The Steady-State Distribution of the $M_t/M/\infty$ Queue with a Sinusoidal Arrival Rate Function

Ward Whitt

Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027-6699, USA

Abstract

The number of customers in a stable $M_t/GI/n$ queue with a periodic arrival rate function and n servers has a proper steady-state limiting distribution if the initial place within the cycle is chosen uniformly at random. Insight is gained by examining the special case with infinitely many servers, exponential service times and a sinusoidal arrival rate function. Heavy-traffic limits help explain an unexpected bimodal form. The peakedness (ratio of the variance to the mean) can be used for approximations with finitely many servers.

Keywords: queues with periodic arrival rates, periodic steady state, heavy traffic, fluid models, peakedness

1. Introduction

The purpose of this paper is to gain insight into the steady-state distribution associated with stable $M_t/G/n$ queues having a nonhomogeneous Poisson process (NHPP) as its arrival process and a periodic arrival rate function. It is known that, under regularity conditions, the number of customers in the system, $Q(t)$, has a limiting periodic distribution as t increases; e.g., see [1, 2]. The limiting periodic distribution can be defined by the limiting distribution of the subsequence obtained by looking at successive cycles at the same place within the cycle. If the initial place within the cycle is chosen uniformly at random, then there is a proper limiting steady-state distribution. That steady-state distribution describes the long-run (time) average performance. (See Remark 2.1 for an arrival view.)

In order to gain insight into that steady-state distribution, we will focus on a special case that is remarkably tractable: the $M_t/M/\infty$ model with a sinusoidal arrival rate function. Many concrete results for this model are contained in [3], and we will exploit them. We will obtain new results primarily by establishing heavy-traffic limits for the steady-state distribution of the $M_t/M/\infty$ model. Similar results can be established for more general models, as we explain in §6.

When queues have time-varying arrival rates, we are usually most interested in time-dependent descriptions of the performance. However, we may also be interested in the long-term average performance, as captured by the steady-state distribution. As we will show here, the steady-state distribution for the relatively simple $M_t/M/\infty$ model is somewhat complicated, having a form that might not be anticipated, even given a good understanding of the time-varying performance (as available from [3]). We will illustrate in §2.

There is likely to be greater interest in the steady-state distribution of a periodic queue when the periodic cycles are relatively short. Then customer times in the system might well extend across several cycles. Periodic NHPP arrival processes with short cycles are natural models in service systems with arrivals generated by appointments. At first we might think that an arrival process associated with appointments necessarily should be deterministic, but extra randomness leads to a periodic NHPP as a candidate arrival process model. In particular, the deterministic appointment pattern is often disrupted by the random arrivals of customers about their scheduled appointment times, by random no-shows and by random extra non-scheduled arrivals. Thus a periodic NHPP with short cycles is a natural candidate arrival process model for systems with appointments. As in [4], an explicit formula for the peakedness (ratio of variance to the mean of the steady-state distribution) can be useful for approximations for systems with finitely many servers; see (8).

We find that the steady-state distribution with short cycles tends to be quite different from the steady-state distribution with long cycles. To capture the behavior of the periodic $M_t/M/\infty$ model with short and long cycles, we establish double limits in §4 and §5. Previous work on periodic birth and death processes and periodic queues is contained in [1, 2, 5, 6, 7] and references therein.

2. The Steady-State Distribution

Consider the $M_t/M/\infty$ queueing model with the sinusoidal arrival rate function

$$\lambda(t) \equiv \bar{\lambda}\,(1 + \beta \sin(\gamma t))\,. \quad (1)$$

There are three parameters: (i) the average arrival rate $\bar{\lambda}$, (ii) the relative amplitude β and (iii) the time scaling factor γ or, equivalently the cycle length $c = 2\pi/\gamma$. Let the service times be i.i.d. and independent of the arrival process, having mean service time $1/\mu$. Without loss of generality, by choosing the measuring units of time, we assume that $\mu = 1$. If we want to consider $\mu \neq 1$, then we must replace λ and γ by λ/μ and γ/μ in the formulas below. We will be considering the limiting behavior as $\gamma \downarrow 0$ and $\gamma \uparrow \infty$. These are equivalent to limits as $\mu \uparrow \infty$ and $\mu \downarrow 0$, respectively.

