Chapter 3

Preservation of Pointwise Convergence

3.1. Introduction

With the continuous-mapping approach to stochastic-process limits, we are concerned about limits $x_n \to x$ and $f_n(x_n) \to f(x)$ for a sequences of functions $\{x_n : n \geq 1\}$ in D and $f_n, f : D \to D$; see Section 3.5 and Chapter 13 in the book. However, in many applications we actually are interested in the pointwise limits

$$x(t)/\phi(t) \to \gamma \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad t \to \infty \tag{1.1}$$

and

$$f(x)(t)/\phi(t) \to \eta \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad t \to \infty \tag{1.2}$$

for single functions $x \in D$ and $f : D \to D$, where ϕ is a suitable scaling function. In particular, we may want to show that the pointwise limit (1.1) implies the associated pointwise limit (1.2) and identify the limit η.

It is significant that we often can obtain such limits in $\mathbb{R}$ as consequences of function-space limits by setting

$$y_s(t) \equiv x(st)/\phi(s), \quad s > 0 \ . \tag{1.3}$$

As a regularity condition, we will assume that the scaling function ϕ is a homeomorphism of $\mathbb{R}_+$, i.e., $\phi \in \Lambda(\mathbb{R}_+)$. That implies that $\phi(0) = 0$, ϕ is increasing and $\phi(t) \to \infty$ as $t \to \infty$. If we can show that

$$y_s \to y \quad \text{in} \quad D \quad \text{as} \quad s \to \infty \tag{1.4}$$

for y_s in (1.3), where $1 \notin Disc(y)$, then we can apply the projection map π_1 taking x into $x(1)$ to obtain

$$y_s(1) = x(s)/\phi(s) \to y(1) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad s \to \infty\,, \tag{1.5}$$

which implies the desired convergence in (1.1) and identifies the limit γ in (1.1) as $y(1)$ in (1.5). Moreover, if we can show that

$$f(x)(t)/\phi(t) = g_t(y_t) \quad \text{for each} \quad t > 0\,, \tag{1.6}$$

where $g_s, g : D \to \mathbb{R}$ and

$$g_s(y_s) \to g(y) \quad \text{in} \quad \mathbb{R} \tag{1.7}$$

whenever $y_s \to y$ in D, then we can obtain (1.2) from (1.4) as well, and we identify the limit η in (1.2) as $g(y)$.

For example, this reasoning applies to the supremum function: $f(x)(t) = x^{\uparrow}(t)$ for $t > 0$. Then $g(y) = g_s(y) = f(y)(1) = y^{\uparrow}(1)$ for all $y \in D$ and $s > 0$. As a consequence, the limit in (1.2) holds with $\eta = g(y) = y^{\uparrow}(1)$.

Even though many pointwise limits for single functions can be subsumed as special cases of function-space limits, it is interesting to consider what can be obtained directly without resorting to the function-space construction in (1.3). In particular, it is natural to ask how pointwise limits for single functions are preserved under the composition, supremum, and inverse maps. We investigate that question in this chapter.

For queues and related applied probability models, this convergence-preservation issue for single sample paths corresponds to sample-path analysis, which is commonly associated with the fundamental relations (conservation laws) $L = \lambda W$ and Arrivals See Time Averages (ASTA); see El-Taha and Stidham (1999); see the chapter notes at the end of the chapter.

3.2. From Pointwise to Uniform Convergence

Clearly, the pointwise limit in (1.1) is more elementary than the function-space limit (1.4) but, surprisingly, (1.4) is not much stronger than (1.1). Indeed, under minor regularity conditions, (1.1) actually implies (1.4). Recall that ϕ in $\Lambda(\mathbb{R}_+)$ is regularly varying with index $p > 0$, denoted by $\phi \in \mathcal{R}(p)$, if

$$\phi(tx)/\phi(t) \to x^p \quad \text{as} \quad t \to \infty \tag{2.1}$$

for all $x > 0$; see Appendix A at the end of the book.

Theorem 3.2.1. (from pointwise to uniform convergence) *Let* $x \in D$ *and* $\phi \in \Lambda(\mathbb{R}_+)$ *with* $\phi \in \mathcal{R}(p)$ *for* $p > 0$. *If the limit* (1.1) *holds in* $\mathbb{R}$, *then*

$$\|y_s - y\|_T \to 0 \quad as \quad s \to \infty \quad for\ each \quad T > 0 \tag{2.2}$$

for y_s *in* (1.3) *and*

$$y(t) = \gamma t^p, \quad t \geq 0 \ .$$

Proof. Under the conditions, for any $\epsilon > 0$, there is a t_0 such that

$$|x(t)/\phi(t) - \gamma| < \epsilon \quad \text{for all} \quad t \geq t_0 \ . \tag{2.3}$$

and an s_0 such that

$$\sup_{0 \leq t \leq T} \left| \frac{\phi(st)}{\phi(s)} - t^p \right| < \epsilon \quad \text{for all} \quad s \geq s_0 \ ; \tag{2.4}$$

see Theorem A.5 in Appendix A in the book. For $t \leq t_0/s$,

$$|y_s(t) - y(t)| \leq |y_s(t)| + |y(t)| \leq \frac{\|x\|_{t_0}}{\phi(s)} + \gamma \left(\frac{t_0}{s}\right)^p \ , \tag{2.5}$$

which is less than ϵ for all sufficiently large s, say $s \geq s_1 \geq s_0$. Since

$$y_s(t) - y(t) = \frac{x(st)}{\phi(st)} \left(\frac{\phi(st)}{\phi(s)} - t^p \right) + t^p \left(\frac{x(st)}{\phi(st)} - \gamma \right) \ , \tag{2.6}$$

for $s \geq s_1$,

$$\begin{aligned} \|y_s - y\|_T &\leq \epsilon + \sup_{t \geq t/s} \left\{ \left| \frac{x(st)}{\phi(st)} \right| \left| \frac{\phi(st)}{\phi(s)} - t^p \right| + t^p \left| \frac{x(st)}{\phi(st)} - \gamma \right| \right\} \\ &\leq \epsilon + (\gamma + \epsilon)\epsilon + T^p \epsilon \ , \end{aligned} \tag{2.7}$$

which can be made arbitrarily small with an appropriate choice of ϵ. ■

For the special case in which $\phi(t) = t$, condition (1.1) corresponds to a strong law of large numbers (SLLN) for a stochastic process, while the conclusion (2.2) corresponds to a functional strong law of large numbers (FSLLN). The following corollary is Theorem 4 from Glynn and Whitt (1988).

Corollary 3.2.1. (from a SLLN to a FSLLN) *Let* $\{X(t) : t \geq 0\}$ *be a real-valued stochastic process and let*

$$\hat{\mathbf{X}}_n(t) \equiv n^{-1} X(nt), \quad t \geq 0, \quad n \geq 1 \ . \tag{2.8}$$

If a SLLN holds, i.e., if

$$t^{-1}X(t) \to \gamma \text{ w.p.1 in } \mathbb{R} \text{ as } t \to \infty\,, \tag{2.9}$$

then a FSLLN holds, i.e.,

$$\|\hat{\mathbf{X}}_n - \gamma\mathbf{e}\|_T \to 0 \quad \text{w.p.1 in} \quad D([0,T],\mathbb{R}) \quad \text{as} \quad n \to \infty \tag{2.10}$$

for all $T > 0$.

3.3. Supremum

In this section we consider the supremum map. The following elementary convergence-preservation result is referred to as the "fundamental lemma of maxima" in Section 2.5 of El-Taha and Stidham (1999).

