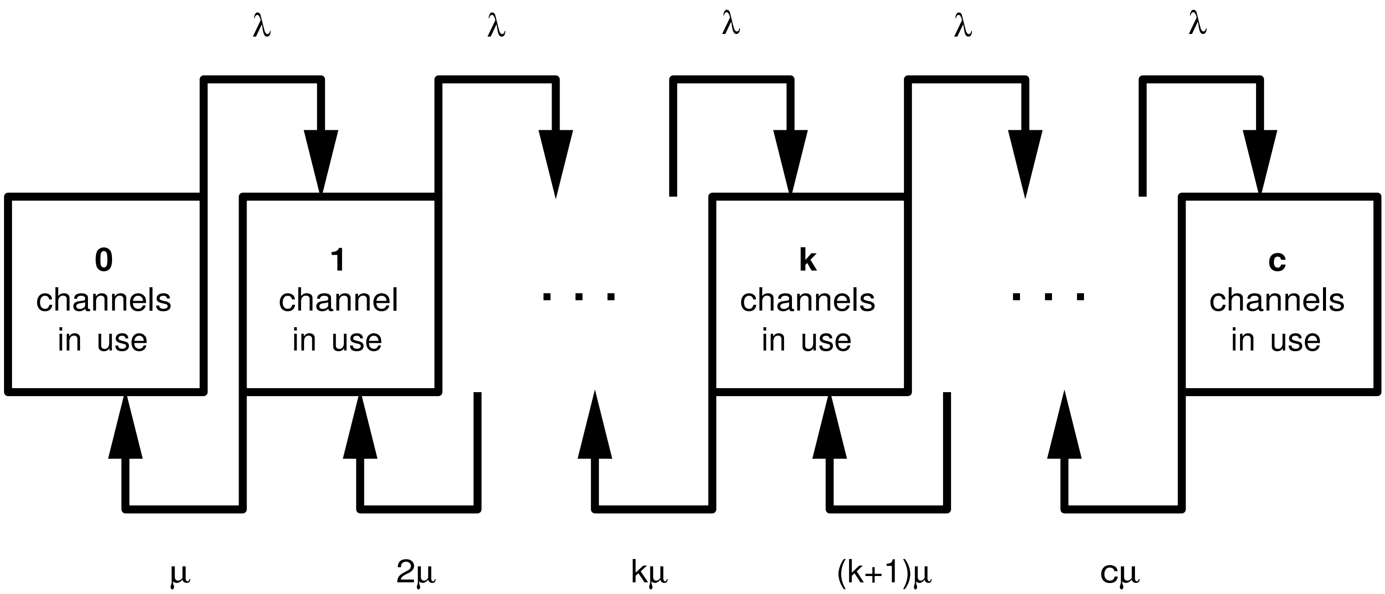

Figure 5.1: State Space Diagram for the Erlang Model.

Before we discuss how to approximate these blocking probabilities, we consider what we *mean* by blocking probability in the context of nonstationary systems. Let $\lambda(t)$ be the arrival rate and let $Q(t)$ be the number of busy trunklines at time t for the $M_t/G/c/0$ system. Now let (t_0, t_1) denote an interval of time of interest, and let

$$\overline{\beta}(t_0, t_1) \equiv \int_{t_0}^{t_1} \lambda(\tau) \mathsf{P}(Q(\tau) = c) d\tau \Big/ \int_{t_0}^{t_1} \lambda(\tau) d\tau \ , \tag{5.2}$$

which is the expected number of blocked calls during the interval (t_0, t_1), divided by the expected number of arriving calls during the same interval. Note that for the $M/G/c/0$ model, we have $\overline{\beta}(t_0, t_1) = \beta_c(\lambda \mathsf{E}[S])$. For may applications, it seems more appropriate to measure the proportion of arriving customers that are blocked than the proportion of time that the system is available. This makes the weighted average $\overline{\beta}(t_0, t_1)$ a more suitable metric for blocking in the $M_t/G/c/0$ system than the simple time average

$$\widehat{\beta}(t_0, t_1) \equiv \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} P(Q(\tau) = c) d\tau. \tag{5.3}$$

However, we can use the time-dependent blocking formula $P(Q(\tau) = c)$ to compute either $\overline{\beta}(t_0, t_1)$ in (5.2) or $\widehat{\beta}(t_0, t_1)$ in (5.3). (Although at first glance $\overline{\beta}(t_0, t_1)$ appears to be the ratio of expectations instead of an expectation of a ratio and not exactly the expected proportion of customers blocked in the interval (t_0, t_1), the latter interpretation is *valid* with a minor modification as we show in Lemma 10.3.)

The first approximate analysis method we present is the *pointwise stationary approximation* (PSA), which is

$$\mathsf{P}(Q(t) = c) \approx \beta_c(\lambda(t) \mathsf{E}[S]). \tag{5.4}$$

This is an approximation technique that is applicable to any time-inhomogeneous process. The PSA approximation for the blocking metric in (5.2) is

$$\overline{\beta}(t_0, t_1) \approx \int_{t_0}^{t_1} \lambda(\tau) \beta_c(\lambda(\tau) \mathsf{E}[S]) d\tau \Big/ \int_{t_0}^{t_1} \lambda(\tau) d\tau \ . \tag{5.5}$$

There are many limitations to PSA. It does not properly capture the history and it is insensitive to the distribution of S. An alternative to PSA comes from an exact analysis of the $M_t/G/\infty$ queue. We can obtain a tractable nonstationary analysis of a system in exchange for assuming that we have an infinite number of trunklines. Let $Q^*(t)$ be the number of busy trunklines for the $M_t/G/\infty$ queue. If the system starts empty in the past, then $Q^*(t)$ has the Poisson distribution with mean

$$m(t) \equiv \mathsf{E}[Q^*(t)] = \mathsf{E}\left[\int_{t-S}^{t} \lambda(\tau) d\tau\right]; \tag{5.6}$$

see Eick, Massey and Whitt [2]. Going back to the $M_t/G/c/0$ queue, the *modified offered load approximation* (MOL) is

$$\mathsf{P}(Q(t) = c) \approx \beta_c(m(t)), \tag{5.7}$$

for $m(t)$ is (5.6); see Jagerman [6] and Massey and Whitt [10]. Just like PSA, MOL is exact for the stationary $M/G/c/0$ model, because then $m(t) = \lambda(t) \mathsf{E}[S]$. Unlike PSA, MOL captures the effect of the distribution of S beyond its mean; see Davis, Massey, and Whitt [1].

Hence the MOL approximation for the blocking metric in (5.2) is

$$\overline{\beta}(t_0, t_1) \approx \int_{t_0}^{t_1} \lambda(\tau)\beta_c(m(\tau))d\tau \Big/ \int_{t_0}^{t_1} \lambda(\tau)d\tau \ . \tag{5.8}$$

6 Dividing the Highway into Cells

We now illustrate how we can apply the highway PALM to a wireless communication system. Consider the problem of dividing the highway into cells so that the offered load in each cell is at most a specified λ. We might determine λ by specifying a *channel capacity* c for each cell and a target call *blocking probability* b. Then, using Erlang's formula, we can find the unique *offered load* λ such that $\beta_c(\lambda) = b$. Using MOL as inspiration, we define the *offered load* for $(x, y]$ at time t to be the measure $q_t(x, y)$ defined by

$$q_t(x, y) \equiv \int_x^y q(z, t)dz = \mathsf{E}[Q(y, t)] - \mathsf{E}[Q(x, t)]. \tag{6.1}$$

We have shown in Theorem 3.1 that $Q(y, t) - Q(x, t)$ has a Poisson distribution, and is independent of $Q(y', t) - Q(x', t)$ for any other disjoint space interval $(x', y']$.

