PIECEWISE-LINEAR DIFFUSION PROCESSES

by

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Abstract

Diffusion processes are often regarded as among the more abstruse stochastic processes, but diffusion processes are actually relatively elementary, and thus are natural first candidates to consider in queueing applications. To help demonstrate the advantages of diffusion processes, we show that there is a large class of one-dimensional diffusion processes for which it is possible to give convenient explicit expressions for the steady-state distribution, without writing down any partial differential equations or performing any numerical integration. We call these tractable diffusion processes *piecewise linear;* the drift function is *piecewise linear*, while the diffusion coefficient is *piecewise constant.* The explicit expressions for steady-state distributions in turn yield explicit expressions for long-run average costs in optimization problems, which can be analyzed with the aid of symbolic mathematics packages. Since diffusion processes have continuous sample paths, approximation is required when they are used to model discrete-valued processes. We also discuss strategies for performing this approximation, and we investigate when this approximation is good for the steady-state distribution of birth-and-death processes. We show that the diffusion approximation tends to be good when the differences between the birth and death rates are small compared to the death rates.

Keywords: diffusion process; steady-state distribution; diffusion approximation; birth-and-death process.

1. Introduction and Summary

In the natural sciences, diffusion processes have long been recognized as relatively simple stochastic processes that can help describe the first-order behavior of important phenomena. This simplicity is illustrated by the relatively quick way that the model is specified in terms of a drift function and a diffusion function (plus boundary behavior, which here we will take to be standard). However, the analysis of diffusion processes can involve some formidable mathematics, which can reduce the appeal, and evidently has impeded applications to queueing problems. Our purpose here is to circumvent the formidable mathematics, and focus solely on creating the model and obtaining the answer, which here is regarded as the steady-state distribution. From a theoretical standpoint, very little here is new. Our goal is to show that diffusion processes are easier to work with than often supposed.

For accessible introductory accounts of diffusion processes, are Glynn (1990), Harrison (1985), §9.4 and §13.2 of Heyman and Sobel (1982), Chapter 15 of Karlin and Taylor (1981) and Chapter 7 of Newell (1982). For accessible advanced treatments, see Billingsley (1968), Breiman (1968), Ethier and Kurtz (1986), Karatzas and Shreve (1991) and Mandl (1968).

A diffusion process is a continuous-time Markov process $\{X(t) : t \geq 0\}$ with continuous sample paths. We will consider only real-valued time-homogeneous diffusion processes. Such a diffusion process is characterized by its *drift function* or *infinitesimal mean*

$$\mu(x) = \lim_{\varepsilon \downarrow 0} E[X(t+\varepsilon) - X(t) | X(t) = x] \ , \tag{1}$$

its *diffusion function* or *infinitesimal variance*

$$\sigma^2(x) = \lim_{\varepsilon \downarrow 0} E[(X(t+\varepsilon) - X(t))^2 | X(t) = x] \tag{2}$$

and its boundary behavior. We assume that the state space is the subinterval (s_0, s_k), where $-\infty \leq s_0 < s_k \leq +\infty$. If the boundaries s_0 and s_k are finite, then we assume that the boundaries are *reflecting*. It is easy to understand what reflecting means by thinking of what happens with an approximating simple random walk; from the boundary the next step is back into the interior. If the boundary points are not finite, then we assume that they are *inaccessible* (cannot be reached in finite time). The boundary behavior can be subtle, and nonstandard variations can be relevant for applications, e.g., see Harrison and Lemoine (1981), Kella and Whitt (1990) and Kella and Taksar (1993). However, here we consider only the standard case.

We call the diffusion processes that we consider *piecewise-linear diffusions*, because we assume that the drift function $\mu(x)$ is *piecewise-linear* and the diffusion function $\sigma^2(x)$ is *piecewise-constant* in the state x. These piecewise-linear diffusion processes are of interest both as models in their own right and as approximations. The piecewise-linear diffusions can serve as approximations for both non-diffusion processes (e.g., birth-and-death processes, see §2) and diffusion processes with more general piecewise-continuous drift and diffusion functions. In some of the literature on diffusion processes it is assumed that the drift function and diffusion coefficient are continuous, e.g., see p. 159 of Karlin and Taylor (1981), but this stronger assumption is actually not necessary, as can be seen from pp. 13, 25, 90 of Mandl (1968) and other references.

An example of a piecewise-linear diffusion process is the heavy-traffic diffusion approximation for the GI/M/s queue developed by Newell (1973), Halachmi and Franta (1978) and Halfin and Whitt (1981). This diffusion approximation plays an important role in approximations for the general GI/G/s queue in Whitt (1985, 1992a, 1994). In this diffusion process, the drift is constant when all servers are busy and linear otherwise, while the variance is constant throughout. In the context of this GI/M/s example, our purpose is to show that the steady-state distribution can be immediately written down and understood. For this example, it will become evident that the steady-state distribution of the diffusion process has a density that is a piece of an exponential density connected to a piece of a normal density.

Another example of a piecewise linear diffusion process occurs in the diffusion approximation for large trunk groups in circuit-switched networks with trunk reservation; see Reiman (1989, 1991). These papers illustrate optimization applications, which we discuss in §7. All these examples involve queues with state-dependent arrival and service processes; for more examples of this kind see Whitt (1990) and references cited there. A non-queueing example is the two-drift skew Brownian motion in the control problem of Beneš, Shepp and Witsenhausen (1980); see §6.5 of Karatzas and Shreve (1991).

It should be clear that when we use a diffusion approximation for a queueing process, we are assuming that we can disregard the detailed discrete behavior of the queueing process. The diffusion approximation tends to be appropriate when the jumps are relatively small compared to the magnitude of the process, which tends to occur under heavy loads. Formally, diffusion approximations can be justified by heavy-traffic limit theorems, in which we consider a sequence of models with an associated sequence of traffic intensities approaching the critical value for stability from below; e.g., see Halfin and Whitt (1981).