By §5 of [3], the number of customers in the system (or the number of busy servers), $Q(t)$, in the $M_t/M/\infty$ queueing model with the sinusoidal arrival rate function in (1) and mean service time 1, starting empty in the distant past, has a Poisson distribution at each time t with mean

$$m(t) \equiv E[Q(t)] = \bar{\lambda}(1 + s(t)), \quad s(t) = \frac{\beta}{1+\gamma^2}\left(\sin(\gamma t) - \gamma \cos(\gamma t)\right). \quad (2)$$

Moreover,

$$s^U \equiv \sup_{t \geq 0} s(t) = \frac{\beta}{\sqrt{1+\gamma^2}} \quad (3)$$

and

$$s(t_0^m) = 0 \quad \text{and} \quad \dot{s}(t_0^m) > 0 \quad \text{for} \quad t_0^m = \frac{\cot^{-1}(1/\gamma)}{\gamma}. \quad (4)$$

The function $s(t)$ increases from 0 at time t_0^m to its maximum value $s^U = \beta/\sqrt{1+\gamma^2}$ at time $t_0^m + \pi/(2\gamma)$. The interval $[t_0^m, t_0^m + \pi/(2\gamma)]$ corresponds to its first quarter cycle.

Let Z be a random variable with the steady-state probability mass function (pmf) of $Q(t)$; its pmf is a mixture of Poisson pmf's. In particular,

$$P(Z = k) = \frac{\gamma}{2\pi} \int_0^{2\pi/\gamma} P(Q(t) = k)\, dt, \quad k \geq 0, \quad (5)$$

The moments of Z are given by the corresponding mixture

$$E[Z^k] = \frac{\gamma}{2\pi} \int_0^{2\pi/\gamma} E[Q(t)^k]\, dt, \quad k \geq 1, \quad (6)$$

so that $E[Z] = \bar{\lambda}$.

Theorem 2.1. (*the variance and higher moments*) *For the $M_t/M/\infty$ model defined above, starting empty in the distant past,*

$$
\begin{aligned}
E[Z^2] &= (\bar{\lambda} + \bar{\lambda}^2) + \frac{\bar{\lambda}^2\beta^2}{2(1+\gamma^2)}, \quad Var(Z) = \bar{\lambda} + \frac{\bar{\lambda}^2\beta^2}{2(1+\gamma^2)}, \\
E[Z^3] &= (\bar{\lambda} + 3\bar{\lambda}^2 + \bar{\lambda}^3) + \frac{(3\bar{\lambda}^2 + 3\bar{\lambda}^3)\beta^2}{2(1+\gamma^2)}, \\
E[Z^4] &= (\bar{\lambda} + 7\bar{\lambda}^2 + 6\bar{\lambda}^3 + \bar{\lambda}^4) + \frac{(7\bar{\lambda}^2 + 18\bar{\lambda}^3 + 6\bar{\lambda}^4)\beta^2}{2(1+\gamma^2)} + \frac{\bar{\lambda}^4\beta^4(3 + 6\gamma^2 + 3\gamma^4)}{8(1+\gamma^2)^4}.
\end{aligned}
$$

Proof. We give a full proof in an appendix, and here do only the variance:

$$
\begin{aligned}
Var(Z) &= E[Z^2] - \bar{\lambda}^2 = \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} (m(t) + m(t)^2)\, dt - \bar{\lambda}^2 \\
&= \bar{\lambda} + \frac{\bar{\lambda}^2\gamma}{2\pi}\int_0^{2\pi/\gamma} s(t)^2\, dt = \bar{\lambda} + \left(\frac{\bar{\lambda}^2\beta^2}{(1+\gamma^2)^2}\right)\frac{1}{2\pi}\int_0^{2\pi} [\sin u - \gamma\cos u]^2\, du.
\end{aligned}
$$

applying the change of variables $u = \gamma t$ in the last step. Then expand the integrand and apply the power reduction formulas, $2\sin^2\theta = 1 - \cos 2\theta$, $2\cos^2\theta = 1 + \cos 2\theta$, and $2\sin\theta\cos\theta = \sin 2\theta$, noting that the integrals of the trigonometric terms vanish. There are corresponding formulas for higher moments, e.g., $8\sin^4\theta = 3 - 4\cos 2\theta + 4\cos 4\theta$. ∎