Proposition 3.3.1. (preservation of pointwise convergence for the supremum) *Suppose that* $x \in D([0,\infty),\mathbb{R})$, ϕ *is an increasing real-valued function on* $\mathbb{R}_+$ *with* $\phi(t) \to \infty$ *as* $t \to \infty$. *If* $x(t)/\phi(t) \to \gamma \geq 0$ *as* $t \to \infty$, *then* $x^{\uparrow}(t)/\phi(t) \to \gamma$ *as* $t \to \infty$.

Proof. Under the condition, for any $\epsilon > 0$, there exists t_0 such that

$$(\gamma - \epsilon)\phi(t) \leq x(t) \leq (\gamma + \epsilon)\phi(t)$$

for all $t \geq t_0$. Hence,

$$(\gamma - \epsilon)\phi(t) \leq x^{\uparrow}(t) \leq x^{\uparrow}(t_0) \vee (\gamma + \epsilon)\phi(t)$$

for all $t \geq t_0$. Since $\gamma \geq 0$ and $\phi(t) \to \infty$, there is $t_1 \geq t_0$ such that $x^{\uparrow}(t_0) \leq (\gamma + \epsilon)\phi(t)$ for all $t \geq t_1$. Thus, for $t \geq t_1$,

$$|\phi(t)^{-1}x^{\uparrow}(t) - \gamma| \leq \epsilon\,. \quad \blacksquare$$

Under the conditions of Theorem 3.2.1, if $\gamma \geq 0$, then we can apply the continuous mapping theorem to deduce that $x^{\uparrow}(t)/\phi(t) \to \gamma$ as $t \to \infty$; i.e., the conclusion of Proposition (3.3.1) holds by virtue of Theorems 3.2.1 here and 13.4.1 in the book. However, Theorem 3.2.1 here has the extra assumption that ϕ is regularly varying.

Paralleling Proposition (3.3.1), we can also establish a pointwise-convergence result for supremum with centering for a single function.

Proposition 3.3.2. (preservation of pointwise convergence with centering for the supremum) *Suppose that ϕ is an increasing real-valued function such that $\phi(t) \to \infty$ and $\phi(t)/t \to 0$ as $t \to \infty$. If*

$$\phi(t)^{-1}[x(t) - \lambda t] \to \gamma \quad as \quad t \to \infty \tag{3.1}$$

for $\lambda > 0$, then

$$\phi(t)^{-1}[x^{\uparrow}(t) - \lambda t] \to \gamma \quad as \quad t \to \infty\,. \tag{3.2}$$

Proof. Under condition (3.1), for any $\epsilon > 0$, there exists t_0 such that

$$\lambda t - \phi(t)(\gamma - \epsilon) \le x(t) \le \lambda t + \phi(t)(\gamma + \epsilon)$$

for all $t \ge t_0$. Then

$$\lambda t - \phi(t)(\gamma - \epsilon) \le x^{\uparrow}(t) \le x^{\uparrow}(t_0) \vee (\lambda t + \phi(t)(\gamma + \epsilon))\,.$$

However, since $\lambda > 0$ and $\phi(t)/t \to 0$, there is a $t_1 > t_0$ such that $x^{\uparrow}(t_0) \le \lambda t + \phi(t)(\gamma + \epsilon)$ for all $t \ge t_1$. Hence

$$|\phi(t)^{-1}[x^{\uparrow}(t) - \lambda t] - \gamma| < \epsilon$$

for all $t \ge t_1$, so that (3.2) holds. ∎

3.4. Counting Functions

We now turn to counting functions, as in Section 13.8 of the book. A counting function is defined in terms of a sequence $\{s_n : n \ge 0\}$ of nondecreasing nonnegative real numbers with $s_0 = 0$. We can think of s_n as the partial sum

$$s_n \equiv x_1 + \cdots + x_n, \quad n \ge 1\,, \tag{4.1}$$

by simply writing $x_i \equiv s_i - s_{i-1}$, $i \ge 1$. The associated *counting function* $\{c(t) : t \ge 0\}$ is defined by

$$c(t) \equiv \max\{k \ge 0 : s_k \le t\}, \quad t \ge 0\,. \tag{4.2}$$

To have $c(t)$ finite for all $t > 0$, we assume that $s_n \to \infty$ as $n \to \infty$.

To establish limits for counting functions, we use two scaling functions. We again let the scaling functions be elements of $\Lambda(\mathbb{R}_+)$. Note that if $\phi \in \Lambda(\mathbb{R}_+)$, then $\phi(0) = 0$ and ϕ is strictly increasing. Also, ϕ necessarily has an inverse ϕ^{-1} with $\phi \circ \phi^{-1} = \phi^{-1} \circ \phi = e$. Moreover, $(\phi_1 \circ \phi_2)^{-1} = \phi_2^{-1} \circ \phi_1^{-1}$ for two homeomorphisms ϕ_1 and ϕ_2.

The basis for positive results is the basic inverse relation in Lemma 13.8.1 of the book, which we restate here:

Lemma 3.4.1. (basic inverse relation) *For any nonnegative integer n and nonnegative real number t,*

$$s_n \le t \quad \textit{if and only if} \quad c(t) \ge n\,. \tag{4.3}$$

The relation between the limits for s_n as $n \to \infty$ and $c(t)$ as $t \to \infty$ follows easily from the following bounds, which are of independent interest. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x and let $\lceil x \rceil$ be the least integer greater than or equal to x. One-sided bounds are obtained below by either setting $\epsilon = 1$ or setting $\delta = \infty$. Let $1/0 = \infty$ and $1/\infty = 0$.

Lemma 3.4.2. (one-sided bounds) *Suppose that $\phi_1, \phi_2 \in \Lambda(\mathbb{R}_+)$, $0 < \epsilon \le 1$ and $0 < \delta \le \infty$.*

(a) If

$$1 - \epsilon \le \frac{\phi_2(c(t))}{\phi_1(t)} < 1 + \delta \quad \text{for all} \quad t \ge t_0, \tag{4.4}$$

then

$$\frac{1}{1+\delta} < \frac{\phi_1(s_n)}{\phi_2(n)} \le \frac{1}{1-\epsilon} \quad \text{for all} \quad n \ge n_0 \equiv \lceil \phi_2^{-1}(\phi_1(t_0)(\lambda + \delta)) \rceil. \tag{4.5}$$

(b) If

$$1 - \epsilon < \frac{\phi_1(s_n)}{\phi_2(n)} \le 1 + \delta \quad \text{for all} \quad n \ge n_0\,, \tag{4.6}$$

then

$$\frac{\phi_2(c(t))}{\phi_1(t)} \le \frac{1}{1-\epsilon} \tag{4.7}$$

and

$$\frac{\phi_2(c(t)+1)}{\phi_1(t)} \ge \frac{1}{1+\delta}. \tag{4.8}$$

for all $t \ge t_0 \equiv \lceil \phi_1^{-1}(\phi_2(t_0)(1+\delta)) \rceil$. Moreover, there is a sequence of times $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\frac{\phi_2(c(t_k))}{\phi_1(t_k)} \ge \frac{1}{1+\delta} \tag{4.9}$$

for all $t_k \ge t_0$.