We now specify in more detail what we mean by piecewise linear. We assume that there are $k + 1$ real numbers s_i such that $-\infty \leq s_0 < s_1 < \dots < s_k \leq \infty$. Then the state space is (s_0, s_k) with $\mu(x) = a_i x + b_i$ and $\sigma^2(x) = \sigma_i^2 > 0$ on the interval (s_{i-1}, s_i), $1 \leq i \leq k$. (Often the variance function can be regarded as constant overall, but we will consider the general case; motivation is given in §2). As indicated above, if the boundary points s_0 and s_k are finite, then we assume that they are reflecting. Otherwise, we assume that they are inaccessible. Moreover, if $s_0 = -\infty$, then we require that either $a_1 > 0$ or ($a_1 = 0$ and $b_1 > 0$). Similarly, if $s_k = +\infty$, then we require that $a_k < 0$ or ($a_k = 0$ and $b_k < 0$). From pp. 13, 25, 90 of Mandl (1968), these conditions guarantee the existence of a proper steady-state limit (convergence in distribution).

The important point is that the steady-state limit has a density of the form

$$f(x) = p_i f_i(x)\ , \quad s_{i-1} \le x < s_i\ , \tag{3}$$

where $\sum_{i=1}^{k} p_i = 1$, $\int_{s_{i-1}}^{s_i} f_i(x)\,dx = 1$, f_i has a known relatively simple form and p_i can be easily computed. Consequently, the steady-state mean is

$$m \equiv \int_{s_0}^{s_k} x\, f(x)\,dx = \sum_{i=1}^{k} p_i\, m_i\ , \tag{4}$$

where m_i is the mean of f_i, and similarly for higher moments. In particular, in §3 we show that

$$p_i = r_i / \sum_{j=1}^{k} r_j\ , \quad 1 \le i \le k\ , \tag{5}$$

where $r_1 = 1$ and

$$r_i = \prod_{j=2}^{i} \frac{\sigma_{j-1}^2\ f_{j-1}(s_{j-1}-)}{\sigma_j^2\ f_j(s_{j-1}+)}\ , \quad 2 \le i \le k\ . \tag{6}$$

Since the component densities f_i are all continuous, the overall density f is continuous if and only if $\sigma_i^2 = \sigma_1^2$ for all i. In all cases the cumulative distribution function is continuous. Our experience indicates that for most queueing applications it is appropriate to have $\sigma_i^2 = \sigma_1^2$ and thus a continuous steady-state density f.

For piecewise-linear diffusions with $a_i \le 0$ for all i, the component densities f_i in (3) have a relatively simple form, so that it is easy to calculate the component means m_i (and second moments) and the probability weights p_i without performing any integrations. This makes the characterization attractive as an algorithm when k is large, as well as an insightful representation when k is small. In particular, if $a_i \le 0$ for all i, then the component densities are all truncated and renormalized pieces of normal, exponential and uniform densities. The relatively simple form for the steady-state distribution follows quite directly from the general theory, as we indicate in §4, but it does not seem to be well known (among non-experts).

Conceptually, the characterization can be explained by the properties of truncated reversible Markov processes; see §1.6 of Kelly (1979). If the state space of a reversible Markov process is truncated (and given reflecting boundaries), then the truncated process is reversible with a steady-state distribution which is a truncated and renormalized version of the original steady-state distribution, i.e., the truncated steady-state distribution is the conditional steady-state distribution of the unrestricted process given the truncation subset. This property holds for multi-dimensional reversible Markov processes, but we restrict attention here to real-valued processes. For a multi-dimensional diffusion process application, see Fendick and Hernandez-Valencia (1992). This truncation property is also a natural *approximation* more generally, e.g., see Whitt (1984).

For example, if a diffusion process on the real line behaves like an Ornstein-Uhlenbeck (OU) diffusion process over some subinterval of the state space, then its steady-state distribution restricted to that subinterval is a truncation and renormalization of the normal steady-state distribution of the full OU process with those parameters. Moreover, by exploiting basic properties of the normal distribution, it is possible to give explicit expressions for the moments of the conditional distribution restricted to this subinterval; see Proposition 1 below. These explicit expressions in turn help produce closed-form expressions for long-run average costs in optimization problems; see §7. This makes it possible to tackle the optimization problems with symbolic mathematics packages such as Maple V; see Char et al. (1992).

Here is how the rest of the paper is organized. In §2 we discuss diffusion approximations for birth-and-death processes and give some examples showing how piecewise-linear diffusions can naturally arise. In §3 we present the steady-state distribution of a piecewise-continuous diffusion, drawing on the basic theory in Karlin and Taylor (1981) and Mandl (1968). In §4 we present four basic linear diffusion processes whose restrictions will form the pieces of the piecewise-linear diffusion process. In the cases with $a_i \leq 0$ we exhibit the appropriate conditional distribution and its first two moments. In §5 we establish a stochastic comparison that can be used to show that piecewise-linear diffusions which serve as approximations for a more general piecewise-continuous diffusion actually are stochastic bounds. In §6 we investigate when the simple diffusion approximation for birth-and-death processes introduced in §2 should be reasonable. In §7 we discuss optimization. Finally, we state our conclusions in §8.

In this paper, we only consider steady-state distributions. However, it should be noted that diffusion processes can also help us understand transient phenomena, such as arise in simulation experiments; e.g., see Whitt (1989, 1992b).

2. Diffusion Approximations

We often can obtain a diffusion process as an approximation of another process. In this section we briefly discuss how.

2.1 Diffusion Approximations of Birth-and-Death Processes

We first discuss diffusion approximations of *birth-and-death* (BD) processes. As we indicate in §6 below, the steady-state distribution of a BD process is not difficult to calculate directly. However, in some cases it may be desirable to have the closed-form formulas (3)-(6), especially when the number k of pieces is small.