Remark 2.1. (*an alternative arrival view*) In this paper we focus on the random variable Z in (5) describing the time-average performance. If instead we want to describe the average view of arrivals, then we would instead use the random variable Z_a, with the pmf

$$
P(Z_a = k) = \frac{\gamma}{2\pi\bar{\lambda}}\int_0^{2\pi/\gamma} \lambda(t) P(Q(t) = k)\, dt, \quad k \geq 0; \tag{7}
$$

see Proposition A.1 in the Appendix of [8].

Remark 2.2. (*peakedness*) A successful approach for developing performance approximations in stationary multi-server queues with non-Poisson arrival processes is the concept of peakedness; see [4, 9, 10] and references therein. This applies to many-server queues with or without extra waiting space, and with or without customer abandonment from queue. The peakedness is defined as the ratio of the variance to the mean of the number of busy servers in the associated infinite-server queue. With an M_t arrival process having a periodic arrival rate function, a many-server queue becomes a stationary model when we randomize over the starting place in a cycle. Taking that point of view here, we see that Theorem 2.1 yields the peakedness of the M_t arrival process (with the initial position uniformly distributed over the cycle) relative to the exponential service-time distribution; i.e.,

$$z \equiv z(\bar{\lambda}, \beta, \gamma) \equiv \frac{Var(Z)}{E[Z]} = 1 + \frac{\bar{\lambda}\beta^2}{2(1+\gamma^2)}. \quad (8)$$

From (8), we can easily determine when z has a nondegenerate limit as we let the parameters approach limits; we will be considering some of those here. It is common to express the peakedness as a function of the service rate μ. If we let the mean service time be $1/\mu$, then we replace $\bar{\lambda}$ and γ in (8) by $\bar{\lambda}/\mu$ and γ/μ. We see that the peakedness goes to 1, the same as an ordinary Poisson arrival process, when $\gamma \to \infty$ (short cycles) or as $\mu \to 0$ (long service times). We can apply this peakedness expression in (8) to approximate the performance in a multi-server queue with this periodic stationary arrival process, as in [4, 10]. This can be the basis for stationary-process approximations for periodic queues, as in [11].

Example 2.1. (*numerical examples*) Below we show plots of the steady-state pmf. First, Figure 1 shows the steady-state pmf for $\bar{\lambda} = 10$ (left) and $\bar{\lambda} = 1000$ (right), both for $\beta = 10/35 = 0.286$ and three values of γ: 1/8, 1 and 8. As anticipated, for $\bar{\lambda} = 10$, we see that the steady-state

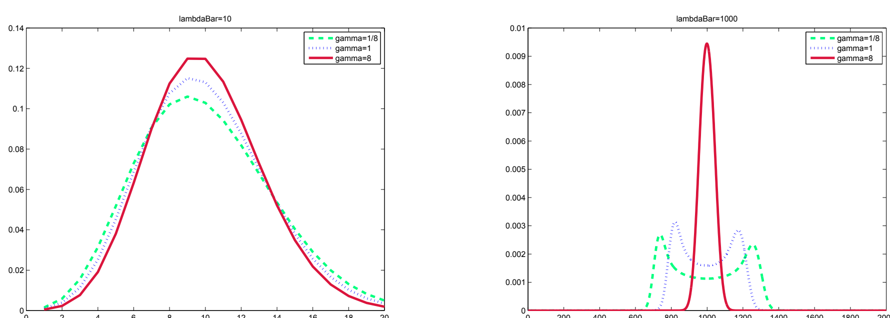

Figure 1: The steady-state pmf in the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) for $\bar{\lambda} = 10$ (left) and $\bar{\lambda} = 1000$ (right), $\beta = 10/35 = 0.286$ and three values of γ: 1/8, 1 and 8.

pmf looks much like the Poisson pmf that holds at each time t, which in turn is approximately a normal probability density function (pdf), using the standard normal approximation for the Poisson distribution. However, being a mixture, the steady-state distribution has extra variability, as can be seen from (8), because $Var(Z) > E[Z]$, whereas $Var(Z) = E[Z]$ for the Poisson distribution. As γ decreases, the cycles get longer, making the steady-state even more variable (as quantified by the variance).