For constructing cell boundaries, there are two natural cases: *dynamic cell boundaries*, which are functions of time, and *fixed cell boundaries*. With dynamic cell boundaries, we consider all time points separately. With fixed cell boundaries, we must handle the worst case. We could also consider boundaries that can be changed at only finitely many time points, but we do not.

Assuming that the average offered load function $q_t(x, y)$ in (6.1) can be determined, for any given λ, we can recursively define dynamic cell boundaries as follows:

1. Set $x_0(t) = 0$.

2. Given $x_i(t)$, set $x_{i+1}(t) \equiv \sup \{ \, y \mid x_i(t) < y \ \text{ and } \ q_t(x_i(t), y) \leq \lambda \, \}$.

We can also define fixed cell boundaries for a finite time interval $[0, T]$ by the following set of recursion relations:

1. Set $\hat{x}_0 = 0$.

2. Given $\hat{x}_i$, for each $0 \leq t \leq T$, set $y_{i+1}(t) \equiv \sup \{ \, y \mid \hat{x}_i < y \ \text{ and } \ q_t(\hat{x}_i, y) \leq \lambda \, \}$.

3. Now set $\hat{x}_{i+1} \equiv \inf_{0 \leq t \leq T} y_{i+1}(t)$.

It is intuitively obvious and easy to prove that fewer dynamic cells are needed than fixed cells for any average offered load function.

Proposition 6.1 *Fewer dynamic cells are needed than fixed cells since for all i we have $\hat{x}_i \leq x_i(t)$.*

Proof: We use induction on i. The result is immediate for $i = 0$. Hence, suppose that $\hat{x}_i \leq x_i(t)$. Since both sequences are increasing, we are done if $\hat{x}_{i+1} \leq x_i(t)$. Therefore, we need only consider the case $\hat{x}_{i+1} > x_i(t)$. By definition, $q_t(\hat{x}_i, \hat{x}_{i+1}) \leq \lambda$ and so $q_t(x_i(t), \hat{x}_{i+1}) \leq \lambda$ follows from $\hat{x}_i \leq x_i(t) < \hat{x}_{i+1}$. By the recursive definition of $x_{i+1}(t)$, we have immediately that $\hat{x}_{i+1} \leq x_{i+1}(t)$. ∎

Let $n(t)$ equal the number of dynamic cells needed at time t. We can compute this by observing that

$$n(t) = \lceil q_t(0, \infty)/\lambda \rceil, \tag{6.2}$$

where $\lceil x \rceil$ is the least integer greater than or equal to x, and $q_t(0, \infty)$ equals the average queue length in the second queue for a two station $M_t/G_t/\infty$-network. The service times at the first station are the think times $T_s^+ - s$, while the service times at the second station are the call holding times $T_s^- - T_s^+$.

Let $\Gamma \equiv (x, y]$ be a region of the highway where we want to partition the region into the optimal number of cells during a given time interval $(s, t]$ under the following conditions:

1. Each cell will carry at most c simultaneous calls.

2. The offered load of calls to the cells will be evenly distributed.

3. Each cell will tolerate no more than some ϵ_B fraction of the initiated calls blocked.

4. Each cell will tolerate no more than some ϵ_D fraction of the handoff calls dropped.

We then choose for some N and some time τ a partition of cells $\Gamma_1(\tau), \ldots, \Gamma_N(\tau)$ where $\Gamma_i(\tau) \equiv (x_{i-1}(\tau), x_i(\tau)]$, $x = x_0(\tau) < x_1(\tau) < \cdots < x_N(\tau) = y$. If the offered load is evenly distributed, then $q_\tau(\Gamma_i(\tau)) = q_\tau(\Gamma(\tau))/N$. Given c, ϵ_B, and ϵ_D, we will say that the optimal number of cells is $N(\Gamma)$, which is defined to be the minimal integer N such that

$$\max_{1 \leq i \leq N} \frac{\int_s^t c_\tau^+(\Gamma_i(\tau)) \beta_c(q_\tau(\Gamma)/N) d\tau}{\int_s^t c_\tau^+(\Gamma_i(\tau)) d\tau} \leq \epsilon_B \tag{6.3}$$

and

$$\max_{1 \leq i \leq N} \frac{\int_s^t h_\tau(x_{i-1}(\tau), \tau) \beta_c(q_\tau(\Gamma)/N) d\tau}{\int_s^t h_\tau(x_{i-1}(\tau), \tau) d\tau} \leq \epsilon_D. \tag{6.4}$$

Note that $N(\Gamma) \leq N_0(\Gamma)$, where

$$N_0(\Gamma) \equiv \min \left\{ N \,\middle|\, \beta_c \left(\sup_{s \leq \tau \leq t} q_\tau(\Gamma)/N \right) \leq \min(\epsilon_B, \epsilon_D) \right\}. \tag{6.5}$$

We intend to study methods for effectively finding desirable time-dependent cells $\{\Gamma_i(\tau)\}$ in future work.

7 Approximate Blocking in Cells

In this section we indicate how we can obtain approximate blocking probabilities for cells once they have been constructed. These blocking probabilities exploit MOL and PSA heuristics plus exact results for the highway PALM.

Using MOL-like estimation methods, we can define for any cell $(x, y]$ a metric $\overline{\beta}^C(t_0, t_1)$ for the fraction of *blocked originating calls* during some interval (t_0, t_1) as approximately

$$\overline{\beta}^C(t_0, t_1) \approx \int_{t_0}^{t_1} c^+_{(x,y]}(s)\beta_c(q_s(x, y))ds \Big/ \int_{t_0}^{t_1} c^+_{(x,y]}(s)ds\ , \tag{7.1}$$

where $c^+_{(x,y]}(s) \equiv \int_x^y c^+(z, s)dz$. Similarly, we can define a metric $\overline{\beta}^H(t_0, t_1)$ for the fraction of *dropped handoff calls* entering cell $(x, y]$ during the busy hour (t_0, t_1) to be

$$\overline{\beta}^H(t_0, t_1) \approx \int_{t_0}^{t_1} h(x, s)\beta_c(q_s(x, y))ds \Big/ \int_{t_0}^{t_1} h(x, s)ds \tag{7.2}$$