We begin by showing how a diffusion process can arise as a limit of a sequence of BD processes. To express the limiting behavior, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For each positive integer n, let $\{B_n(t) : t \geq 0\}$ be a BD process on the integers from $\lfloor c_n + l_n \sqrt{n} \rfloor$ to $\lfloor c_n + u_n \sqrt{n} \rfloor$ with state-dependent BD rates $\beta_n(j)$ and $\delta_n(j)$, respectively. Let the boundary behavior be the same as assumed for the diffusion processes. Let

$$X_n(t) = \frac{B_n(t) - c_n}{\sqrt{n}} , \quad t \geq 0 . \tag{7}$$

In the context of (7), the drift and diffusion functions of $X_n(t)$ are

$$\mu_n(x) \equiv \lim_{\varepsilon \downarrow 0} E[X_n(t+\varepsilon) - X_n(t) | X_n(t) = x]$$
$$= \frac{\beta_n(\lfloor c_n + x\sqrt{n} \rfloor) - \delta_n(\lfloor c_n + x\sqrt{n} \rfloor)}{\sqrt{n}} \tag{8}$$

and

$$\sigma_n^2(x) \equiv \lim_{\varepsilon \downarrow 0} E[(X_n(t+\varepsilon) - X_n(t))^2 | X_n(t) = x]$$
$$= \frac{\beta_n(\lfloor c_n + x\sqrt{n} \rfloor) + \delta_n(\lfloor c_n + x\sqrt{n} \rfloor)}{n} \ . \tag{9}$$

If $l_n \to l$, $u_n \to u$, $\mu_n(x) \to \mu(x)$ and $\sigma_n^2(x) \to \sigma^2(x)$ as $n \to \infty$, then $X_n(t)$ can be said to converge to the diffusion process on (l,u) with drift function $\mu(x)$ and diffusion function $\sigma^2(x)$; see Stone (1963) and Iglehart (1965). This convergence is in a strong sense, including the finite-dimensional distributions of the stochastic processes and more, see Billingsley (1968), but we will consider only the steady-state distributions. Convergence of the steady-state distributions can be shown directly by a modification of the argument in §6 below.

Example 2.1 The M/M/s queue.

The number of customers in the system in the classical M/M/s queue is a BD process with birth (arrival) rate $\beta(j) = \beta_0$ and death rate $\delta(j) = \eta \min\{j,s\}$ in state j, where η is the individual service rate. For states in the interval $[0,s]$, we have $\delta(j) = \eta j$, while for states in the interval (s,∞) we have $\delta(j) = \eta s$. Consider a sequence of M/M/s queueing models indexed by n. In model n, let the number of servers be $s_n = n$, let the arrival rate be $\beta_n(j) = n - a\sqrt{n}$ for all j, and let the individual service rate be 1, so that the death rate is $\delta_n(j) = \min\{j,n\}$. Then it is natural to let $c_n = n$, so that $l_n = -\sqrt{n}$ and $u_n = +\infty$ Then we have convergence to a diffusion process, as shown in Halfin and Whitt (1981). ■

Of course, in applications we typically have only *one* BD process. Then we can form the diffusion approximation by letting $l = l_n$, $u = u_n$, $\mu(x) = \mu_n(x)$ and $\sigma^2(x) = \sigma_n^2(x)$ where $\mu_n(x)$ and $\sigma_n^2(x)$ are defined by (6) and (7) for some given n, which we can take as $n = 1$. Setting $n = 1$ corresponds to simply matching the infinitesimal means and variances. Based on Berger and Whitt (1992), §8.5, we suggest refining this direct diffusion approximation by making the state space for the diffusion process $(l - 1/2,\ u + 1/2)$ instead of (l, u). This corresponds to the familiar refinement when a continuous (e.g., the normal) distribution is used to approximate an integer-valued probability distribution; see p. 185 of Feller (1968).

Henceforth here we will concentrate on the direct approximation for the steady-state distribution of a BD process based on $n = 1$. We hasten to point out that a user should check whether the accuracy of the approximation is adequate for the intended application. We investigate when the crude direct approximation for the steady-state distribution is reasonable in §6.

Suppose that the BD parameters β and δ are both *linear*; i.e., $\beta(j) = \beta_0 + \beta_1 j$ and $\delta(j) = \delta_0 + \delta_1 j$ for $l \le j \le u$. Instead of (8) and (9), we can use the *linear approximations*

$$\mu(x) \approx \beta_0 + \beta_1 x - \delta_0 - \delta_1 x \tag{10}$$

and

$$\sigma^2(x) \approx \beta_0 + \beta_1 x + \delta_0 + \delta_1 x \tag{11}$$

for $l - 1/2 \le x \le u + 1/2$. Furthermore, assuming that the process will mostly be in the region of x_0 in which $\mu(x_0) \approx 0$, we can further approximate the variance by

$$\sigma^2(x) \approx \beta_0 + \delta_0 \ + \ (\beta_1 + \delta_1) x_0 \ , \tag{12}$$

provided that $\mu(x_0) \approx 0$ for some x_0 with $l - 1/2 \le x_0 \le u + 1/2$. Otherwise we let $\sigma^2(x)$ be either $\sigma^2(l)$ or $\sigma^2(u)$, whichever is closer.

Finally, even when β and δ are not linear, we may be able to produce (10) and (12) over subintervals by making a piecewise-linear approximation.

Example 2.1 continued.

Returning to the M/M/s queue, we apply (10) and (12) to obtain $\mu(x) = \beta_0 - \eta x$ and $\sigma^2(x) = 2\beta_0$ over $(-1/2,\ s+1/2)$, and $\mu(x) = \beta_0 - \eta s$ and $\sigma^2(x) = \beta_0 + \eta s$ over $(s+1/2, \infty)$. To have constant variance overall, we argue that $\mu(x) \approx 0$ for $x \approx s$, so that $\beta_0 \approx \eta s$; thus we have the further approximation $\sigma^2(x) = 2\beta_0$ for $x \in (s, \infty)$ as well as for $x \in [0, s]$. The relevant values of x are $s + c\sqrt{s}$ for some constant c. For this example, the exact steady-state distribution of the BD process combines a truncated Poisson distribution below s with a geometric distribution above s, while the diffusion approximation yields a truncated normal distribution below s and an exponential distribution above s; see Halfin and Whitt (1981). These approximations often tend to be good, as is well known.

Example 2.2. Secondary Servers with a Buffer.

We now consider an example of a BD process with three linear regions. There is a service facility with one primary server plus a buffer of capacity c_1. There are s secondary servers that accept overflows from the primary buffer. There is an additional buffer of capacity c_2 to hold arrivals when all servers are busy. The secondary system is costly, so that whenever space opens up in the primary buffer, a customer in service in the secondary system immediately leaves and enters the primary buffer. With this last feature, the number of customers in the system can be modelled as a BD process.