However, we see a radically different picture for higher arrival rates. When $\bar{\lambda} = 1000$, the plots for $\gamma = 1$ and $\gamma = 1/8$ look radically different from the plot for $\gamma = 8$, which is reminiscent of the previous plots for $\bar{\lambda} = 10$. For $\gamma = 1$ and $\gamma = 1/8$, we see that these pmf's $\bar{\lambda} = 1000$ are bimodal, showing that the steady-state distribution places more weight on the extremes than it does on the mean. In order to better explain these results, we next establish heavy-traffic limits.

3. Heavy-Traffic Limits

In order to explain the plots for $\gamma = 1$ and $\gamma = 1/8$ when $\bar{\lambda} = 1000$ in Figure 1, we now establish heavy-traffic limits by letting $\bar{\lambda} \to \infty$. In the appendix we give a heavy-traffic limit for the moments in Theorem 2.1. As a consequence, we obtain the following revealing limits for the skewness and the kurtosis of Z. The limiting kurtosis of -1.5 should be compared with the least possible value of -2 obtained by a Bernoulli random variable attaching probability $1/2$ to each of ± 1. The variance $Var(Z)$ is one half of the variance of that two-point distribution.

Corollary 3.1. (*heavy-traffic limit for the kurtosis*) *As $\lambda \to \infty$, the skewness and kurtosis of Z approach the simple limits*

$$\begin{aligned} \gamma_1(Z) &\equiv \frac{E[(Z - E[Z])^3]}{E[(Z - E[Z])^2]^{3/2}} \to 0, \\ \gamma_2(Z) &\equiv \frac{E[(Z - E[Z])^4]}{E[(Z - E[Z])^2]^2} - 3 \to -1.5. \end{aligned}$$

We now establish a limit for the entire distribution of Z by considering a sequence of $M_t/M/\infty$ models indexed by n, where n is the average arrival rate. In particular, we assume that (1) holds with $\bar{\lambda}_n = n$. We again let the mean service time be $1/\mu = 1$ and assume that the system starts empty in the distant past, so that the system is in periodic steady state at time 0.

By letting $\bar{\lambda}_n = n$, we are using the familiar many-server heavy-traffic scaling, as in many previous studies, e.g., [12, 13, 14] and references therein. We will apply the functional weak law of large numbers (FWLLN), which is a special case of Theorem 3.21 of [14], but for the $M_t/M/\infty$ model there is more history, e.g., [12]. The ordinary LLN version of the fluid limit concludes that

$$\frac{Q_n(t)}{n} \Rightarrow 1 + s(t) \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad n \to \infty \tag{9}$$

for each t. The associated deterministic fluid approximation is

$$Q_n(t) \approx n(1 + s(t)), \quad t \geq 0. \tag{10}$$

Let Z_n be a random variable with the steady-state distribution associated with the n^{th} model, defined as in (5). Let $\bar{Z}_n \equiv Z_n/n$ be the associated scaled steady-state random variable, again using the usual fluid scaling. This application of the fluid limit in (9) is interesting mathematically because it leads to a stochastic limit for the scaled steady-state random variable $\bar{Z}_n$. That can be explained because the variability, as partially characterized by the variance, is an order of magnitude larger in the present setting than in the usual stationary setting. Let $\Rightarrow$ denote convergence in distribution.