Proof. (a) If (4.4) holds, then

$$n_1(t) \equiv \lfloor \phi_2^{-1}(\phi_1(t)(1-\epsilon)) \rfloor \le c(t) < \lceil \phi_2^{-1}(\phi_1(t)(1+\delta)) \rceil \equiv n_2(t)$$

for all $t \ge t_0$ and, by Lemma 3.4.1,

$$s_{n_1(t)} \le t < s_{n_2(t)} \quad \text{for all} \quad t \ge t_0 . \tag{4.10}$$

Let t_1 and t_2 be functions of n defined by

$$t_1(n) \equiv \phi_1^{-1}(\phi_2(n)/(1-\epsilon)) \quad \text{and} \quad t_2(n) \equiv \phi_1^{-1}(\phi_2(n)/(1+\delta)) ,$$

and note that $n_1(t_1(n)) = n_2(t_2(n)) = n$ for all n. Hence, for all $n \ge n_0 \equiv \lceil \phi_2^{-1}(\phi_1(t_0)(1+\delta)) \rceil$, we have $t_1(n_0) \ge t_2(n_0) \ge t_0$ and, by (4.10),

$$t_2(n) < s_{n_2(t_2(n))} = s_n = s_{n_1(t_1(n))} \le t_1(n)$$

or, equivalently,

$$\phi_2(n) \left(\frac{1}{1+\delta} - 1 \right) < \phi_1(s_n) - \phi_2(n) \le \phi_2(n) \left(\frac{1}{1-\epsilon} - 1 \right)$$

which implies (4.5).

(b) If (4.6) holds, then

$$\tilde{t}_1(n) \equiv \phi_1^{-1}(\phi_2(n)(1-\epsilon)) < s_n \le \phi_1^{-1}(\phi_2(n)(1+\delta)) \equiv \tilde{t}_2(n)$$

for all $n \ge n_0$ and, by Lemma 3.4.1,

$$c(\tilde{t}_1(n)) < n \le c(\tilde{t}_2(n)) \quad \text{for all} \quad n \ge n_0 . \tag{4.11}$$

Let $\tilde{n}_1$ and $\tilde{n}_2$ be functions of t defined by

$$\tilde{n}_1(t) \equiv \lceil \phi_2^{-1}(\phi_1(t)/(1-\epsilon)) \rceil \quad \text{and} \quad \tilde{n}_2(t) \equiv \lfloor \phi_2^{-1}(\phi_1(t)/(1+\delta)) \rfloor$$

and note that

$$\tilde{t}_2(\tilde{n}_2(t)) \le t \le \tilde{t}_1(\tilde{n}_1(t)) .$$

Hence, by (4.11),

$$\tilde{n}_2(t) \le c(\tilde{t}_2(\tilde{n}_2(t))) \le c(t) \le c(\tilde{t}_1(\tilde{n}_1(t))) < \tilde{n}_1(t)$$

and

$$\phi_2^{-1}(\phi_1(t)/(1+\delta)) - 1 \le c(t) \le \phi_2^{-1}(\phi_1(t)/(1-\epsilon))$$

for all $t \ge t_0 \equiv \phi_1^{-1}\ (\phi_2(n_0)(1+\delta))$, because $\tilde{n}_1(t_0) \ge \tilde{n}_2(t_0) = n_0$, which implies (4.7) and (4.8) by the reasoning for part (a). For (4.9), choose the sequence $\{t_k\}$ so that $\phi_2^{-1}(\phi_1(t_k)/(1+\delta))$ is an integer. Then we have the lower bound $c(t_k) \ge \phi_2^{-1}(\phi_1(t_k)/(1+\delta))$ for all k, which implies (4.9). ∎

We now apply Lemma 3.4.2 to characterize the asymptotic behavior.

Theorem 3.4.1. (implications for pointwise convergence) *Suppose that* ϕ_1, $\phi_2 \in \Lambda(\mathbb{R}_+)$ *and* $0 \le \lambda \le \infty$.

(a) If $\phi_2(c(t))/\phi_1(t) \to \lambda$ *as* $t \to \infty$, *then* $\phi_1(s_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$.

(b) If $\phi_1(s_n)/\phi_2(n) \to \lambda^{-1}$ *as* $n \to \infty$, *then*

$$\overline{\lim_{t\to\infty}}\ \phi_2(c(t))/\phi_1(t) = \lambda. \tag{4.12}$$

(c) If, in addition to the condition for (b), either

$$\frac{\phi_2(c(t)+1) - \phi_2(c(t))}{\phi_1(t)} \to 0 \quad as \quad t \to \infty \tag{4.13}$$

or

$$\frac{\phi_2(n+1)}{\phi_2(n)} \to 1 \quad as \quad n \to \infty, \tag{4.14}$$

then $\phi_2(c(t))/\phi_1(t) \to \lambda$ *as* $t \to \infty$.

(d) If $\phi_1(s_n)/\phi_2(n) \to 0$ *as* $n \to \infty$ *and either*

$$\overline{\lim_{t\to\infty}}\ \frac{\phi_2(c(t)+1) - \phi_2(c(t))}{\phi_1(t)} < \infty \tag{4.15}$$

or

$$\underline{\lim_{n\to\infty}}\ \frac{\phi_2(n)}{\phi_2(n+1)} > 0, \tag{4.16}$$

then $\phi_2(c(t))/\phi_1(t) \to \infty$ *as* $t \to \infty$.

Proof. (a) First suppose that $0 < \lambda < \infty$. Then incorporate λ into $\phi_1(t)$ by dividing by λ. The condition implies that for all appropriate ϵ and δ there exists t_0 such that (4.4) holds. By Lemma 3.4.2(a), (4.5) holds. Since ϵ and δ are arbitrary in (4.5), it implies the desired conclusion. To treat the cases $\lambda = 0$ and $\lambda = \infty$, use the one-sided bounds in Lemma 3.4.2. For example, if $\phi_2(c(t))/\phi_1(t) \to 0$ as $t \to \infty$, then for all positive ϵ and δ there exists t_0 such that $\phi_2(c(t))/\epsilon\phi_1(t)) < 1 + \delta$ for all $t \ge t_0$. By Lemma 3.4.2(a), $\epsilon\phi_1(s_n)/\phi_2(n) > 1/(1+\delta)$ for all $n \ge n_0$. Since ϵ can be arbitrarily small, $\phi_1(s_n)/\phi_2(n) \to \infty$ as $n \to \infty$.

(b) Reason as in (a) using (4.6), (4.7) and (4.9).

(c) Use (4.8), (4.13) and (4.14), noting that

$$\frac{1}{1-\epsilon} - \frac{\phi_2(c(t)+1) - \phi_2(c(t))}{\phi_1(t)} \le \frac{\phi_2(c(t))}{\phi_1(t)} \le \frac{1}{1-\epsilon} \tag{4.17}$$

and

$$\frac{\phi_2(c(t))}{\phi_2(c(t)+1)(1+\epsilon)} \le \frac{\phi_2(c(t))}{\phi_1(t)} \le \frac{1}{1-\epsilon}. \tag{4.18}$$

(d) Reason as in (c), using (4.15) and (4.16) with (4.17) and (4.18). ■

Remark 3.4.1. Note that $\phi_2(c(t))/\phi_1(t) \to \lambda$ as $t \to \infty$ if and only if $\phi_2(c(\phi_1^{-1}(t)))/t \to \lambda$ as $t \to \infty$; i.e., the spatial normalization $\phi_1(t)$ is equivalent to the standard normalizing function e after making a time transformation by ϕ^{-1}. ■

Example 3.4.1. *The need for an extra condition.* To see that an extra condition is needed in Theorem 3.4.1(c), let $s_n = n$ for all n, so that $c(t) = \lfloor t \rfloor$ for all t. Also let $\phi_1(t) = \phi_2(t) = e^t$ for all t. Then $\phi_1(s_n)/\phi_2(n) = 1$ for all n, while

$$\phi_2(c(t))/\phi_1(t) = e^{\lfloor t \rfloor - t},$$

which has limit supremum 1 and limit infimum e^{-1}. Also note that neither (4.13) nor (4.14) is satisfied.