Let the arrival rate be constant, so that $\beta(j) = \beta_0$ for all j. The service rate is linear in the three regions

$$\delta(k) = \begin{cases} \eta_1 , & 1 \le k \le c_1 + 1 \\ \eta_1 + (k - c_1 - 1)\eta_2 , & c_1 + 2 \le k \le c_1 + s \\ \eta_1 + s\eta_2 , & c_1 + s + 1 \le k \le c_1 + c_2 + s + 1 . \end{cases} \quad (13)$$

The resulting direct diffusion approximation has drift function

$$\mu(x) = \begin{cases} \beta_0 - \eta_1 , & -1/2 \le x < c_1 + 3/2 \\ \beta_0 - \eta_1 - (x - c_1 - 1)\eta_2 , & c_1 + 3/2 \le x < c_1 + s + 3/2 \\ \beta_0 - \eta_1 - s\eta_2 , & c_1 + s + 3/2 \le x \le c_1 + c_2 + s + 3/2 \end{cases} \quad (14)$$

and diffusion function

$$\sigma^2(x) = \begin{cases} \beta_0 + \eta_1 , & -1/2 \le x < c_1 + 3/2 \\ \beta_0 + \eta_1 + (x_0 - c_1 - 1)\eta_2 , & c_1 + 3/2 \le x < c_1 + s + 3/2 \\ \beta_0 + \eta_1 + s\eta_2 , & c_1 + s + 3/2 \le x \le c_1 + c_2 + s + 3/2 \end{cases} \quad (15)$$

provided that

$$\mu(x_0) = \beta_0 - \eta_1 - (x_0 - c_1 - 1)\eta_2 \approx 0 . \quad (16)$$

for $c_1 + 3/2 \le x_0 \le c_1 + s + 3/2$. If $\mu(x) > 0$ (< 0) for all x in this region, then we can set $\sigma^2(x) = \sigma^2(c_1 + s + 3/2)$ $(\sigma^2(x) = \sigma^2(c_1 + 3/2))$.

Note that (15) and (16) lead to a piecewise-constant diffusion function. We can further simplify (15) by just letting $\sigma^2(x) \approx 2\beta_0$, assuming that $\mu(x) \approx 0$ over the entire range of relevant values.

2.2 Diffusion Approximations for General Integer-Valued Processes

Diffusion approximations are even more important when the stochastic process being approximated is not a BD process, because then there may be no alternative formula for the steady-state distribution. The crude direct approximation above easily generalizes; we just match the infinitesimal means and variances as in (8) and (9). However, the infinitesimal means and variances are often hard to determine. An alternative approach is to match the large-time behavior, as discussed in Whitt (1982) and references cited there.

To match the large time behavior, let $\{X(t):\ t \geq 0\}$ be a given integer-valued stochastic process and let $X_j(t)$ represent the sum of the jumps from state j during the period that X has spent t units of time in state j. To formally define $X_j(t)$, let $T_j(t)$ be the time when X has spent t units of time in state j, defined by setting

$$t = \int_0^{T_j(t)} 1_{\{X(u)=j\}}\, du \ , \tag{17}$$

where 1_A is the indicator function of the set A. Let J_i be the time of the i^{th} jump of X and let $N(t)$ be the number of jumps of X in $[0,t]$. Then

$$X_j(t) = \sum_{i=1}^{N(T_j(t))} (X(J_i)-j)\ 1_{\{X(J_i-)=j\}}\ , \quad t \geq 0\ . \tag{18}$$

We typically can only approximately determine $\{X_j(t):\ t \geq 0\}$, but even an estimate can serve as the basis for the diffusion approximation.

We assume that $\{X_j(t):\ t \geq 0\}$ obeys a central limit theorem, i.e.,

$$\frac{X_j(t) - \lambda_j t}{\sqrt{\lambda_j c_j^2 t}} \Rightarrow N(0,1) \quad \text{as} \quad t \to \infty\ , \tag{19}$$

where $N(0,1)$ is a standard (zero mean, unit variance) normal random variable and $\Rightarrow$ denotes convergence in distribution. We then create the diffusion approximation by first setting

$$\mu(j) = \lambda_j \quad \text{and} \quad \sigma^2(j) = \lambda_j c_j^2 \tag{20}$$

and then fitting continuous functions to $\mu(j)$ and $\sigma^2(j)$. It is easy to see that this procedure coincides with (8) and (9) with $n=1$ when X is a BD process, but it also applies more generally.

2.3 BD Approximations

Since BD processes are also relatively easy to work with, we could consider constructing approximating BD processes instead of approximating diffusion processes. This might be convenient for looking at the time-dependent behavior, e.g., for doing simulation or optimization via Markov programs in the spirit of Kushner and Dupuis (1992). However, it is not as easy to approximate by a BD process as it is by a diffusion process.

Starting from a diffusion process, we can obtain an approximating BD process by solving (8) and (9) for the birth and death rate functions β and δ. In particular, we get

$$\beta(j) \;=\; \frac{\sigma^2(j)+\mu(j)}{2} \quad \text{and} \quad \delta(j) \;=\; \frac{\sigma^2(j)-\mu(j)}{2} \;. \tag{21}$$

Obviously, this BD construction works only when $\sigma^2(j) \geq \mu(j)$ for all j. When $\sigma^2(j)$ is significantly less than $\mu(j)$, we should not anticipate that a BD approximation will be good.

We also note that piecewise-linear BD processes can be considered. The geometric, Poisson and discrete uniform distributions play the role of the exponential, normal and continuous uniform distributions below. The truncation property holds because the BD process is also a reversible Markov process.