Theorem 3.1. (*the fluid limit*) *For the sequence of $M_t/M/\infty$ models indexed by n defined above,*

$$\bar{Z}_n \Rightarrow Z \quad in \quad \mathbb{R} \quad as \quad n \to \infty, \tag{11}$$

with $E[\bar{Z}_n] = E[Z] = 1$ for all n and

$$Var(\bar{Z}_n) \to Var(Z) \equiv \frac{\beta^2}{2(1+\gamma^2)} \quad as \quad n \to \infty, \tag{12}$$

where Z has support on the interval $[1 - s^U, 1 + s^U]$ and non-degenerate cdf

$$F(x) \equiv P\left(Z \leq 1 + s^U x\right) = 1 - F(-x) = \frac{1}{2} + \frac{\gamma}{2\pi}[s^{-1}(xs^U) - t_0^m], \tag{13}$$

for $0 \leq x \leq 1$, s in (2), s^U in (3) and t_0^m in (4).

Proof. The convergence in (11) follows from the LLN stated in (9). In particular,

$$\begin{aligned} P(\bar{Z}_n \le 1+x) &= P(Z_n \le n(1+x)) = \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} \Pi_{n(1+x)}(nm_1(t))\,dt, \quad (14) \\ &\Rightarrow \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} 1_{(-\infty,x]}(s(t))\,dt \equiv P(Z \le 1+x) \quad \text{as} \quad n\to\infty, \end{aligned}$$

where $\Pi_x(m)$ is the cdf of a Poisson distribution with mean m and $1_A(x)$ is the indicator function of the set A, equal to 1 when $x \in A$ and 0 otherwise. We also use the bounded convergence theorem to get associated convergence of the integrals, using the fact that, for each x, the indicator function appearing in the limit is continuous in t for all but finitely many t. We then see that (13) is an equivalent expression for this limiting cdf, using the fact that $m(t)$ is a continuous strictly increasing function over its first quarter cycle, starting where it is 0 and increasing. The variance result in (12) follow from Theorem 2.1. ∎

Theorem 3.1 establishes convergence in distribution $\bar{Z}_n \Rightarrow Z$, which means convergence of the cdf's. To directly see that convergence, Figure 2 shows the cdf's of the scaled random variables $\bar{Z}_n$ for Example 2.1 with four values of n ranging from 10 to 1000. The limiting random variable Z has support on the interval $[0.798, 1.202]$. Thus, we see that the case $n = 1000$ is close to the limiting form.

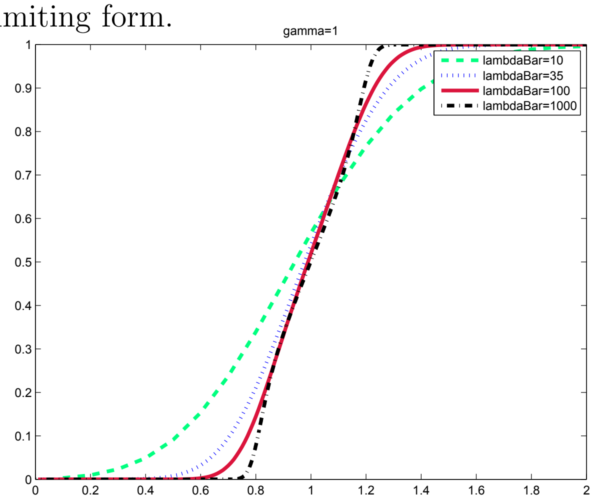

Figure 2: The cdf of the scaled steady-state random variable $\bar{Z}_n$ in the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) for $\beta = 10/35 = 0.286$, $\gamma = 1$ and four values of $n = \bar{\lambda} = 10$, 35, 100 and 1000.

We will now derive the probability density function (pdf) of Z. For that purpose, it is convenient to introduce

$$g(x) \equiv \frac{s(t_0^m + [x\pi/(2\gamma)])}{s^U}, \quad 0 \le x \le 1. \tag{15}$$

From (3) and (4), we see that $g : [0,1] \to [0,1]$ is strictly increasing and continuous with $g(0) = 0$, $g(1) = 1$ and $\dot{g}(1) = 0$, where $\dot{g}$ is the derivative. Hence g has a unique inverse $g^{-1} : [0,1] \to [0,1]$, which also is strictly increasing and continuous, with $g^{-1}(0) = 0$, $g^{-1}(1) = 1$ and $g(g^{-1}(x)) = g^{-1}(g(x)) = x$ for $0 \le x \le 1$.