The analysis above assumes that handoffs and originating calls are treated the same, but it may be desirable to given handoffs preferential treatment. One way to do this is with trunk reservation, as discussed in Kelly [7] and Mitra, Gibbens and Huang [13]. We can apply PSA-like estimation methods to analyzing handoffs with trunk reservations. Suppose that each base station provides c channels but admits new originating calls only when at least $r + 1$ channels are free. The time-homogeneous Markovian version of such a system is specified by the state space diagram in Figure 7.1 (assuming a constant service rate μ for all calls). From basic birth-and-death process theory, we see that the steady-state probability for dropped handoff traffic is equal to

$$\beta^H_{r,c}(\lambda_H, \lambda_C) \equiv \frac{(\lambda_H + \lambda_C)^{c-r}\lambda_H^r}{c!} \Bigg/ \left(\sum_{i=0}^{c-r} \frac{(\lambda_H + \lambda_C)^i}{i!} + \sum_{j=1}^{r} \frac{(\lambda_H + \lambda_C)^{c-r}\lambda_H^j}{(c - r + j)!} \right) \tag{7.3}$$

and the steady state probability for all blocked traffic is

$$\beta^C_{r,c}(\lambda_H, \lambda_C) \equiv \sum_{j=0}^{r} \frac{(\lambda_H + \lambda_C)^{c-r}\lambda_H^j}{(c - r + j)!} \Bigg/ \left(\sum_{i=0}^{c-r} \frac{(\lambda_H + \lambda_C)^i}{i!} + \sum_{j=1}^{r} \frac{(\lambda_H + \lambda_C)^{c-r}\lambda_H^j}{(c - r + j)!} \right) . \tag{7.4}$$

We will use $\beta^H_{r,c}$ and $\beta^C_{r,c}$ in (7.3) and (7.4) to develop a PSA approximation in the nonstationary model. We have the time-dependent arrival rates in $h(x, t)$ and $c^+(x, t)$, but we need to know an appropriate average service time. For this purpose, for $s < \sigma(x, t)$, let $R_s(y, t)$ be the *residual conversation time* after time t in cell $(x, y]$ for a call that arrives to the highway at time s where $\sigma(y, t) < s < \sigma(x, t)$, or

$$R_s(y, t) = \left(\min(T_s^-, \tau(y, s)) - t\right) \cdot 1_{\{T_s^+ \le t < T_s^-\}}. \tag{7.5}$$

We can express the sum of residual times in cell $(x, y]$ at time t as $\int_{\sigma(y,t)}^{\sigma(x,t)} R_s(y, t)dA(s)$. By the properties of Poisson integration (see the Appendix), we can express the expectation of this random variable as

$$\mathsf{E}\left[\int_{\sigma(y,t)}^{\sigma(x,t)} R_s(y, t)dA(s)\right] = \int_{\sigma(y,t)}^{\sigma(x,t)} \mathsf{E}\left[R_s(y, t)\right] \alpha(s)ds. \tag{7.6}$$

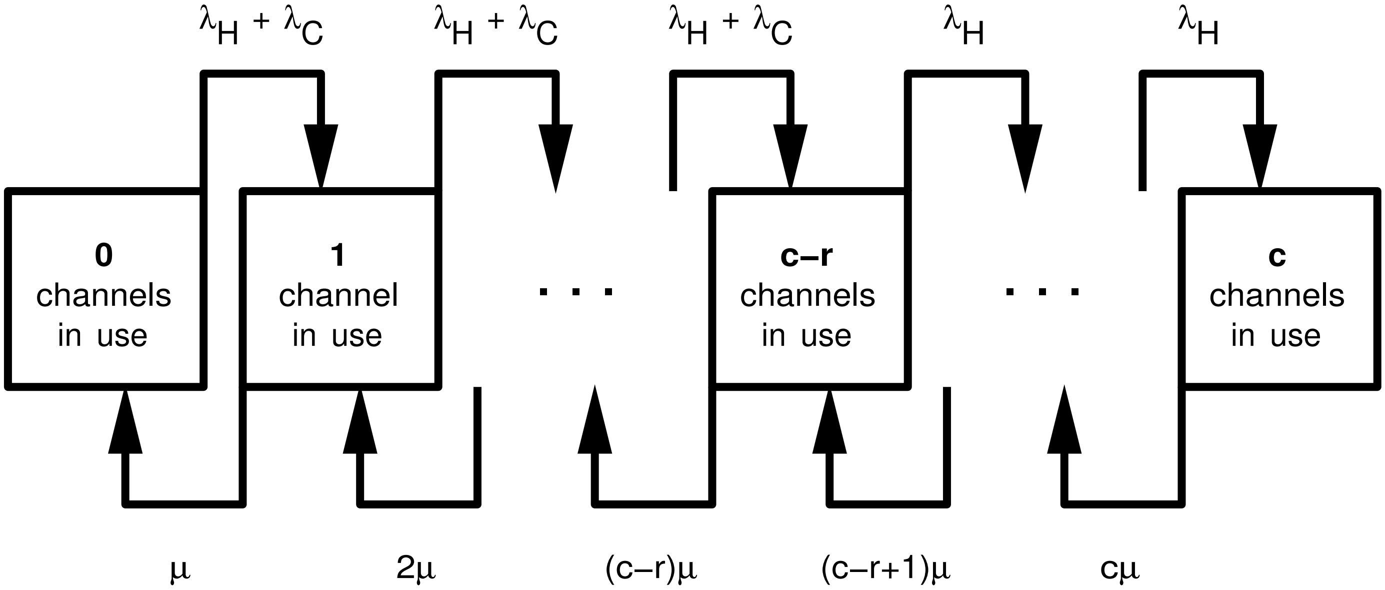

Figure 7.1: State Space Diagram for Trunk Reservation Model.

Now let $\overline{S}_{(x,y]}(t)$ be the *average residual service time* after time t for cell $(x, y]$, where

$$\overline{S}_{(x,y]}(t) \equiv \mathsf{E}\left[\frac{1}{Q_{(x,y]}(t)} \int_{\sigma(y,t)}^{\sigma(x,t)} R_s(y,t) dA(s) \,\middle|\, Q_{(x,y]}(t) > 0 \right] . \tag{7.7}$$

We give a simple expression for $\overline{S}_{(x,y]}(t)$ below.

Proposition 7.1 *For all times t and cell intervals $(x, y]$, we have*

$$\overline{S}_{(x,y]}(t) = \int_{\sigma(y,t)}^{\sigma(x,t)} \mathsf{E}\left[R_s(y,t)\right] \alpha(s) ds \Big/ \mathsf{E}\left[Q_{(x,y]}(t)\right] . \tag{7.8}$$

Proof: Define the processes X and Y where for all $\tau > \sigma(y,t)$, we have

$$X(\tau) \equiv \int_{\sigma(y,t)}^{\tau} 1_{\{T_s^+ \le t < T_s^-\}} dA(s) \text{ and } Y(\tau) \equiv \int_{\sigma(y,t)}^{\tau} \Big(\min(T_s^-, \tau(y,s)) - t\Big) \cdot 1_{\{T_s^+ \le t < T_s^-\}} dA(s). \tag{7.9}$$

Note that $X(\sigma(x,t)) = Q_{(x,y]}(t)$ and $Y(\sigma(x,t)) = \int_{\sigma(y,t)}^{\sigma(x,t)} R_s(y,t) dA(s)$. Moreover, X is a Poisson process as a function of τ and that

$$Y(\tau) = \int_{\sigma(y,t)}^{\tau} \Big(\min(T_s^-, \tau(y,s)) - t\Big) \, dX(s). \tag{7.10}$$