3. Piecewise-Continuous Diffusions

We now exhibit the steady-state distribution for a (time-homogeneous) diffusion with piecewise-continuous drift and diffusion functions $\mu(x)$ and $\sigma^2(x)$, with $\sigma^2(x) > 0$. As before, we use the $k+1$ points s_i and assume that the drift and diffusion coefficients are continuous on (s_{i-1}, s_i) with limits from the left and right at each s_i for each i; see pp. 13, 25 and 90 of Mandl (1968). We also assume that the boundary points s_0 and s_k are reflecting if finite and inaccessible if infinite. We assume there is a proper time-dependent distribution which converges to a proper steady-state distribution with density $f(x)$. (For the piecewise-linear case, this follows from the extra structure.) The general theory implies that

$$f(x) \ = \ \frac{m(x)}{M(s_k)} \ , \quad s_0 \le x \le s_k \ , \tag{22}$$

where

$$m(x) \ = \ \frac{2}{\sigma^2(x) s(x)} \tag{23}$$

is the *speed density*,

$$s(x) \ = \ \exp\left\{ - \int_{\theta}^{x} \frac{2\mu(y)}{\sigma^2(y)} \, dy \right\} \tag{24}$$

is the *scale density* with θ arbitrary satisfying $s_0 \ < \ \theta \ < \ s_k$, and

$$M(x) \ = \ \int_{s_0}^{x} m(y)\, dy \ , \quad s_0 \le x \le s_k \ , \tag{25}$$

provided that all integrals are finite; see pp. 13, 25, 90 of Mandl (1968) and §15.3 and 15.5 of Karlin and Taylor (1981).

From (22)-(25), we see that the density $f(x)$ can easily be calculated by numerical integration. Our object is to obtain more convenient explicit expressions. From (24) and (25), we see that $s(x)$ and $M(x)$ are continuous on (s_0 , s_k), so that m and f are continuous everywhere in the interval (s_0 , s_k) except perhaps at the points s_i, $1 \le i \le k - 1$, where $\sigma^2(x)$ is discontinuous. Indeed, since $\sigma^2(x)$ has positive limits from the left and the right at s_i for each i, $1 \le i \le k - 1$, so will the density f and we can relate the right and left limits. In particular,

$$f(s_i +) \ = \ \frac{\sigma^2(s_i -)}{\sigma^2(s_i +)} f(s_i -) \ . \tag{26}$$

From (26) and (3), we easily obtain the formula for the probability weights in (5).

From (22) – (25), we also directly deduce that the conditional density, conditioning on a subinterval is $Kf(x)$ for x in this subinterval. Moreover, this conditional density is the steady-state density of the diffusion process obtained by restricting the original diffusion process to this subinterval, using reflecting boundaries at all finite boundary points.

4. Four Basic Linear Diffusion Processes

We construct the component densities f_i in (3) from the steady-state densities of four basic diffusion processes.

4.1 The Ornstein-Uhlenbeck Process

If

$$\mu(x) = -a(x - m) \quad \text{and} \quad \sigma^2(x) = \sigma^2 > 0 \tag{27}$$

for $a > 0$ and $-\infty < x < \infty$, then we have the Ornstein-Uhlenbeck (OU) process, for which the steady-state limit is normally distributed with mean m and variance $\sigma^2/2a$.

Let $N(m, b^2)$ denote a normally distributed random variable with mean m and variance b^2. Let Φ be the cumulative distribution function (cdf) and ϕ the density of $N(0, 1)$. If X is the steady-state distribution of the OU process in (27) restricted to the interval (s_{i-1}, s_i), then X has the distribution of $N(m, \sigma^2/2a)$ conditioned to be in the interval (s_{i-1}, s_i); i.e., X has the density

$$f(x) = \frac{b^{-1}\phi\left[\dfrac{x-m}{b}\right]}{\Phi\left[\dfrac{s_i - m}{b}\right] - \Phi\left[\dfrac{s_{i-1} - m}{b}\right]}, \quad s_{i-1} < x < s_i , \tag{28}$$

where $b^2 = \sigma^2/2a$.

Of course, the cdf Φ appearing in (28) involves an integral, but it can be calculated approximately without integrating using rational approximations; see §26.2 of Abramowitz and Stegun (1972).

Note that we can easily infer the shape of f from (28). For example, f is unimodal; and the mode is in the interior of (s_{i-1}, s_i), and thus at m, if and only if $s_{i-1} < m < s_i$. In general, $f(x)$ increases as x moves toward m.

The following proposition gives the first two moments of X.

Proposition 1. If $-\infty \le s_{i-1} < s_i \le \infty$, then

$$E[N(m, b^2) | s_{i-1} \le N(m, b^2) \le s_i] = m + b \frac{\left[\phi\left[\frac{s_{i-1} - m}{b}\right] - \phi\left[\frac{s_i - m}{b}\right] \right]}{\Phi\left[\frac{s_i - m}{b}\right] - \Phi\left[\frac{s_{i-1} - m}{b}\right]} \quad (29)$$

and

$$E[N(m, b^2)^2 | s_{i-1} \le N(m, b^2) \le s_i] =$$

$$m^2 + 2mb \frac{\left[\phi\left[\frac{s_{i-1} - m}{b}\right] - \phi\left[\frac{s_i - m}{b}\right] \right]}{\Phi\left[\frac{s_i - m}{b}\right] - \Phi\left[\frac{s_{i-1} - m}{b}\right]} + b^2$$

$$+ b^2 \frac{\left[\left[\frac{s_{i-1} - m}{b}\right] \phi\left[\frac{s_{i-1} - m}{b}\right] - \left[\frac{s_i - m}{b}\right] \phi\left[\frac{s_i - m}{b}\right] \right]}{\Phi\left[\frac{s_i - m}{b}\right] - \Phi\left[\frac{s_{i-1} - m}{b}\right]} . \quad (30)$$

Proof. First note that $x\phi(x) = -\phi'(x)$ for all x, so that

$$E[N(0, 1) | s_{i-1} \le N(0, 1) \le s_i] = \frac{\phi(s_{i-1}) - \phi(s_i)}{\Phi(s_i) - \Phi(s_{i-1})} .$$

Consequently,

$$E[N(m, b^2) | s_{i-1} \le N(m, b^2) \le s_i] = m + bE\left[\frac{N(m, b^2) - m}{b} | s_{i-1} \le N(m, b^2) \le s_i \right]$$

$$= m + bE\left[N(0, 1) | \frac{s_{i-1} - m}{b} \le N(0, 1) \le \frac{s_i - m}{b} \right] .$$