Example 3.4.2. *The extra conditions are not necessary.* To see that the specific extra conditions in Theorem 3.4.1(c) are not necessary, let $s_n = e^n$ for all n, so that $c(t) = \lfloor \log t \rfloor$. Let $\phi_2(t) = e^t$ and $\phi_1(t) = t$ for all t. Then $\phi_1(s_n)/\phi_2(n) = 1$ for all n and

$$\frac{\phi_2(c(t))}{\phi_1(t)} = \frac{e^{\lfloor \log t \rfloor}}{t} \to 1 \quad \text{as} \quad t \to \infty,$$

but $\phi_2(n+1)/\phi_2(n) = e$ for all n and

$$\frac{\phi_2(c(t)+1) - \phi_2(c(t))}{\phi_1(t)} = \frac{(e-1)e^{\lfloor \log t \rfloor}}{t} \to e-1 \quad \text{as} \quad t \to \infty\,. \quad ■$$

A special case of interest is when the homeomorphisms are of the form $\phi(t) = t^p$ for $p > 0$. Of course, the case of greatest interest is $p = 1$; then we have simple averages.

Corollary 3.4.1. (the special case of powers) *Suppose that* $0 < p < \infty$ *and* $0 \le \lambda \le \infty$. *The following are equivalent:*

(i) $c(t)/t^p \to \lambda$ *as* $t \to \infty$,

(ii) $(c(t))^{1/p}/t \to \lambda^{1/p}$ *as* $t \to \infty$,

(iii) $s_n/n^{1/p} \to \lambda^{-1}/p$ *as* $n \to \infty$,

(iv) $(s_n)^p/n \to \lambda^{-1}$ *as* $n \to \infty$.

Proof. Apply Theorem 3.4.1 with $\phi_2(t) = t$ and $\phi_1(t) = t^p$ to relate (i) and (iv). Note that (4.14) holds. To relate (i) and (ii), note that $(c(t))^{1/p}/t = (c(t)/t^p)^{1/p}$, and similarly for (iii) and (iv).■

We used the property that $\phi(x/y) = \phi(x)/\phi(y)$ for $\phi(x) = x^p$ in Corollary 3.4.1. The following classic lemma shows that this does not hold more generally.

Lemma 3.4.3. *A homeomorphism ϕ of $\mathbb{R}_+$ satisfies $\phi(xy) = \phi(x)\phi(y)$ for all nonnegative x and y if and only if $\phi(t) = t^p$ for some $p > 0$.*

Proof. The sufficiency is immediate. For the necessity, suppose that $\phi(xy) = \phi(x)\phi(y)$ for all nonnegative x and y. If we let $\psi(x) = \log \phi(e^x)$, then $\psi(x+y) = \psi(x) + \psi(y)$ for all real x and y. It is well known and easy to see that $\psi(x) = px$ for some real number p, which implies that $\phi(x) = e^{\psi(\log x)} = e^{p \log x} = x^p$. Since ϕ is strictly increasing, we must have $p > 0$. ■

The Corollary to Theorem 3.4.1 is useful because it enables us to replace $\phi_2(c(t))/\phi_1(t)$ and $\phi_1(s_n)/\phi_2(n)$ by $c(t)/\phi_2^{-1}(\phi_1(t))$ and $s_n/\phi_1^{-1}(\phi_2(n))$ respectively. The following lemma shows that we can do this more generally.

Lemma 3.4.4. *Suppose that $\phi \in \Lambda(\mathbb{R}_+)$, $a_n \to \infty$ and $a_n/b_n \to 1$ as $n \to \infty$. If there is a t_0 such that $\log \phi(e^t)$ is uniformly continuous in (t_0, ∞), then $\phi(a_n)/\phi(b_n) \to 1$ as $n \to \infty$.*

Proof. Since $a_n \to \infty$ and $a_n/b_n \to 1$ as $n \to \infty$, $\log a_n - \log b_n \to 0$, $\log a_n \to \infty$ and $\log b_n \to \infty$ as $n \to \infty$. If $\log(\phi(e^t))$ is uniformly continuous in (t_0, ∞), then

$$\begin{aligned} &\log \phi(e^{\log a_n}) - \log \phi(e^{\log b_n}) = \log \phi(a_n) - \log \phi(b_n) \\ &= \log(\phi(a_n)/\phi(b_n)) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

so that $\phi(a_n)/\phi(b_n) \to 1$ as $n \to \infty$. ■

The following Corollary to Lemma 3.4.4 indicates how Lemma 3.4.4 can be applied in our context.

Corollary 3.4.2. *If $\phi_2(c(t))/\phi_1(t) \to \lambda$ as $t \to \infty$, where ϕ_1, $\phi_2 \in \Lambda(\mathbb{R}_+)$ and $\log \phi_2^{-1}(e^t)$ is uniformly continuous in (t_0, ∞) for some t_0, then $c(t)/\phi_2^{-1}(\lambda\phi_1(t)) \to 1$ as $t \to \infty$.*

Remark 3.4.2. Lemma 3.4.4 implies Corollary 3.4.1 because $\log \phi(e^t) = \log \lambda + pt$ when $\phi(t) = \lambda t^p$. Another function covered by Lemma 3.4.4 is

$\phi(t) = a \log bt$; then $\log \phi(e^t)) = \log a + \log(\log b + t)$. However, $\log \phi(e^t) = \log a + be^t$ when $\phi(t) = ae^{bt}$, so that the uniform continuity does not hold when $\phi(t) = ae^{bt}$. ∎

The following result is also useful to characterize the normalizing functions.

Lemma 3.4.5. *Suppose that $\phi \in \Lambda(\mathbb{R}_+)$, $0 < \lambda < \infty$ and $a_n \to \infty$ as $n \to \infty$. If there is a t_0 such that $\log \phi(e^t)$ is uniformly continuous in (t_0, ∞), then*

$$\overline{\lim_{n\to\infty}} \left| \frac{\phi(a_n)}{\phi(\lambda a_n)} \right| < \infty.$$

Proof. Recall that if a function ψ is uniformly continuous in (t_0, ∞), then

$$\sup\{|\psi(t + x) - \psi(t)| : t \geq t_0\} < \infty$$

for any positive x. Since

$$\log \lambda a_n - \log a_n = \log \lambda \ ,$$

$$\begin{aligned} \overline{\lim_{n\to\infty}} \ \{|\log \phi(e^{\log \lambda a_n}) - \log \phi(e^{a_n})|\} &= \overline{\lim_{n\to\infty}} \ \{|\log \phi(\lambda a_n) - \log \phi(a_n)|\} \\ &= \overline{\lim_{n\to\infty}} \ \{|\log(\phi(\lambda a_n)/\phi(a_n))| < \infty \ , \end{aligned}$$

which implies the desired conclusion.∎

We are thinking of $\{s_n : n \geq 1\}$ being the points in a point process sample path, so it is natural to assume that $\{s_n\}$ is nondecreasing. However, we could start with a general sequence of real numbers $\{t_n : n \geq 1\}$ and obtain $\{s_n\}$ as the successive maxima, i.e.,

$$s_n \equiv t_n^{\uparrow} \equiv \max\{t_k : 0 \leq k \leq n\}, \quad n \geq 1, \tag{4.19}$$

where $t_0 = 0$. A similar result holds for $c(t)$. The following result closely parallel Proposition 3.3.1.