On the other hand, after letting $h(x) \equiv F(x) - (1/2)$, from (13) and (15), we have

$$\begin{aligned} h(g(x)) &\equiv F(g(x)) - \frac{1}{2} = P\left(Z \le 1 + s^U g(x)\right) - \frac{1}{2} = \frac{\gamma}{2\pi}[s^{-1}(g(x)s^U) - t_0^m], \\ &= \frac{\gamma}{2\pi}[s^{-1}(s(t_0^m + x\pi/\gamma)) - t_0^m] = x \quad \text{for all} \quad x, \quad 0 \le x \le 1, \end{aligned} \tag{16}$$

so that $h = g^{-1}$.

Corollary 3.2. *The limiting random variable Z in (11) has interval of support $[1 - s^U, 1 + s^U]$, cdf*

$$P\left(Z \le 1 + s^U g(x)\right) = P\left(S > 1 - s^U g(x)\right) = x + \frac{1}{2}, \quad 0 \le x \le 1, \tag{17}$$

and pdf

$$f_Z(1 + s^U g(x)) = f_S(1 - s^U g(x)) = \frac{1}{s^U \dot{g}(x)}, \quad 0 \le x \le 1, \tag{18}$$

for g in (15), or

$$f_Z(1 + s^U y) = f_S(1 + s^U y) = \frac{1}{s^U \dot{g}(g^{-1}(y))}, \quad 0 \le y \le 1. \tag{19}$$

The pdf $f_Z(y)$ is bimodal, approaching $+\infty$ at $y = 1 \pm s^U$, and has a unique minimum value at $y = 1$.

Proof. The expression for the cdf in (17) follows directly from (13), (15) and (16), noting that

$$x = h(g(x)) = F(g(x)) - \frac{1}{2} = P\left(S \le 1 + s^U g(x)\right) - \frac{1}{2}.$$

Expressions (18) and (19) follow from (17), the chain rule and the inverse function theorem. From (15) and (2), we see that $g(x)$ is increasing over $[0,1]$ with $g(1) = 1$, while $\dot{g}(x)$ is decreasing with $\dot{g}(1) = 0$. (The maximum of $g(x)$ for $x \ge 0$ occurs at $x = 1$.) ■

The limiting steady-state random variable Z associated with the deterministic fluid approximation for $Q_n(t)$ in (10) is not itself deterministic. The non-degenerate steady-state pdf explains the shape we see for $\gamma \le 1$ in Figure 1 when $n = 1000$.

4. A Double Limit for Short Cycles

From the plot of the steady-state pmf when $\bar{\lambda} = 1000$ on the right in Figure 1, we see that the fluid approximation performs very poorly when $\gamma = 8$. Indeed, the plot is so different from the conclusion of Theorem 3.1 that we are inclined to question the validity of Theorem 3.1. However, Figure 3 below addresses that concern by displaying the steady-state distribution for even larger values of $\bar{\lambda}$. In particular, Figure 3 shows the steady-state pmf when $\bar{\lambda} = 10^4$ (left) and $\bar{\lambda} = 10^6$ (right). These are computed by approximating the Poisson pmf at each time t by the associated normal pdf with the same mean and variance. From these plots, we see that Theorem 3.1

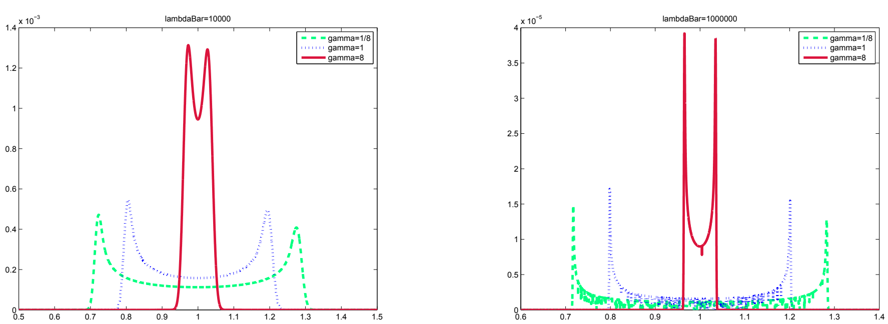

Figure 3: The steady-state pmf in the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) for $\bar{\lambda} = 10^4$ (left) and $\bar{\lambda} = 10^6$ (right), $\beta = 10/35 = 0.286$ and three values of γ: 1/8, 1 and 8.