Next note that $x^2 \phi(x) = \phi(x) + \phi''(x)$, so that

$$E[N(0, 1)^2 | s_{i-1} \le N(0, 1) \le s_i] = 1 + \frac{s_{i-1}\phi(s_{i-1}) - s_i \phi(s_i)}{\Phi(s_i) - \Phi(s_{i-1})} .$$

Consequently,

$$\begin{aligned} E[N(m, b^2)^2 | s_{i-1} \le N(m, b^2) \le s_i] = m^2 \\ + 2mbE[N(0, 1) | s_{i-1} \le N(m, b^2) \le s_i] \\ + b^2 E[N(0, 1)^2 | s_{i-1} \le N(m, b^2) \le s_i] . \quad \blacksquare \end{aligned}$$

4.2 Reflected Brownian Motion with Zero Drift

If

$$\mu(x) = 0 \quad \text{and} \quad \sigma^2(x) = \sigma^2 > 0 \tag{31}$$

on (s_{i-1}, s_i) for $-\infty < s_{i-1} < s_i < \infty$, then we have the reflected Brownian motion (RBM) process with zero drift, for which the steady-state limit X is uniformly distributed on (s_{i-1}, s_i) with mean $(s_{i-1} + s_i)/2$ and second moment $(s_i^3 - s_{i-1}^3)/3(s_i - s_{i-1})$. The conditional distribution on a subinterval is again uniform with the new endpoints playing the role of s_{i-1} and s_i.

4.3 Reflected Brownian Motion with Drift

If

$$\mu(x) = -a \quad \text{and} \quad \sigma^2(x) = \sigma^2 > 0 \tag{32}$$

for $a > 0$ on (s, ∞), then we have RBM with negative drift, for which the steady-state limit is distributed as s plus an exponential with mean $\sigma^2/2a$. This case also covers RBM with positive drift a on $(-\infty, -s)$, say $\{R(t) : t \geq 0\}$, because $\{-R(t) : t \geq 0\}$ is then the RBM with negative drift above. Hence, if f and g are the steady-state densities with negative and positive drift, respectively, then $g(-s-x) = f(s+x)$ for $x \geq 0$. Hence, it suffices to focus only on the negative drift case.

It is well known and easy to see that the conditional distribution of s plus an exponential given that it is contained in the interval (s_{i-1}, s_i), where $s_{i-1} > s$, is the same as an exponential on $(0, s_i - s_{i-1})$; i.e., the conditional density is

$$f(x) = \frac{\lambda e^{-\lambda(x - s_{i-1})}}{1 - e^{-\lambda(s_i - s_{i-1})}}\ , \quad s_{i-1} < x < s_i\ , \tag{33}$$

where λ^{-1} is the mean of the exponential random variable; here $\lambda^{-1} = \sigma^2/2a$.

Let X be a random variable with the density f in (33). Then elementary calculations yield

$$E[X] = s_{i-1} + \lambda^{-1}\frac{[1 - \lambda e^{-\lambda(s_i - s_{i-1})}(1 + \lambda(s_i - s_{i-1}))]}{1 - e^{-\lambda(s_i - s_{i-1})}} \tag{34}$$

and

$$E[X^2] = s_{i-1}^2 + \frac{2s_{i-1}\lambda^{-1}[1 - \lambda e^{-\lambda(s_i - s_{i-1})}\left[1 + \lambda(s_i - s_{i-1})\right]]}{1 - e^{-\lambda(s_i - s_{i-1})}} \tag{35}$$
$$+ \lambda^{-2}\frac{\left[1 - e^{-\lambda(s_i - s_{i-1})}\left[1 + \lambda(s_i - s_{i-1}) + \frac{\lambda^2(s_i - s_{i-1})^2}{2}\right]\right]}{1 - e^{-\lambda(s_i - s_{i-1})}}\ ,$$

where $\lambda^{-1} = \sigma^2/2a$.

4.4 Positive Linear Drift

A relatively difficult case occurs if

$$\mu(x) = a(x - m) \quad \text{and} \quad \sigma^2(x) = \sigma^2 > 0 \tag{36}$$

for $a > 0$ and $s_{i-1} < x < s_i$. Then there is positive linear drift away from m. By partitioning the interval into two subintervals and performing a change of variables, it suffices to consider the case

$$\mu(x) = ax \quad \text{and} \quad \sigma^2(x) = \sigma^2 > 0 \tag{37}$$

on $(0, s)$. However, even (37) is difficult. Indeed, no nice explicit form is available for (37). In particular, from (22) – (25) we see that the steady-state density for (37) is of the form

$$f(x) = Ke^{ax^2/\sigma}, \quad 0 \le x \le s, \tag{38}$$

and the mean is

$$EX = \frac{\sigma K}{2a} (e^{as^2/\sigma} - 1) \tag{39}$$

for a constant K such that $\int_0^s f(x)\,dx = 1$. Except for the constant K, the form of (38) and (39) is quite simple and thus easily understood. However, K does not have a simple expression. The constant K can be found from Dawson's integral $D(y) \equiv e^{-y^2} \int_0^y e^{x^2}\,dx$, whose values appear in Table 7.5 of Abramowitz and Stegun (1972). The maximum value is $D(y) = 0.541$ occurring at $y = 0.924$; see 7.1.17 of Abramowitz and Stegun.

Since the constant K in (38) is relatively intractable, if this case is present, then we would resort to either direct numerical integration in the setting of §3 or approximation of the drift coefficient in (36) by piecewise-constant drift coefficients as in §4.2 and §4.3 over several subintervals.

Example 4.1. Insurance Fund.

We now give a (non-queueing) example with a positive state-dependent drift. As in Harrison (1977), consider an insurance firm with an asset process that is a diffusion with state-dependent drift $\mu(x) = \alpha x$ for positive x where $\alpha > 0$ and constant variance function, but let the process have a reflecting barrier at zero instead of the absorbing barrier. Moreover, combine this with DeFinetti's model of an insurance fund as discussed on pages 146-147 of Gerber (1979), in which all proceeds above some level b are paid out as dividends. Then the asset process is a linear diffusion on $[0,b]$ with drift function $\mu(x) = \alpha x$, where $\alpha > 0$.