Proposition 3.4.1. *Suppose that ϕ_1, $\phi_2 \in \Lambda(\mathbb{R}_+)$ and $0 \leq \lambda \leq \infty$. If $\phi_1(t_n)/\phi_2(n) \to \lambda^{-1}$ as $n \to \infty$, then $\phi_1(s_n)/\phi_2(n) \to \lambda^{-1}$ as $n \to \infty$ for s_n in (4.19).*

Proof. First assume that $0 < \lambda < \infty$. Given the assumed convergence, for all $\epsilon > 0$, there is an n_0 such that

$$\phi_1^{-1}(\phi_2(n)/\lambda(1+\epsilon)) \le t_n \le \phi_1^{-1}(\phi_2(n)/\lambda(1-\epsilon)) \quad \text{for all} \quad n \ge n_0,$$

which implies

$$\phi_1^{-1}(\phi_2(n)/\lambda(1+\epsilon)) \le s_n \le \max\{s_{n_0}, \phi_1^{-1}(\phi_2(n)/\lambda(1-\epsilon))\} \quad \text{for all} \quad n \ge n_0.$$

Let n_1 be such that

$$\phi_1^{-1}(\phi_2(n)/\lambda(1-\epsilon)) \ge s_{n_0}\,.$$

Then, for all $n \ge n_1$,

$$\frac{1}{\lambda(1+\epsilon)} \le \frac{\phi_1(s_n)}{\phi_2(n)} \le \frac{1}{\lambda(1-\epsilon)},$$

which implies the conclusion. For $\lambda = 0$ and $\lambda = \infty$ use associated one-sided inequalities. ■

3.5. Counting Functions with Centering

We now turn to counting functions with centering. Due to the results for the inverse map with centering in Section 13.7 of the book, Theorem 13.8.2 in the book yields FCLTs for stochastic counting processes with centering given FCLTs for associated sequences of nondecreasing nonnegative random variables with centering, by an application of the continuous mapping theorem. We now show that we can also exploit the monotonicity to obtain *ordinary* CLTs for stochastic counting processes from associated *ordinary* CLTs for sequences of nondecreasing nonnegative random variables. The resulting CLT for stochastic counting process is the same as can be obtained from the FCLT by projection, but the condition is weaker. In both cases, we rely on an existing limit rather than specific stochastic assumptions. For this purpose, let $\{S_n : n \ge 0\}$ be a sequence of nondecreasing nonnegative random variables with $S_0 = 0$ and let $\{C(t) : t \ge 0\}$ be the associated stochastic counting process, defined as before by

$$C(t) \equiv \max\{k \ge 0 : S_k \le t\}, \quad t \ge 0\,. \tag{5.1}$$

We again use regularly varying functions.

Theorem 3.5.1. (CLT equivalence) *Suppose that* $m > 0$ *and* $\psi \in \mathcal{R}(p)$ *for* $0 < p < 1$. *Then*

$$\psi(n)^{-1}[S_n - nm] \Rightarrow L \quad in \quad \mathbb{R} \quad as \quad n \to \infty\ , \tag{5.2}$$

where $\{S_n : n \geq 0\}$ *is a sequence of nondecreasing nonnegative random variables with* $S_0 = 0$ *if and only if*

$$\psi(t)^{-1}[C(t) - m^{-1}t] \Rightarrow -m^{-(1+p)}L \quad in \quad \mathbb{R} \quad as \quad n \to \infty\,, \tag{5.3}$$

where $\{C(t) : t \geq 0\}$ *is the associated stochastic counting process.*

We obtain Theorem 3.5.1 from a more general theorem which allows more general scalings, which are of value when analyzing nonstationary point processes.

Theorem 3.5.2. (CLT implications with more general scaling functions) *Suppose that* $\phi_1,\ \phi_2 \in \Lambda(\mathbb{R}_+)$, $\psi \in C_\uparrow$ *and*

$$\psi(t)/\psi(t + x\psi(t)) \to 1 \quad as \quad t \to \infty \tag{5.4}$$

for all x.

(a) If

$$X(t) \equiv \psi(\phi_1(t))^{-1}[\phi_2(C(t)) - \phi_1(t)] \Rightarrow L \quad in \quad \mathbb{R} \quad as \quad n \to \infty\,, \tag{5.5}$$

then

$$Y(n) = \psi(\phi_2(n))^{-1}[\phi_1(S_n) - \phi_2(n)] \Rightarrow -L \quad in \quad \mathbb{R} \quad as \quad n \to \infty\,. \tag{5.6}$$

(b) If (5.6) *above holds, then there exists an increasing sequence of positive real numbers* $\{t_n : n \geq 1\}$ *with* $t_n \to \infty$ *as* $n \to \infty$ *such that* $X(t_n) \Rightarrow L$ *for* $X(t)$ *in* (5.5) *above.*

(c) If, in addition to (5.6) *above,*

$$[\phi_2(n+1) - \phi_2(n)]/\psi(\phi_2(n)) \to 1 \quad as \quad n \to \infty \tag{5.7}$$

and

$$\psi(\phi_2(n+1))/\psi(\phi_2(n)) \to 1 \quad as \quad n \to \infty\,, \tag{5.8}$$

then (5.5) *above holds.*

We first apply Theorem 3.5.2 to prove Theorem 3.5.1.

Proof of Theorem 3.5.1. We apply Theorem 3.5.2 with $\phi_1(t) = mt$, $\phi_2(t) = t$ and $\psi \in \mathcal{R}(p)$ for $0 < p < 1$. It is easy to see that ψ satisfies (5.4): For any x, there is a t_0 such that

$$\psi((1-\epsilon)t) \leq \psi(t + x\psi(t)) \leq \psi((1 = \epsilon)t) \tag{5.9}$$

for all $t \geq t_0$, from which it follows that

$$\left(\frac{1}{1+\epsilon}\right)^p \leq \frac{\psi(t)}{\psi(t + x\psi(t))} \leq \left(\frac{1}{1-\epsilon}\right)^p \tag{5.10}$$

for all suitably large t. We also apply the regular variation property to deduce that $\psi(\phi_1(t))$ in (5.5) has the asymptotic form

$$\psi(\phi_1(t)) = \psi(mt) \sim m^p \psi(t) \quad \text{as} \quad t \to \infty \; . \tag{5.11}$$

Thus (5.2) is equivalent to (5.6) with the limit in (5.2) changed to $m^p L$. Similarly, (5.3) is equivalent to (5.5) with the limit in (5.3) changed to $-m^{-1}L$. Thus the form of the limits in (5.2) and (5.3) follow from (5.5) and (5.6). Finally, it remains to observe that the assumptions in Theorem 3.5.1 imply that conditions (5.7) and (5.8) hold. ■

We now turn to the proof of Theorem 3.5.2. For that purpose, we use a basic lemma about cumulative distribution functions (cdf's).

Lemma 3.5.1. *Let F_n, $n \geq 0$, be cdf's. The following are equivalent:*

(i) For each $t \in Disc(F_0)^c$, $F_n(t_n) \to F_0(t)$ as $n \to \infty$ for some sequence $\{t_n : n \geq 1\}$ with $t_n \to t$.

(ii) $F_n \Rightarrow F_0$; i.e., for each $t \in Disc(F_0)^c$, $F_n(t) \to F_0(t)$ as $n \to \infty$.