Now we apply (c) of Proposition 10.3 and set $\tau = \sigma(x,t)$. ∎

Thus, approximate instantaneous offered loads of handoffs and originating calls for cell (x, y) are

$$\lambda_H(t) \equiv h(x,t)\overline{S}_{(x,y]}(t) = \frac{h(x,t)}{\mathsf{E}\left[Q_{(x,y]}(t)\right]} \int_{\sigma(y,t)}^{\sigma(x,t)} \mathsf{E}\left[R_s(y,t)\right] \alpha(s) ds \tag{7.11}$$

and

$$\lambda_C(t) \equiv \int_x^y c^+(z,t)dz \cdot \overline{S}_{(x,y]}(t) = \frac{\int_x^y c^+(z,t)dz}{\mathsf{E}\left[Q_{(x,y]}(t)\right]} \int_{\sigma(y,t)}^{\sigma(x,t)} \mathsf{E}\left[R_s(y,t)\right] \alpha(s) ds. \tag{7.12}$$

Finally, we obtain as approximate blocking metrics,

$$\overline{\beta}^C(t_0, t_1) \approx \int_{t_0}^{t_1} c^+_{(x,y]}(s) \beta^C_{r,c}(\lambda_H(s), \lambda_C(s)) ds \Bigg/ \int_{t_0}^{t_1} c^+_{(x,y]}(s) ds \ , \tag{7.13}$$

and

$$\overline{\beta}^H(t_0, t_1) \approx \int_{t_0}^{t_1} h(x,s) \beta^H_{r,c}(\lambda_H(s), \lambda_C(s)) ds \Bigg/ \int_{t_0}^{t_1} h(x,s) ds \ . \tag{7.14}$$

8 The Constant Velocity Case

In this section we further specialize the highway PALM by assuming, first, that the cars all have the same constant velocity and, second, that for each s the think times and call holding times are independent random variables with the same distributions as $T^{(0)}$ and $T^{(1)}$, respectively, that are independent of time s. Since all cars move at some constant velocity v, $\chi(s,t) = v \cdot (t-s)$, $\tau(s,x) = s + x/v$, and $\sigma(x,t) = t - x/v$.

We have seen that all the quantities of interest can be computed given the call density $q(x,t)$. For this example, it is easy to determine this call density. Differentiating (3.8) with respect to x gives

$$\begin{aligned} q(x,t) &= \frac{1}{v}\alpha(t - x/v) \mathsf{P}\left(T^{(0)} \le x/v < T^{(0)} + T^{(1)}\right) && (8.1) \\ &= \frac{1}{v}\alpha(t - x/v)\left[\mathsf{P}\left(T^{(0)} \le x/v\right) - \mathsf{P}\left(T^{(0)} + T^{(1)} \le x/v\right)\right]. && (8.2) \end{aligned}$$

If we assume that $T^{(0)}$ and $T^{(1)}$ are exponentially distributed with rate μ_0 and μ_1, respectively, then

$$q(x,t) = \frac{\mu_0}{v} \cdot \alpha\left(t - x/v\right) \cdot \frac{\exp(-\mu_1 x/v) - \exp(-\mu_0 x/v)}{\mu_0 - \mu_1}. \tag{8.3}$$

If, on the other hand, we assume that $T^{(0)}$ is constant, and (to be consistent with the exponential case) $T^{(0)} = 1/\mu_0$, then

$$q(x,t) = \begin{cases} 0 & \text{if } x < v/\mu_0, \\ \alpha(t - x/v)\exp(-\mu_1 x/v + \mu_1/\mu_0)/v & \text{if } v/\mu_0 \le x. \end{cases} \tag{8.4}$$

Now we reverse situations and assume that $T^{(0)}$ is exponential and $T^{(1)}$ is constant with $T^{(1)} = 1/\mu_1$. We now have

$$q(x,t) = \begin{cases} \alpha(t - x/v)\exp(-\mu_0 x/v + \mu_0/\mu_1)/v & \text{if } x < v/\mu_1, \\ \alpha(t - x/v)(\exp(-\mu_1 x/v + \mu_1/\mu_0) - \exp(-\mu_1 x/v))/v & \text{if } v/\mu_1 \le x. \end{cases} \tag{8.5}$$