5. Stochastic Comparisons

Since we may want to approximate a general piecewise-continuous diffusion by a piecewise-linear diffusion, it is useful to have results providing insight into the quality of the approximation.

From §3 we easily can obtain sufficient conditions for a stochastic comparison. We say that one density f_1 is less than or equal to another f_2 on the same interval (s_0, s_k) in the sense of *likelihood ratio,* and we write $f_1 \leq_{lr} f_2$, if $f_2(x)/f_1(x)$ is nondecreasing in x. A likelihood ratio implies that the distribution determined by f_1 is stochastically less than or equal to the distribution determined by f_2; see Ross (1982).

Proposition 2. Consider two piecewise-continuous diffusions on a common interval (s_0, s_k) satisfying (22) – (25). If $\sigma_1^2(x)/\sigma_2^2(x)$ is nondecreasing in x and $\mu_2(x)/\sigma_2^2(x) \geq \mu_1(x)/\sigma_1^2(x)$ for all x, then $f_1 \leq_{lr} f_2$.

Proof. Note that $f_2(x)/f_1(x)$ is nondecreasing if and only if $\sigma_1^2(x)s_1(x)/\sigma_2^2(x)s_2(x)$ is nondecreasing by (22) and (23). Next, by (24), $s_1(x)/s_2(x)$ is nondecreasing if and only if $\mu_2(x)/\sigma_2^2(x) \geq \mu_1(x)/\sigma_1^2(x)$ for all x. ■

Note that the condition in Proposition 2 is satisfied if $\sigma_1^2(x) = \sigma_2^2(x)$ and $\mu_1(x) \leq \mu_2(x)$ for all x.

From (22) – (25) we can also establish continuity results showing that $f_n(x) \to f(x)$ for each x if $\mu_n(x) \to \mu(x)$ and $\sigma_n^2(x) \to \sigma^2(x)$ for each x, plus extra regularity conditions, for a sequence of piecewise-continuous diffusions.

6. On the Quality of Diffusion Approximations for BD Processes

We now investigate when the direct diffusion approximation for BD processes with $n = 1$ in (8) and (9) is reasonable for the stationary distribution of the BD process. For simplicity, we assume that $l > -\infty$. Recall that the steady-state probability mass function for a BD process is

$$\pi_j = \rho_j / \sum_{i=l}^{u} \rho_i , \quad l \le j \le u , \tag{40}$$

where $\rho_l = 1$ and

$$\rho_j = \prod_{i=l+1}^{j} (\beta_{i-1}/\delta_i) = \frac{\beta_l}{\delta_j} \exp \sum_{i=l+1}^{j-1} \log \left[\frac{\beta_i}{\delta_i} \right] , \quad l+1 \le j \le u . \tag{41}$$

To relate (41) to the steady-state distribution of the diffusion, we exploit the expansion of the logarithm, i.e.,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots . \tag{42}$$

From (41) and (42), we obtain a condition for the diffusion approximation to be good. *The condition is that* $(\beta_i - \delta_i)/\delta_i$ *is suitably small for the i of interest.* Assuming this is the case, we have

$$\rho_j \approx \frac{\beta_l}{\delta_j} \exp \sum_{i=l+1}^{j-1} \left[\left[\frac{\beta_i - \delta_i}{\delta_i} \right] + O \left[\frac{\beta_i - \delta_i}{\delta_i} \right]^2 \right] . \tag{43}$$

From (8) and (9) with $n = 1$, $\beta_l \approx \sigma^2(l)$, $\delta_j \approx \sigma^2(j)/2$, $\delta_i \approx \sigma^2(i)/2$ and

$$\rho_j \approx \frac{2\sigma^2(l)}{\sigma^2(j)} \exp \sum_{i=l+1}^{j-1} \frac{2\mu(i)}{\sigma^2(i)} . \tag{44}$$

If, in addition, $2\mu(i)/\sigma^2(i)$ is suitably smooth, e.g., linear, then

$$\rho_j \approx \frac{2\sigma^2(l)}{\sigma^2(j)} \exp \int_{l+1/2}^{j-1/2} \frac{2\mu(y)}{\sigma^2(y)}\, dy \tag{45}$$

and indeed, by (22), (23), (24) and (45),

$$\pi_j \approx \int_{j-1/2}^{j+1/2} f(y)\, dy \approx f(j) \ , \ l \le j \le u \ , \tag{46}$$

where f is the diffusion process density in (22). Formula (46) shows that the steady-state BD probability mass function values π_j are reasonably approximated by the steady-state diffusion density $f(j)$.

7. Optimization

It can be rather straightforward to handle costs in a piecewise linear diffusion process. Suppose a cost is charged to the system at rate $g_i(x)$ per unit time when $x \in [s_{i-1}, s_i)$. Then standard renewal-reward theory tells us that the expected average cost per unit time is

$$\sum_{i=1}^{k} p_i \int_{s_{i-1}}^{s_i} g_i(x) f_i(x)\, dx \ . \tag{47}$$

Example 2.2 Revisited.

Suppose that we consider the secondary service with buffers again. The piecewise linear diffusion process approximation is given in (14) and (15). By the results of section 3 and 4, we find that in regions 1 and 3, the stationary distribution is truncated exponential, and in region 2 it is truncated normal. To simplify notation, we will let $\beta_0 - \eta_1 = -\mu \le 0$ and $\beta_0 + \eta_1 = \delta$. We also let $(\beta_0 - \eta_1)/\eta_2 = -\alpha \le 0$ and $\sqrt{\beta_0/\eta_2} = \gamma$. Then, from section 4 we find

$$f_1(x) = \frac{\lambda_1 e^{-\lambda_1(x+1/2)}}{1 - e^{-\lambda_1(c_1 + 2)}} \ , \tag{48}$$

$$f_2(x) = C(s) \cdot \phi \left[\frac{x + \alpha - (c_1 + 1)}{\gamma} \right] \ , \tag{49}$$

$$f_3(x) = \frac{\lambda_2 e^{-\lambda_2(x-(c_1+1+s+1/2))}}{1-e^{-\lambda_2 c_2}} \ , \tag{50}$$

where

$$\lambda_1 = \frac{\delta}{2\mu} \ , \ \frac{1}{C(s)} = \gamma \left[\Phi \left[\frac{1/2 + \alpha}{\gamma} \right] - \Phi \left[\frac{1/2 + s + \alpha}{\gamma} \right] \right] , \ \lambda_2 = \frac{\delta + s\eta_2}{2(\mu - s\eta_2)} \ . \tag{51}$$