(iii) For each $t \in Disc(F_0)^c$ and all sequences $\{t_n : n \geq 1\}$ with $t_n \to t$ as $n \to \infty$, $F_n(t_n) \to F_0(t)$ as $n \to \infty$.

Proof. Clearly (iii)→(ii)→(i), so it suffices to show that (i)→(iii). Let $t \in Disc(F_0)^c$. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$F_0(t) - \epsilon \leq F_0(t - \delta) \leq F_0(t) \leq F_0(t + \delta) \leq F_0(t) + \epsilon \; . \tag{5.12}$$

Since F_0 is nondecreasing, it has at most countably many discontinuities. Let t', $t'' \in Disc(F_0)^c$ be such that $t - \delta < t' < t < t'' < t + \delta$. Given (i), there exist sequences $\{t'_n : n \geq 1\}$ and $\{t''_n : n \geq 1\}$ such that $t'_n \to t'$, $t''_n \to t''$, $F_n(t'_n) \to F_0(t')$ and $F_n(t''_n) \to F_0(t'')$ as $n \to \infty$. Let $\{t_n : n \geq 1\}$ be any sequence such that $t_n \to t$ as $n \to \infty$. Hence, there is an n_0 such that

$$F_0(t - \delta) < F_n(t'_n) \leq F_n(t_n) \leq F_n(t''_n) \leq F_0(t + \delta) \tag{5.13}$$

for all $n \geq n_0$. Combining (5.12) and (5.13), we see that

$$F_0(t) - \epsilon \leq F_n(t_n) \leq F_0(t) + \epsilon \; . \quad \blacksquare \tag{5.14}$$

Proof of Theorem 3.5.2. (a) Suppose that (5.5) holds. Then

$$F_t(x-) \equiv P(X(t) < x) \to P(L < x) \equiv F(x) \quad \text{as} \quad t \to \infty \tag{5.15}$$

for each $x \in Disc(F)^c$. However,

$$\begin{aligned} F_t(x-) &= P(\phi_2(C(t)) < \phi_1(t) + x\psi(\phi_1(t))) \\ &= P(C(t) < \phi_2^{-1}(\phi_1(t) + x\psi(\phi(t))) \end{aligned}$$

so that, by Lemma 3.4.1, $F_t(x) = P(S_{n(t)} > t)$ for any t such that

$$n(t) \equiv \phi_2^{-1}(\phi_1(t) + x\psi(\phi_1(t))) \tag{5.16}$$

is an integer. For such t,

$$F_t(x-) = P(\psi(\phi_2(n(t)))^{-1}[\phi_1(S_{n(t)}) - \phi_2(n(t))] > -x(t)) \, , \tag{5.17}$$

where

$$\begin{aligned} x(t) &= -[\phi_1(t) - \phi_2(n(t))]/\psi(\phi_2(n(t))) \\ &= x\psi(\phi_1(t))/\psi(\phi_1(t) + x\psi(\phi_1(t))) \to x \quad \text{as} \quad t \to \infty \end{aligned} \tag{5.18}$$

by (5.16) and (5.4). Note that, for each positive integer n, we can find t_n such that $n(t_n) = n$, because ϕ_1, ϕ_2 and ψ are nondecreasing and continuous, and $n(t) \to \infty$ as $t \to \infty$. Hence

$$G_n(x_n) \equiv P(\psi(\phi_2(n))^{-1}[\phi_1(S_n) - \phi_2(n)] < x_n) = F_{t_n}(x_n) \tag{5.19}$$

where $x_n = x(t_n) \to x$ as $n \to \infty$. Since $x \in Disc(F)^c$, $F_{t_n}(x_n) \to F(x)$ as $n \to \infty$. By Lemma 3.5.1, $G_n \Rightarrow F$, so that $Y(n) \Rightarrow -L$.

(b) Let the cdf G_n be defined by (5.19). Then

$$G_n(x) = P(S_n > \phi_1^{-1}(\phi_2(n) - x\psi(\phi_2(n)))) = P(A(t_n) < n)$$

for

$$t_n = \phi_1^{-1}(\phi_2(n) - x\psi(\phi_2(n))) \tag{5.20}$$

by Lemma 3.4.1. Thus, for F_t in (5.15) and t_n in (5.20), $F_{t_n}(x_n-) = G_n(x)$ for

$$\begin{aligned} x_n &= [\phi_2(n) - \phi_1(t_n)]/\psi(\phi_1(t_n)) \\ &= x\psi(\phi_2(n))/\psi(\phi_2(n) - x\psi(\phi_2(n)) \to x \quad \text{as} \quad n \to \infty \end{aligned} \tag{5.21}$$

by (5.20) and (5.4). Hence, if $G_n(x) \to F(x)$ for $x \in Disc(F)^c$, then $F_{t_n}(x_n) \to F(x)$ as $n \to \infty$. By Lemma 3.5.1, $F_{t_n} \Rightarrow F$, so that $X(t_n) \Rightarrow L$.

(c) For any t, let n be such that $t_n \le t < t_{n+1}$ for t_n in (5.20). Since $C(t_n) \le C(t) \le C(t_{n=1})$, it suffices to show that $C(t_n)$ and $C(t_{n+1})$ have the same limits with the normalization. It suffices to show that

$$\psi(\phi_1(t_{n+1}))^{-1}[\phi_2(C(t_n)) - \phi_1(t_{n+1})] \Rightarrow L \,, \tag{5.22}$$

which in turn holds if

$$\psi(\phi_1(t_{n+1}))/\psi(\phi_1(t_n)) \to 1 \tag{5.23}$$

and

$$[\phi_1(t_{n+1}) - \phi_1(t_n)]/\psi(\phi_1(t_n)) \to \quad \text{as} \quad n \to \infty\,. \tag{5.24}$$

By (5.4) and (5.20), (5.23) is equivalent to (5.8). By (5.20) and (5.23), (5.24) is equivalent to (5.7): Applying (5.20) and dividing numerator and denominator by $\psi(\phi_2(n))$, we see that (5.21) becomes A_n/B_n, where

$$B_n = \psi(\phi_2(n) - x\psi(\phi_2(n)))/\psi(\phi_2(n)) \to 1 \quad \text{as} \quad n \to \infty \tag{5.25}$$

by (5.4) and

$$A_n = [\phi_2(n+1) - x\psi(\phi_2(n+1)) - \phi_2(n) + x\psi(\phi_2(n)]/\psi(\phi_2(n)) \tag{5.26}$$

by (5.8) and (5.7). ∎

Example 3.5.1. *It is possible that only subsequences converge.* To see that we can have $X(t_n) \Rightarrow L$ as $n \to \infty$ in Theorem 3.5.2 (b) without $X(t) \Rightarrow L$ as $t \to \infty$ for $X(t)$ in (5.5), let $\phi_1(t) = t$, $\phi_2(t) = t^2$ and $\psi(t) = 1$ for $t \ge 0$. Then (5.4) and (5.8) hold, but (5.7) does not:

$$\frac{\phi_2(n+1) - \phi_2(n)}{\psi(\phi_2(n))} = \frac{(n+1)^2 - n^2}{n^2} \to 0\ . \tag{5.27}$$

Let $P(L = 0) = 1$ and let $S_n = n^2$, so that

$$\psi(\phi_2(n))^{-1}[\phi_1(S_n) - \phi_2(n)] = 0 = -L \quad \text{w.p.1 for all } n. \tag{5.28}$$

However, $C(t) = \sqrt{\lfloor t \rfloor}$ and $\phi_2(C(t)) = \lfloor t \rfloor$, so that

$$\psi(\phi_1(t))^{-1}[\phi_2(C(t)) - \phi_1(t)] = \lfloor t \rfloor - 1\,, \tag{5.29}$$

from which we see that

$$\underline{\lim}_{t \to \infty}\ X(t) = -1 < 0 = \overline{\lim}_{t \to \infty}\ X(t)$$

for $X(t)$ in (5.5). ∎

A major theme here is obtaining probabilistic limits directly from deterministic limits. Thus it is natural to ask if there is a deterministic analog of Theorem 3.5.2 that implies Theorem 3.5.2. We show that there is. In particular, the following result implies parts (a) and (c) of Theorem 3.5.2.