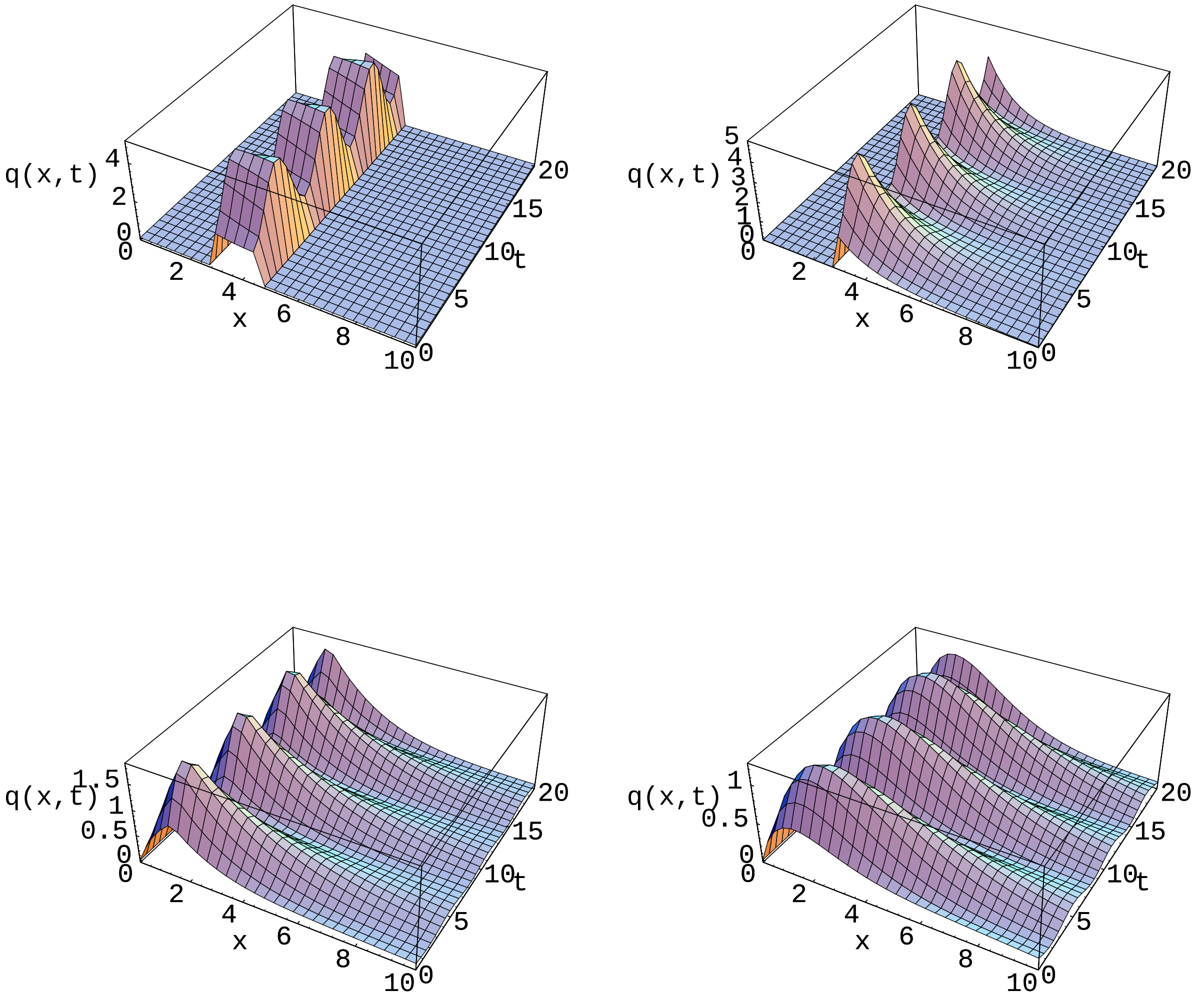

Figure 8.1: Graphs of the Effect of Deterministic versus Exponential Think and Calling Time Distributions on the Average Calling Density.

Finally, if we assume that both $T^{(0)}$ and $T^{(1)}$ are constants, we have

$$q(x,t) = \begin{cases} 0 & \text{if } x < v/\mu_0, \\ \alpha(t - x/v)/v & \text{if } v/\mu_0 \leq x < v(1/\mu_0 + 1/\mu_1), \\ 0 & \text{if } v(1/\mu_0 + 1/\mu_1) \leq x. \end{cases} \tag{8.6}$$

To illustrate the effect of randomness on the call density, we plot these four densities for the case $\alpha(t) = 10 + 5\sin t$, $\mu_0 = 1$, $\mu_1 = 2$, and $v = 3$; see the graphs in Figure 8.1. The two graphs in the top row have constant think times and the two graphs in the left column have constant call holding times. Otherwise the times are exponentially distributed with the same mean.

It is significant that these graphs do not all look alike. Given exponential think times (the bottom row of graphs), the call holding time distribution evidently does not matter much. However, the three graphs that exclude the lower right graph are quite different.

More generally, we see how we can study various PALM models through graphs.

9 Extensions of the Basic Model

In this section, we discuss the ways in which our basic highway PALM is not *simple* but *primitive*; i.e., the basic PALM model is a building block for constructing more elaborate highway models. In the next subsections, we indicate how to construct models with the following features:

- One-Way Traffic with Passing
- Highway with Two-Way Traffic
- Highway Network in $\mathbb{R}^2$
- Cars Making Multiple Calls.

9.1 One-Way Traffic with Passing

As illustrated in Figure 9.1, we can model a highway with one-way traffic having cars that pass each other as a superposition of the basic highway model. Let I be some (at most countable) index of independent highway PALM systems. We define the aggregate processes Q and H as

$$Q(x,t) \equiv \bigoplus_{i \in I} Q_i(x,t) \quad \text{and} \quad H(x,t) \equiv \bigoplus_{i \in I} H_i(x,t). \tag{9.1}$$

Similarly, let C^+ and C^- be defined by

$$C^+(x,t) \equiv \bigoplus_{i \in I} C_i^+(x,t) \quad \text{and} \quad C^-(x,t) \equiv \bigoplus_{i \in I} C_i^-(x,t). \tag{9.2}$$

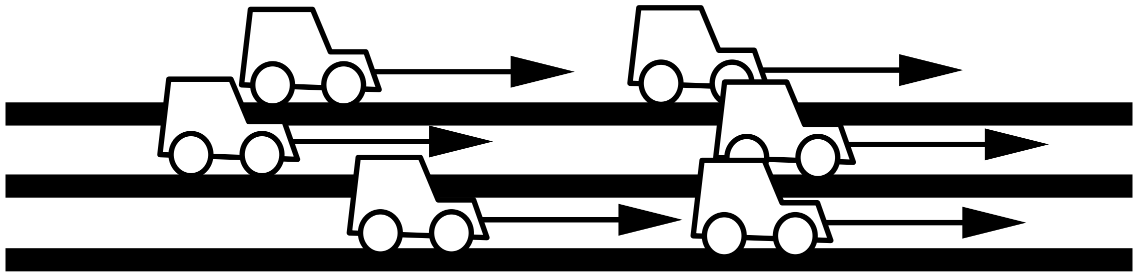

Figure 9.1: One-Way Traffic with Passing as Superpositions of the Basic Highway PALM.

9.2 Highway with Two-Way Traffic

We can model a highway with two-way traffic as simply the superposition of two independent replicas of our one-way highway, going in opposite directions. The eastbound traffic would move according to a velocity field $v_{>}(x,t)$, and the westbound traffic by a velocity field $v_{<}(x,t)$, as depicted in Figure 9.2.

For a given cell $(x,y]$, we can define the processes

$$Q_{(x,y]}(t) \equiv Q^{>}_{(x,y]}(t) \ominus Q^{<}_{(x,y]}(t) \tag{9.3}$$

where $Q^{>}_{(x,y]}(t) \equiv Q_{>}(y,t) \ominus Q_{>}(x,t)$ and $Q^{<}_{(x,y]}(t) \equiv Q_{<}(y,t) \ominus Q_{<}(x,t)$. We can define $C^{+}_{(x,y]}$ and $C^{-}_{(x,y]}$ in a similar fashion. To this we add the process $H^{+}_{(x,y]}$, which counts the number of handoffs for calls *entering* cell $(x,y]$, and $H^{-}_{(x,y]}$, which counts the number of handoffs for calls *departing* cell $(x,y]$. Formally, we define them as

$$H^{+}_{(x,y]} \equiv H_{>}(x,t) \oplus H_{<}(y,t) \quad \text{and} \quad H^{-}_{(x,y]} \equiv H_{<}(x,t) \oplus H_{>}(y,t). \tag{9.4}$$

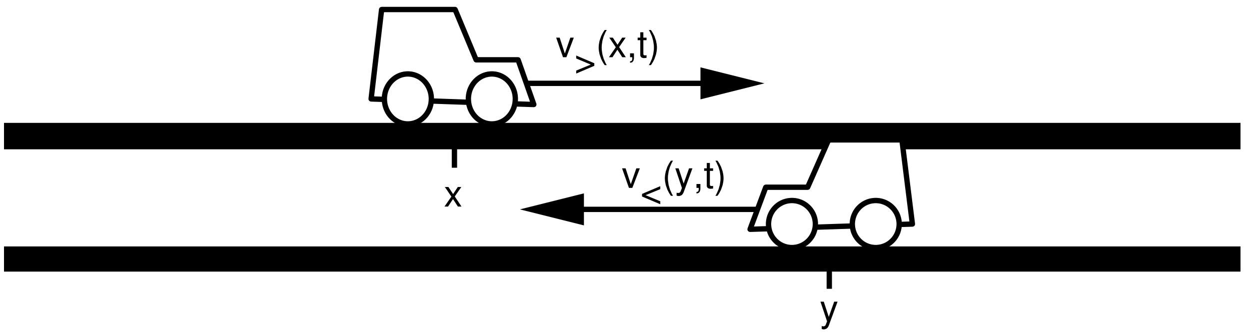

Figure 9.2: Two-Way Traffic as Superpositions of the Basic Highway PALM.