(It follows from §4.1 that for the normal part, we have the mean $m = c_1 + 1 - \alpha$.) From (5) and (6) we also get $r_1 = 1$,

$$r_2 = \frac{\delta\lambda_1 e^{-\lambda_1(c_1+2)}}{(1-e^{-\lambda_1(c_1+2)} 2\beta_0 C(s)\phi((\alpha+1/2)/\gamma)} \tag{52}$$

$$r_3 = r_2 \cdot \frac{(1-e^{-\lambda_2 c_2})2\beta_0 C(s)\phi((\alpha+s+1/2)/\gamma)}{(\delta+s\eta_2)\lambda_2} \ . \tag{53}$$

Now we consider optimization problems. Even if we restrict attention to choosing the parameters c_1, s and c_2, there are quite a few possibilities. For example, the s secondary servers could be a given, as would be the number of buffer spaces, $c_1 + c_2$. In this case, the decision problem would be *how to split the buffers,* and where to place the secondary servers (if we restrict ourselves to using them as a *dedicated* group). In extreme cases, we might want to place all of the buffer spaces in between the single server and the secondary servers (if e.g., $h_1 = h_3 < h_2$), or in the other extreme (e.g., if $h_2 < \min\{h_1, h_3\}$), we may want to place all of the servers together at the head of the system, thus effectively working as a (partially) *ranked* $M/M/s + 1/c_1 + c_2$ system, with a strange cost structure. (In both of these cases, there would only be 2 regions.) We will call this Problem 1.

Alternatively, the buffer spaces as well as their positions might be fixed externally, and the decision variable might simply be how many excess servers, *s*, to hire, within a given budget constraint. We will call this Problem II.

In both cases, since *queueing* occurs only in the regions 1 and 3, costs should be *quadratic* in those 2 regions, and *linear* in the region where service is in *parallel;* i.e., we will take $g_i(x) = h_i \cdot (x - s_{i-1})^2$, $i = 1,3$, and $g_2(x) = h_2 \cdot (x - s_1) \equiv h_2 \cdot (x - (c_1 + 1 + 1/2))$. Let $B(c_1, s, c_2)$ denote the cost function for the system. Then we have

$$E(B(c_1,s,c_2)) = p_1 h_1 E(X_1 - s_0)^2 + p_2 h_2 E(X_2 - s_1) + p_3 h_3 E(X_3 - s_2)^2 , \tag{54}$$

where X_i has density f_i. These values are then easily obtained from (29) and (35), yielding

$$E(X_1 - s_0)^2 = \frac{1}{4} - \frac{1 - \lambda_1 e^{-\lambda_1 (c_1 + 2)} (1 + \lambda_1 (c_1 + 2))}{\lambda_1 (1 - e^{-\lambda_1 (c_1 + 2)})} + \frac{1}{4} \frac{1 - e^{-\lambda_1 (c_1 + 2)} (1 + \lambda_1 (c_1 + 2) + \lambda_1^2 (c_1 + 2)^2 / 2)}{1 - e^{-\lambda_1 (c_1 + 2)}} , \tag{55}$$

$$E(X_2 - s_1) = c_1 + 1 - \alpha + \frac{\gamma^2}{C(s)} \left[\phi \left[\frac{\alpha + 1/2}{\gamma} \right] - \phi \left[\frac{s + \alpha + 1/2}{\gamma} \right] \right] , \tag{56}$$

$$E(X_3 - s_2)^2 = (c_1 + 1 + s + 1/2)^2 + \left[\frac{2(c_1 + 1 + s + 1/2)}{\lambda_2} \right] \left[\frac{1 - \lambda_2 e^{-\lambda_2 c_2} (1 + \lambda_2 c_2)}{1 - e^{-\lambda_2 c_2}} \right] + \frac{1 - e^{-\lambda_2 c_2} (1 + \lambda_2 c_2 + \lambda_2^2 c_2^2 / 2)}{\lambda_2^2 (1 - e^{-\lambda_2 c_2})} . \tag{57}$$

It should of course be recalled that $p_2 = p_2(c_1, s)$, $p_3 = p_3(c_1, s, c_2)$ $\lambda_2 = \lambda_2(s)$.

Standard numerical optimization techniques can now be used to optimize the system, for example, for Problem I, suppose that $c_1 + c_2 + K$, then let $c_1 = c$, and $c_2 = K - c$ in (55)–(57), and just optimize $E(B(c, s, K - c))$ with respect to *c*. The two extreme cases referred to above correspond respectively to the cases $c = 0$, $c = K$. For Problem II, we would try to maximize *s* subject to $E(B(c_1, s, c_2)) \leq l$, where *l* is our budget per unit time.

For example, we applied the symbolic mathematical package Maple V to differentiate $EB(c, s, K - c)$ with respect to c in order to find the optimal solution for Problem I; see Char et al. (1992). Using piecewise linear diffusion processes together with symbolic mathematics packages seems like a promising approach.

8. Conclusions

In Sections 1, 3 and 4 we showed that the steady-state distribution of a one-dimensional piecewise-linear diffusion can be expressed conveniently in closed form, in a way that is insightful. It remains to obtain corresponding results for multi-dimensional diffusions.

In Sections 2 and 6 we discussed diffusion approximations for birth-and-death processes and other integer-valued processes. It remains to further evaluate the quality of these approximations.

In Sections 3 and 5 we discussed piecewise-linear diffusion approximations for more general diffusion processes with piecewise-continuous drift and diffusion functions. In §7 we showed how the piecewise linear diffusion processes can be used effectively for optimization, especially when combined with a symbolic mathematics package such as Maple V. It remains to exploit the use of symbolic mathematics packages further.

Overall, we have tried to support the idea that diffusion processes can be useful for queueing and other applied problems.

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