Theorem 3.5.3. (deterministic analog of Theorem 3.5.2) *Let* $\phi_1, \phi_2 \in \Lambda(\mathbb{R}_+)$ *and let* ψ *be a continuous positive real-valued function on* $[0,\infty)$ *for which (5.4) holds*

(a) If

$$x(t) \equiv \psi(\phi_1(t))^{-1}[\phi_2(c(t)) - \phi_1(t)] \to \alpha \quad in \quad \mathbb{R} \quad as \quad t \to \infty\ , \tag{5.30}$$

then

$$y(n) \equiv \psi(\phi_2(n))^{-1}[\phi_1(s_n) - \phi_2(n)] \to -\alpha \quad in \quad \mathbb{R} \quad as \quad n \to \infty\ . \tag{5.31}$$

. *(b) If, in addition to* (5.31) *here,* (5.7) *and* (5.8) *above hold, then* (5.30) *here holds.*

Proof. (a) If (5.30) holds, then for all $\epsilon > 0$ there exists t_0 such that $\alpha - \epsilon \le x(t) < \alpha + \epsilon$ for all $t \ge t_0$. Given that $x(t) < \alpha + \epsilon$,

$$\phi_2^{-1}(\phi_1(t) + (\alpha - \epsilon)\psi(\phi_1(t)) \le c(t) < \phi_2^{-1}(\phi_1(t) + (\alpha + \epsilon)\psi(\phi_1(t)))\,. \tag{5.32}$$

Let t be such that

$$n(t) \equiv \phi_2^{-1}(\phi_1(t) + (\alpha + \epsilon)\psi(\phi_1(t))) \tag{5.33}$$

is an integer. By Lemma 3.4.1,

$$s_{n(t)} > t\ . \tag{5.34}$$

Given (5.34),

$$y(n(t)) \equiv \psi(\phi_2(n(t)))^{-1}[\phi_1(s_{n(t)}) - \phi_2(n(t))] > -\alpha(t) \tag{5.35}$$

where

$$\alpha(t) \equiv \frac{\phi_1(t) - \phi_2(n(t))}{\psi(\phi_2(n(t)))} = \frac{-(\alpha + \epsilon)\psi(\phi_1(t))}{\psi(\phi_1(t) + (\alpha + \epsilon)\psi(\phi_1(t)))} \to -(\alpha + \epsilon) \tag{5.36}$$

by (5.4), so that $y(n(t)) > -(\alpha + 2\epsilon)$ for all $t \ge t_1 \ge t_0$. Since for each positive integer n, we can find t such that $n(t) = n$, there is an n_0 such that $y(n) > -(\alpha + 2\epsilon)$ for all $n \ge n_0$. Similarly, from the lower bound in (5.32), we can conclude that $y(t) \leq -(\alpha - 2\epsilon)$ for all $n \geq n_1$. Since ϵ was arbitrary, the proof is complete.

(b) For any $\epsilon > 0$, there exists n_0 such that $-\alpha - \epsilon < y(n) \leq -\alpha + \epsilon$ for $n > n_0$. As a consequence,

$$\phi_1^{-1}(\phi_2(n) - (\alpha + \epsilon)\psi(\phi_2(n))) < s_n \leq \phi_1^{-1}(\phi_2(n) - (\alpha - \epsilon)\psi(\phi_2(n))) \quad (5.37)$$

for all $n \geq n_0$. Now let

$$t_n \equiv \phi_1^{-1}(\phi_2(n) - (\alpha - \epsilon)\psi(\phi_2(n))) \, . \quad (5.38)$$

By Lemma 3.4.1,

$$c(t_n) \geq n \quad (5.39)$$

and

$$\begin{aligned} x(t_n) &= \frac{\phi_2(c(t_n)) - \phi_1(t_n)}{\psi(\phi_1(t_n))} \\ &\geq \frac{\phi_2(n) - \phi_1(t_n)}{\psi(\phi_1(t_n))} \\ &= \frac{(\alpha - \epsilon)\psi(\phi_2(n))}{\psi(\phi_2(n) - (\alpha - \epsilon)\psi(\phi_2(n)))} \geq \alpha - 2\epsilon \end{aligned} \quad (5.40)$$

for $n \geq n_1 \geq n_0$ by (5.4). We now want to show that there is t_0 such that $x(t) \geq \alpha - 3\epsilon$ for all $t \geq t_0$. Consider t with $t_n \leq t < t_{n+1}$. Notice that

$$\phi_2(c(t_n)) - \phi_1(t_{n+1)} \leq \phi_2(c(t)) - \phi_1(t) \leq \phi_2(c(t_{n+1})) - \phi_1(t_n)) \, . \quad (5.41)$$

Since (5.41) holds, (5.40), (5.7) and (5.8) imply that there is t_0 such that $x(t) > \alpha - 3\epsilon$ for $t > t_0$. Similarly, using the lower bound in (5.37), we can deduce that for any $\epsilon > 0$ there exists t_0 such that $x(t) < \alpha + 3\epsilon$ for $t > t_0$. Since ϵ was arbitrary, the proof is complete. ■

3.6. Composition

We now turn to the composition map. We first state a preliminary lemma.

Lemma 3.6.1. *If $\phi \in \mathcal{R}(p)$ with $p > 0$, and $t^{-q}y(t) \to \mu > 0$, then*

$$\phi(y(t))/\phi(t^q) \to \mu^p \quad \textit{as} \quad t \to \infty \, . \quad (6.1)$$

Proof. For any $\epsilon > 0$, there is t_0 such that $t^q(\mu - \epsilon) < y(t) < t^q(\mu + \epsilon)$. Since ϕ is regularly varying with index p,

$$\overline{\lim_{t\to\infty}} \frac{\phi(y(t))}{\phi(t^q)} \le \lim_{t\to\infty} \frac{\phi(t^q(\mu+\epsilon))}{\phi(t^q)} \le (\mu+\epsilon)^p$$

and

$$\underline{\lim_{t\to\infty}} \frac{\phi(y(t))}{\phi(t^q)} \ge \lim_{t\to\infty} \frac{\phi(t^q(\mu-\epsilon))}{\phi(t^q)} \ge (\mu-\epsilon)^p . \quad \blacksquare$$

Proposition 3.6.1. *Suppose that $\phi \in \mathcal{R}(p)$ with $p > 0$. If*

$$\phi(t)^{-1}X(t) \Rightarrow U \quad \textit{and} \quad t^{-1}Y(t) \Rightarrow \mu \quad \textit{in} \quad \mathbb{R} , \tag{6.2}$$

then

$$\phi(t)^{-1}X(Y(t)) \Rightarrow \mu^p U \quad \textit{in} \quad \mathbb{R} . \tag{6.3}$$

Proof. Since the limit μ in (6.2) is deterministic, we have the joint limit

$$(\phi(t)^{-1}X(t), t^{-1}Y(t)) \Rightarrow (U, \mu) \quad \text{in} \quad R^2 . \tag{6.4}$$