9.3 Highway Network in $\mathbb{R}^2$

Now suppose that on some subregion of $\mathbb{R}^2$, we superimpose a collection of highways that may physically overlap, as depicted in Figure 9.3, but the resulting traffic for one highway acts independently of another. We construct this model by initially viewing the i-th highway as the basic highway PALM defined on $\mathbb{R}_+$. We now associate a mapping $\phi_i : \mathbb{R}_+ \to \mathbb{R}^2$ that we will call an *embedding* if

1. The mapping ϕ_i is continuously differentiable and one-to-one.

2. For all $x \in \mathbb{R}_+$, the euclidean norm of $\phi_i'(x)$ is one.

Such a transformation will preserve the *speed* of each car, but not necessarily the direction of the velocity.

The i-th highway will then be $\phi_i(\mathbb{R}_+)$. Now let Ω be some bounded, open region of $\mathbb{R}^2$ with a connected boundary γ. In Figure (9.3) the region Ω is a hexagon. We assume that every highway will eventually leave such a region, i.e., $\Omega \cap \phi_i([x,\infty)) = \emptyset$ for sufficiently large x. Moreover, we assume that every highway can move in and out of a region no more than a finite number of times. So for the i-th embedding ϕ_i, we associate a finite set of disjoint intervals

$$\left\{ (x_j^+(i), x_j^-(i)] \mid j = 1, \ldots, J(i) \right\} \tag{9.5}$$

such that $x_j^+(i) \leq x_j^-(i)$ and

$$x_1^+(i) \leq x_1^-(i) \leq \cdots \leq x_{J(i)}^+(i) \leq x_{J(i)}^-(i), \tag{9.6}$$

where

$$\phi_i\left(\bigcup_{j=1}^{J(i)} \left(x_j^+(i), x_j^-(i)\right)\right) \subset \Omega \tag{9.7}$$

and

$$\phi_i\left(\bigcap_{j=1}^{J(i)} \left[x_j^+(i), x_j^-(i)\right]^c\right) \cap \Omega = \emptyset \tag{9.8}$$

By our definition, $(x_j^+(i), x_j^-(i))$ is mapped by the embedding ϕ_i into the j-th subroute of the i-th highway that passes through Ω. The point $\phi_i(x_j^+(i))$ will be an *entrance point* to Ω, and $\phi_i(x_j^-(i))$ will be an *exit point.*

To simplify notation, for x and y in $\mathbb{R}_+$ let $Q_{(x,y]}(t;i) \equiv Q_i(y,t) \ominus Q_i(x,t)$ denote the number of calls in progress along the subroute of the i-th highway which is $\phi_i((x,y])$. We can define C^+ and C^- in a similar fashion. Now for any given cell Ω, we define

$$Q_\Omega(t) \equiv \bigoplus_{i \in I} \bigoplus_{j=1}^{J(i)} Q_{(x_j^+(i), x_j^-(i)]}(t;i). \tag{9.9}$$

Similarly, we can define

$$C_\Omega^+(t) \equiv \bigoplus_{i \in I} \bigoplus_{j=1}^{J(i)} C^+_{(x_j^+(i), x_j^-(i)]}(t;i) \quad \text{and} \quad C_\Omega^-(t) \equiv \bigoplus_{i \in I} \bigoplus_{j=1}^{J(i)} C^-_{(x_j^+(i), x_j^-(i)]}(t;i). \tag{9.10}$$

We now construct

$$H_\gamma^+(t) \equiv \bigoplus_{i\in I} \sum_{j=1}^{J(i)} H_i(x_j^+(i),t) \quad \text{and} \quad H_\gamma^-(t) \equiv \bigoplus_{i\in I} \sum_{j=1}^{J(i)} H_i(x_j^-(i),t). \tag{9.11}$$

9.4 Cars Making Multiple Calls

Let $T_s^+(i)$ be the time that the car that arrives to the highway at time s begins its i-th call, and let $T_s^-(i)$ be time that this car terminates its i-th call. We then have

$$s \le T_s^+(1) \le T_s^-(1) \le T_s^+(2) \le T_s^-(2) \le \cdots. \tag{9.12}$$

If we define $Q_i(x,t)$ to equal the number of i-th calls in cell $(0,x]$ before time t, then

$$Q_i(x,t) = \int_{\sigma(x,t)}^t 1_{\left(T_s^+(i)\le t< T_s^-(i)\right)} dA(s) \tag{9.13}$$

and each $Q_i(x,t)$ behaves like the basic highway PALM. In terms of H_i, C_i^+, and C_i^-, we have

$$Q(x,t) \equiv \bigoplus_{i=1}^{\infty} Q_i(x,t) \quad \text{and} \quad H(x,t) \equiv \bigoplus_{i=1}^{\infty} H_i(x,t) \tag{9.14}$$

but

$$C^+(x,t) \equiv \sum_{i=1}^{\infty} C_i^+(x,t) \quad \text{and} \quad C^-(x,t) \equiv \sum_{i=1}^{\infty} C_i^-(x,t). \tag{9.15}$$

Note that $Q(x,t)$ and $H(x,t)$ decompose into independent summands just like the "passing" model in Section 9.1, but $C^+(x,t)$ and $C^-(x,t)$ do not. It follows that $Q(x,t)$ and $H(x,t)$ will still have Poisson distributions, but this will not necessarily be true for $C^+(x,t)$ and $C^-(x,t)$.

10 Appendix: Poisson Integration

Let A be a non-homogeneous Poisson process with rate function α and let $\{Z_k\}$ be an i.i.d. sequence of random elements of some Polish space (complete separable metric space) Σ distributed as Z. We will say that $\phi : \Sigma \times \mathbb{R} \to \mathbb{R}$ is an *integrand* with respect to A and Z if it is a measurable function.

Given such an integrand ϕ, we define $Z_\phi(t)$ to be

$$Z_\phi(t) \equiv \int_{-\infty}^t \phi(Z_{A(s)}, s)\, dA(s) = \sum_{n=1}^{A(t)} \phi(Z_n, \hat{A}_n), \tag{10.1}$$

where $\hat{A}_n$ is the n-th jump time of A, counting backwards from time t.

