Internet Supplement

to

Stochastic-Process Limits

An Introduction to Stochastic-Process And their Application to Queues

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Preface

0.1. Why is there an Internet Supplement?

This Internet Supplement has three purposes: First, it is intended to maintain a list of corrections for errors found after the book has been published. Second, it is intended to provide supporting details, such as proofs, for material in the book. Third, it is intended to provide supplementary material related to the subject of the book.

As indicated in the Preface to the book, in order to avoid excessive length, material was deleted from the book and placed in this Internet Supplement. The first choice for cutting was the more technical material. Thus, the Internet Supplement contains many proofs for theorems in the book. Specifically, missing proofs for results stated in the book are contained here in Chapter 1 (all but Section 1.4), Section 5.3 and Chapters 6–8.

It was also considered necessary to cut some entire discussions. Hence the book also contains supplementary material related to, but going beyond, what is in the book. Such material is contained here in Section 1.4, Chaprers 2–5 (all but Section 5.3) and Chapter 9.

In addition to making corrections as errors are discovered, the Internet Supplement provides an opportunity to add other material after the book has been published. We would like to add additional material on the spaces E and F, going beyond the brief introduction in Chapter 15 of the book.

0.2. Organization

We now indicate how the Internet Supplement is organized.

Chapter 1 here complements Chapter 3 of the book on the framework for stochastic-process limits. Sections 1.2 and 1.3 provide proofs for the Prohorov metric properties and the Skorohod representation theorem from

Section 3.2 of the book. Section 1.4 explains the adjective "weak" in "weak convergence" from a Banach-space perspective. Finally, Section 1.5 gives proofs of the continuous-mapping theorems and the Lipschitz-mapping theorem in Section 3.4 of the book.

Chapter 2 here complements Chapter 4 of the book on basic stochastic-process limits. Section 2.2 complements Section 4.3 of the book on Donsker's theorem by providing an introduction to strong approximations and their application to establish rates of convergence in the setting of Donsker's theorem, using the Prohorov metric on the space of probability measures $\mathcal{P}$ on the function space D. Section 2.3 complements Section 4.4 of the book on Brownian limits with weak dependence by presenting FCLT's exploiting Markov, regenerative and martingale structure. Section 2.4 complements Section 4.5 in the book on convergence to stable Lévy motion by discussing FCLT's in the framework of double sequences (or triangular arrays) of random variables; with an IID assumption, the scaled partial sums converge to general Lévy processes. Finally, Section 2.5 complements Section 4.6 of the book on strong dependence by showing that the linear-process representation in equation (6.6) of the book arises naturally in the framework of time-series models.

Chapter 3 here complements Chapter 13 of the book on useful functions that preserve convergence by showing how pointwise convergence in $\mathbb{R}$ is preserved under mappings. Section 3.2 shows that in some settings pointwise convergence directly implies uniform convergence over bounded intervals. As a consequence, an ordinary strong law of large numbers (SLLN) directly implies the more general functional strong law of large numbers (FSLLN). The remaining sections in Chapter 3 discuss the preservation of pointwise convergence under the supremum, inverse and composition maps. With the inverse map, attention is focused on counting processes, with and without centering.

Chapter 4 here complements Sections 5.9 and 10.4.4 of the book by discussing another application of stochastic-process limits to simulation. Sections 5.9 and 10.4.4 of the book show how heavy-traffic stochastic-process limits for queues can be used to help plan queueing simulations. In particular, they determine the approximate required simulation run length, as a function of model parameters, in order to achieve desired statistical precision. Drawing upon and extending Glynn and Whitt (1992a), Chapter 4 shows how FCLT's and the continuous-mapping approach can be used to establish general criteria for sequential stopping rules for simulations to be asymptotically valid.

Chapter 5 here complements Chapters 5, 8 and 9 of the book on singleserver queues. Section 5.2 here discusses general reflected-Lévy-process approximations for queues that arise when there is a sequence of queueing models with net-input processes satisfying the FCLT's discussed here in Section 2.4. Section 5.3 here provides the proof of Theorem 8.3.1 in the book, which establishes a FCLT for the cumulative busy time of a single on-off source. Finally, following Puhalskii (1994), Section 5.4 here shows how the continuous-mapping approach with the inverse map and nonlinear centering in Theorem 13.7.4 of the book can be used to convert stochastic-process limits for arrival, departure and queue-length processes into associated stochastic-process limits for waiting-time and workload processes in quite general queueing models.

Chapters 6, 7 and 8 here provide proofs for theorems in Chapters 12, 13 and 14, respectively, in the book. The numbering within the chapters here closely parallels the numbering within the corresponding chapter in the book, so the desired proof here should be easy to find. In addition, there is an extra section in Chapter 8 here on queueing networks. Drawing on and extending Kella and Whitt (1996), Section 8.9 establishes general conditions for a multidimensional reflected process to have a limiting stationary version.

Chapter 9 here continues the study of useful functions begun in Chapter 13 of the book. In particular, drawing upon and extending Mandelbaum and Massey (1995), Chapter 9 here studies convergence preservation of the supremum, (one-sided, one-dimensional) reflection and inverse maps with nonlinear centering. Under regularity conditions, the limit for the scaled functions after applying these maps can be identified with an appropriate "directional" derivative of the map.

Finally, Chapter 10 here is intended to contain corrections for errors found after the book has been published.

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