Use the Skorohod representation theorem to replace convergence in distribution in (6.4) with convergence w.p.1 (for special versions). By Lemma 3.6.1, $\phi(Y(t))/\phi(t) \to \mu^p$. Then

$$\frac{X(Y(t))}{\phi(t)} = \frac{\phi(Y(t))}{\phi(t)} \frac{X(Y(t))}{\phi(Y(t))} \to \mu^p U . \tag{6.5}$$

Finally, (6.5) implies (6.3). $\blacksquare$

Proposition 3.6.2. *Suppose that $\phi \in \mathcal{R}(p)$ with $0 < p < 1$. If*

$$\phi(t)^{-1}[X(t) - \lambda t, Y(t) - \mu t] \Rightarrow (U, V) \quad \textit{in} \quad \mathbb{R}^2 \tag{6.6}$$

then

$$\phi(t)^{-1}[X(Y(t)) - \lambda\mu t] \Rightarrow \mu^p U + \lambda V \quad \textit{in} \quad \mathbb{R}^2 . \tag{6.7}$$

Proof. From (6.6) the regular variation condition, we have $t^{-1}Y(t) \Rightarrow \mu$ and $\phi(Y(t))/\phi(t) \to \mu^p$ as $t \to \infty$. Now replace convergence in distribution by convergence w.p.1 for special versions. Then

$$\frac{\phi(Y(t))}{\phi(t)} \frac{X(Y(t)) - \lambda Y(t)}{\phi(Y(t))} + \frac{\lambda Y(t) - \lambda \mu t}{\phi(t)} \to \mu^p U + \lambda V \text{ w.p.1 as } t \to \infty\,, \tag{6.8}$$

which implies (6.7). ■

There are two difficulties with Propositions (3.6.1) and (3.6.2) for applications. An obvious difficulty is that we may actually need the stronger conclusions giving limits in D in applications. The other difficulty is that it may be difficult to obtain the conditions. The joint limit in (6.6) holds if the component limits hold in $\mathbb{R}$ when $X(t)$ and $Y(t)$ are independent, but in most applications $X(t)$ and $Y(t)$ are actually dependent. A critical step then is to establish condition (6.6).

To illustrate, we may start with the sequence $\{(A_n, B_n) : n \geq 1\}$ of ordered pairs of nonnegative random variables. We may be able to determine that

$$\phi(n)^{-1}[A_n - n\lambda, B_n - n\mu^{-1}] \Rightarrow (U', V') \quad \text{as} \quad n \to \infty \quad \text{in} \quad \mathbb{R}^2 \tag{6.9}$$

and be interested in the asymptotic behavior of $A_{C(t)}$, where

$$C(t) = \max\{k \geq 1 : B_k \leq t\}, \quad t \geq 0\,. \tag{6.10}$$

From the second component of (6.9), we can determine that

$$\phi(t)^{-1}[C(t) - \mu t] \Rightarrow -\mu^{(1+p)} V' \tag{6.11}$$

from Theorem 3.5.1. However, we have difficulty directly expressing the joint limits of

$$\phi(n)^{-1}(A_n - n\lambda) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \phi(t)^{-1}[C(t) - \mu t] \quad \text{as} \quad t \to \infty\,. \tag{6.12}$$

The extension of (6.9) in D offers a resolution. We can hopefully extend (6.9) to

$$(\mathbf{A}_n, \mathbf{B}_n) \Rightarrow (\mathbf{U}, \mathbf{V}) \quad \text{in} \quad D^2 \tag{6.13}$$

where

$$\begin{aligned} \mathbf{A}_n(t) &\equiv \phi(n)^{-1}[A_{\lfloor nt \rfloor} - \lambda nt] \\ \mathbf{B}_n(t) &\equiv \phi(n)^{-1}[B_{\lfloor nt \rfloor} - \mu^{-1} nt] \\ (\mathbf{U}(1), \mathbf{V}(1)) &\stackrel{\mathrm{d}}{=} (U', V')\,. \end{aligned} \tag{6.14}$$

From (6.13), we can get

$$(\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n) \Rightarrow (\mathbf{U}, \mathbf{V}, -\mu\mathbf{V} \circ \mu\mathbf{e}) \tag{6.15}$$

where

$$\mathbf{C}_n(t) \equiv \phi(n)^{-1}[C(nt) - \mu nt] \ . \tag{6.16}$$

We can then apply the composition map in D. In particular, letting

$$\mathbf{\Phi}_n(t) \equiv n^{-1}C(nt) \ , \tag{6.17}$$

and

$$\mathbf{X}_n \equiv \phi(n)^{-1}[A_{C(nt)} - \lambda\mu nt] \, , \tag{6.18}$$

we obtain

$$\mathbf{X}_n = \mathbf{A}_n \circ \mathbf{\Phi}_n + \mu\mathbf{C}_n \Rightarrow \mathbf{U} \circ \lambda\mathbf{e} + \mu\mathbf{C} \quad \text{in} \quad D \tag{6.19}$$

under regularity conditions, by Theorem 13.3.2 in the book. As a consequence,

$$\phi(n)^{-1}[A_{C(n)} - \lambda\mu n] \Rightarrow \mathbf{U}(\lambda) + \mu\mathbf{C}(1) \quad \text{in} \quad \mathbb{R}, \tag{6.20}$$

assuming that $P(1 \in Disc(\mathbf{U} \circ \lambda\mathbf{e} + \mu\mathbf{C})) = 0$.

Alternative approaches have been developed for dealing with this problem directly, starting with the Anscombe (1952, 1953) condition, see Gut (1988), but those conditions are essentially equivalent to $\mathbf{A}_n \Rightarrow \mathbf{U}$ with $P(\mathbf{U} \in C) = 1$.

3.7. Chapter Notes

The main results in this chapter are so basic that they no doubt have a long history, but we are unable to trace that history beyond our own work. Much related material, with emphasis on the classical case of partial sums of i.i.d. random variables, appears in Gut (1988).

We have primarily drawn upon Glynn and Whitt (1986, 1988) and Massey and Whitt (1994). Those papers contain further applications to queues related to the conservation law $L = \lambda W$. El-Taha and Stidham (1999) is closely related from that perspective. El-Taha and Stidham demonstrate the far-reaching implications possible from pointwise limits for single functions (sample-path analysis). Baccelli and Bremaud (1994) provide an alternative treatment of many of the same topics in the context of stationary processes. An overview of $L = \lambda W$ appears in Whitt (1991, 1992).

The strengthening of pointwise convergence to uniform convergence in Theorem 3.2.1 extends Theorem 4 of Glynn and Whitt (1988), which was in the form of Corollary 3.2.1. For the case $\phi(t) = t$, Proposition 3.3.1 is implication (iii)$\rightarrow$(v) in Theorem 2(b) of Glynn and Whitt (1986). The more general version appears in Section 2.5 of El-Taha and Stidham (1999).

Theorem 3.5.1 here extends Theorem 4.2 of Massey and Whitt (1994) by allowing the space scaling function ψ to be regularly varying instead of a simple power. Lemma 3.5.1 is an improved statement of Lemma 4.1 of Massey and Whitt (1994). The deterministic basis for Theorem 3.5.2 in Theorem 3.5.3 is new here.

An extensive treatment of the composition map and convergence in distribution under a random time change appears in Gut (1988). The first few sections there provide useful perspective. A related result is the conservation law $Y = \lambda X$ in El-Taha and Stidham (1999).