We proved in [11] that:

1. For all bounded or non-negative integrands ϕ

$$\mathsf{E}\left[Z_\phi(t)\right] = \int_{-\infty}^t \mathsf{E}\left[\phi(Z,s)\right] \alpha(s) ds. \tag{10.2}$$

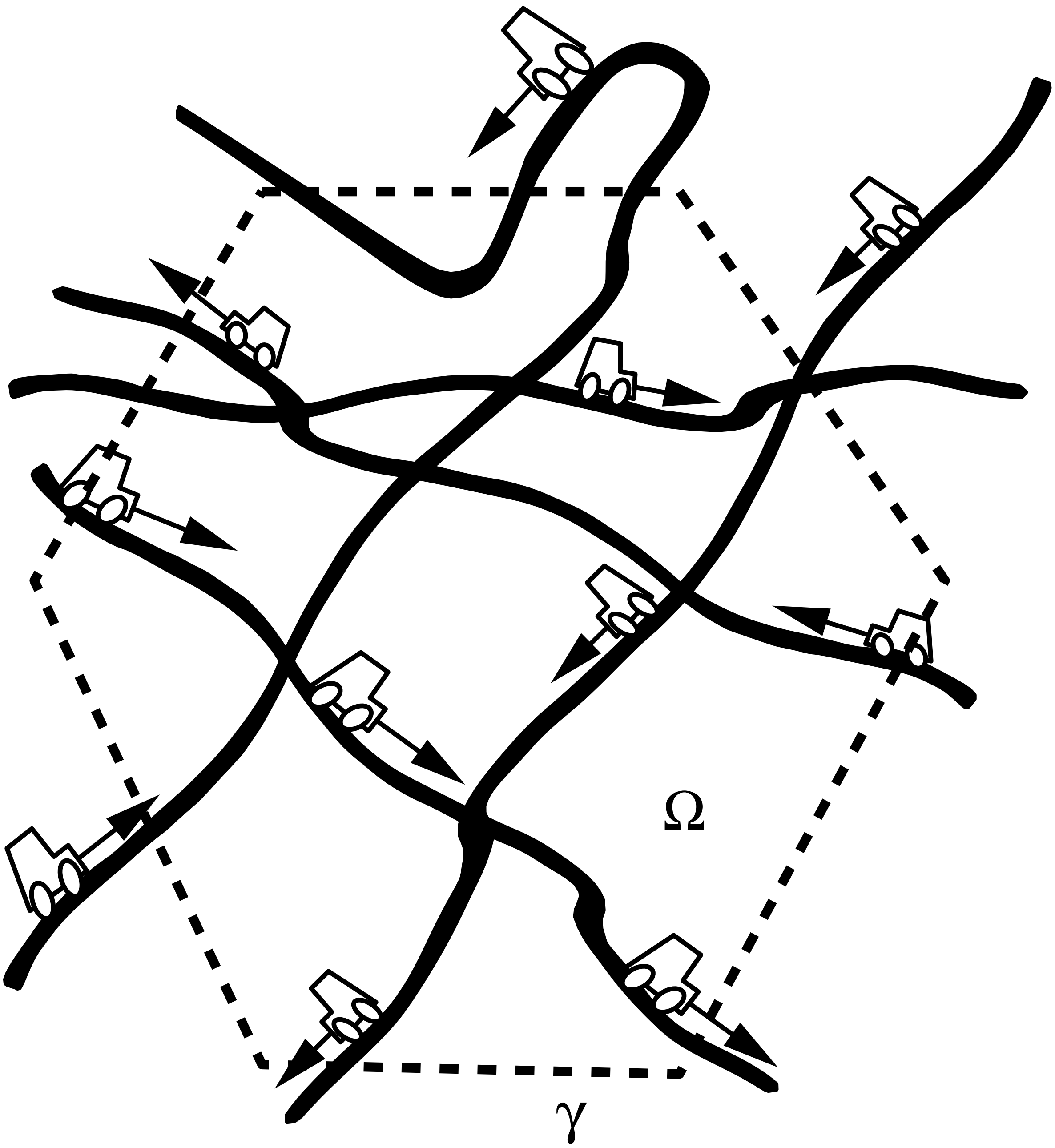

Figure 9.3: Highway Network as Superpositions of Basic Highway PALMs.

2. For all bounded or non-negative integrands ϕ and ψ

$$\mathsf{Cov}\left[Z_\phi(t), Z_\psi(t)\right] = \int_{-\infty}^{t} \mathsf{E}\left[\phi(Z,s)\psi(Z,s)\right]\alpha(s)ds. \tag{10.3}$$

3. If ϕ is a binary function, then Z_ϕ is a Poisson process.

4. If ϕ, ψ, and $\phi + \psi$ are all binary, then Z_ϕ and Z_ψ are independent Poisson processes.

To these results we add:

Proposition 10.1 *For all t, we have*

(a) For any two binary integrands ϕ and ψ and any given t, $Z_\phi(t)$ and $Z_\psi(t)$ are positively correlated Poisson random variables. In addition, their covariance is zero if and only if they are independent.

(b) Moreover, if η is a third binary integrand, then $Z_\eta(t)$ is a Poisson random variable that is independent of $Z_\phi(t) + Z_\psi(t)$ if and only if $Z_\eta(t)$ is independent of both $Z_\phi(t)$ and $Z_\psi(t)$.

(c) Finally, if ϕ is a bounded, real-valued integrand, then

$$\mathsf{E}\left[\left.\frac{Z_\phi(t)}{A(t)}\right| A(t) > 0\right] = \frac{\mathsf{E}\left[Z_\phi(t)\right]}{\mathsf{E}\left[A(t)\right]} = \frac{\int_0^t \mathsf{E}\left[\phi(Z,\tau)\right]\alpha(\tau)d\tau}{\int_0^t \alpha(\tau)d\tau}. \tag{10.4}$$

Proof: For (a), we use (10.3) to show non-negative correlation. Now observe that zero covariance implies $\phi\psi \equiv 0$ up to time t. This means that $\phi + \psi$ is binary up to time t, and so $Z_\phi(t)$ is independent of $Z_\psi(t)$.

For (b), if $Z_\eta(t)$ is independent of $Z_\phi(t) + Z_\psi(t)$, then their covariance is zero. Using the bilinearity of the covariance, we have

$$\mathsf{Cov}[Z_\eta(t), Z_\phi(t)] = -\mathsf{Cov}[Z_\eta(t), Z_\psi(t)]. \tag{10.5}$$

It follows by (a) that both covariances are zero, $Z_\eta(t)$ is independent of $Z_\phi(t)$, and $Z_\eta(t)$ is independent of $Z_\psi(t)$.

For (c), let $\Lambda(t) = \int_0^t \alpha(\tau)d\tau$. If N is a standard Poisson process (rate 1) then we can construct the sample paths of A by using $N \circ \Lambda$ so that $N(\Lambda(t))$ has the same distribution as $A(t)$. Similarly, if $\hat{N}_n$ is the n-th jump for the standard Poisson process N, then $\hat{A}_n$ has the same distribution as $\Lambda^{-1}(\hat{N}_n)$. Using the fact that $\hat{N}_1, \ldots, \hat{N}_m$ are distributed like the order statistics for m i.i.d. random variables that are uniformly distributed on $[0,t]$ when we condition on the event $\{N(t) = m\}$, we have

$$\begin{aligned}
&\mathsf{E}\left[\left.\frac{1}{A(t)}\int_{-\infty}^{t} \phi(Z_{A(\tau)}, \tau)dA(\tau)\right| A(t) > 0\right] \\
&\quad = \mathsf{E}\left[\left.\frac{1}{A(t)}\sum_{n=1}^{A(t)} \phi(Z_n, \hat{A}_n)\right| A(t) > 0\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathsf{P}(A(t)>0)} \mathsf{E}\left[\frac{1}{A(t)} \sum_{n=1}^{A(t)} \phi(Z_n, \hat{A}_n); A(t)>0\right] \\
&= \frac{1}{\mathsf{P}(A(t)>0)} \sum_{m=1}^{\infty} \frac{1}{m} \mathsf{E}\left[\sum_{n=1}^{m} \phi(Z_n, \hat{A}_n); A(t)=m\right] \\
&= \frac{1}{\mathsf{P}(A(t)>0)} \sum_{m=1}^{\infty} \frac{1}{m} \mathsf{E}\left[\sum_{n=1}^{m} \phi(Z_n, \hat{A}_n) \middle| A(t)=m\right] \mathsf{P}(A(t)=m) \\
&= \frac{1}{\mathsf{P}(A(t)>0)} \sum_{m=1}^{\infty} \frac{1}{m} \mathsf{E}\left[\sum_{n=1}^{m} \phi\left(Z_n, \Lambda^{-1}(\hat{N}_n)\right) \middle| N(\Lambda(t))=m\right] \mathsf{P}(A(t)=m) \\
&= \frac{1}{\mathsf{P}(A(t)>0)} \sum_{m=1}^{\infty} \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} \mathsf{E}\left[\phi\left(Z, \Lambda^{-1}(\tau)\right)\right] d\tau \; \mathsf{P}(A(t)=m) \\
&= \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} \mathsf{E}\left[\phi\left(Z, \Lambda^{-1}(\tau)\right)\right] d\tau \\
&= \frac{\int_0^t \mathsf{E}[\phi(Z,\tau)]\alpha(\tau)d\tau}{\int_0^t \alpha(\tau)d\tau}.
\end{aligned}
$$

Proof of Theorem 3.1: This can be regarded as a consequence of Theorem 2.1 of [11]. Given the properties of Poisson integration, it is clear that each $Q(x,t)$ has a Poisson distribution and as a function of x, these random variables have the independent increment property. By a similar argument, we see that the $H(x,t)$'s form a Poisson process as a function of t.

Both integrands use $\sigma(x,t)$ since by definition any arrival after this time must still be in the region $(0,x]$ at time t, and any arrival before this time must be past position x by time t. Taking expectations of these Poisson integrals gives us the formulas for the expectations of $Q(x,t)$ and $H(x,t)$ as in (2.11) of [11]. ∎

Proof of Theorem 3.2: Using the representations (3.2) and (3.5) for $Q(x,t)$ and $H(x,t)$ respectively, the first statement follows immediately from the properties of Poisson integration. To prove the second and third statements, we observe that

$$
C^+(x,t) = \int_{-\infty}^{\sigma(x,t)} 1_{\{T_s^+ \le \tau(s,x)\}} dA(s) + \int_{\sigma(x,t)}^{t} 1_{\{T_s^+ \le t\}} dA(s) \tag{10.6}
$$

and

$$
C^-(x,t) = \int_{-\infty}^{\sigma(x,t)} 1_{\{T_s^- \le \tau(s,x)\}} dA(s) + \int_{\sigma(x,t)}^{t} 1_{\{T_s^- \le t\}} dA(s). \tag{10.7}
$$

The rest follows from using:

$$
1_{\{T_s^+ \le t < T_s^-\}} + 1_{\{T_s^- \le t\}} = 1_{\{T_s^+ \le t\}} \tag{10.8}
$$

where we will also substitute $\tau(s,x)$ for t. ∎

Proof of Theorem 4.1: By Theorem 3.1, $H(x,\cdot)$ and $C^+_{(x,y]}$ are Poisson process with respect to time, so it is sufficient to show that they are independent processes. Referring back to Theorem 3.1, we see that $C^+_{(x,y]}$ is independent of $C^+(x,\cdot)$. Using Proposition 10.1 above, we have that $C^+_{(x,y]}$ must be independent of $H(x,\cdot)$ since by Theorem 3.2, the latter is a summand of $C^+(x,\cdot)$.

The argument for the departure process in (b) is easier since, by Theorem 3.2, C^- when holding x or t fixed is correspondingly independent of $H(x,\cdot)$ or $Q(\cdot,t)$ and H and $H(y,t)$ is independent of $Q(y,t)$. We then deduce from Proposition 10.1 that $H(y,t)$ is independent of $Q_{(x,y]}(t)$, since $Q_{(x,y]}(t)$ is a Poisson summand of $Q(y,t)$. ∎

References

[1] Davis, J., Massey, W.A., and Whitt, W., Sensitivity to the Service-Time Distribution in the Nonstationary Erlang Loss Model, *Management Science*, to appear.

[2] Eick, S., Massey, W.A., and Whitt, W., The Physics of the $M_t/G/\infty$ Queue, *Operations Research*, 41 (1993), 731–742.

[3] Gazis, D.C., *Traffic Science*, Wiley, New York, 1974.

[4] Haberman, R., *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow*, Prentice-Hall, Inc., New Jersey, 1977.

[5] Jackson, J. R., *Job-Shop like Queueing Systems*, Mgmt. Sci., 10 (1963), 131–142.

[6] Jagerman, D. L., Nonstationary Blocking in Telephone Traffic, *Bell System Tech. J.* 54 (1975) 625–661.

[7] Kelly, F. P., Loss Networks, *Annals of Applied Probability*, 1 (1991) 319–378.

[8] Lee, W.C.Y., *Mobile Cellular Telecommunications Systems*, McGraw-Hill, New York, 1989.

[9] Leung, K.K., Massey, W.A., and Whitt, W., Traffic Models for Wireless Communications Networks. INFOCOM '94, Toronto, to appear.

[10] Massey, W.A. and Whitt, W., An Analysis of the Modified-Offered-Load Approximation for the Nonstationary Erlang Loss Model, *Annals of Applied Probability*, to appear.

[11] Massey, W.A., and Whitt, W. Networks of Infinite Server Queues with Nonstationary Poisson Input. *Queueing Systems*, 13 (1993), 183–250.

[12] Meier-Hellstern, K.S., Alonso, E. and O'Neill, D. R., The Use of SS7 and GSM to Support High Density Personal Communication Systems, in *Third WINLAB Workshop on Third Generation Wireless Networks*, April 1992.

[13] Mitra, D., Gibbens, R.J., and Huang, B.D., State Dependent Routing on Symmetric Loss Networks with Trunk Reservations, I. *IEEE Trans. Commun.* 40 (1992) 477–482.

[14] Montenegro, G., Sengoku, M., Yamaguchi, Y. and Abe, T., Time-Dependent Analysis of Mobile Communication Traffic in a Ring-Shaped Service Area with Nonuniform Vehicle Distribution. *IEEE Trans. Vehicular Tech.* 41 (1992) 243–254.

[15] Palm, C., Intensity Variations in Telephone Traffic, *Ericsson Technics* 44 (1943) 1–189 (in German). (English translation by North-Holland, Amsterdam, 1988).

[16] Seskar, I., Maric, S. V., Holtzman, J. and Wasserman, J., Rate of Location Area Updates in Cellular Systems, in *Third WINLAB Workshop on Third Generation Wireless Networks*, April 1992.

[17] Symon, K.R., *Mechanics* (Third Edition), Addison-Wesley Publishing Company, Inc. (1971).

[18] Sze, S.M., *Physics of Semiconductor Devices* (Second Edition), John Wiley & Sons (1981).