finally does apply to $\gamma = 8$ for these extremely large values of $\bar{\lambda}$. Nevertheless, Theorem 3.1 does not yield reasonable approximations for typical values of $\bar{\lambda}$. That motivates us to also develop other approximations based limits in which $\gamma_n \to \infty$ as well as $n \to \infty$.

We now consider the new scaled random variable

$$\hat{Z}_n \equiv \frac{Z_n - E[Z_n]}{\sqrt{n}} = \frac{Z_n - n}{\sqrt{n}}, \quad n \geq 1. \tag{20}$$

Let $N(m, \sigma^2)$ denote a random variable with the normal distribution having mean m and variance σ^2.

Theorem 4.1. (*double limit for the steady-state distribution with short cycles*) *For the sequence of $M_t/M/\infty$ models indexed by n with the scaling*

$$\frac{\gamma_n}{\sqrt{n}} \to \infty \quad as \quad n \to \infty, \tag{21}$$

we have the convergence

$$\hat{Z}_n \Rightarrow N(0,1) \quad in \quad \mathbb{R} \quad as \quad n \to \infty, \tag{22}$$

for $\hat{Z}_n$ in (20)*, which is consistent with $E[\hat{Z}_n] = 0$ for all n and*

$$Var(\hat{Z}_n) \to 1 \quad as \quad n \to \infty. \tag{23}$$

Proof. For the stationary processes associated with the associated sequence of stationary $M/M/\infty$ models indexed by n, the limit in (22) is elementary. We will show that the limit is in the same for this sequence of $M_t/M/\infty$ models is the same. It suffices to show that the functional central limit theorem for the arrival process has the same limit. Let $A_n(t)$ be the arrival counting process in system n, We will show that

$$\hat{A}_n(t) \equiv \frac{A_n(t) - nt}{\sqrt{n}} \Rightarrow B(t) \quad \text{in} \quad D \quad \text{as} \quad n \to \infty, \tag{24}$$

where here $\Rightarrow$ denotes convergence in distribution in the functions space D, as in [15], and $\{B(t) : t \geq 0\}$ is standard Brownian motion (BM). From [14], that limit for the arrival process will imply the corresponding limit for $Q_n(t)$.

To establish (24), recall that A_n can be expressed as $A_n(t) = N(\Lambda_n(t))$, $t \geq 0$, where $N(t)$ be a rate-1 Poisson process and $\Lambda_n(t)$ is the cumulative arrival rate function in system n, i.e.,

$$\Lambda_n(t) \equiv \int_0^t \lambda_n(s)\, ds = n\left(t + \frac{\beta[1 - \cos(\gamma_n t)]}{\gamma_n}\right), \quad t \geq 0.$$

.

It is well known that

$$\hat{N}_n(t) \equiv \frac{N(nt) - nt}{\sqrt{n}} \Rightarrow B(t) \quad \text{in} \quad D \quad \text{as} \quad n \to \infty.$$

Moreover, with the scaling in (21) we have $\bar{\Lambda}_n \Rightarrow e$ in D as $n \to \infty$, where $\bar{\Lambda}_n(t) \equiv \Lambda_n(t)/n$ and $e(t) \equiv t$, $t \geq 0$. Since $\{A_n(t) : t \geq 0\}$ is distributed the same as $\{N_n(\bar{\Lambda}_n(t)) : t \geq 0$, we can apply the continuous mapping theorem with the composition map to get the convergence

$$\frac{A_n(t) - \Lambda_n(t)}{\sqrt{n}} = \frac{N_n(\bar{\Lambda}_n(t)) - n\bar{\Lambda}_n(t)}{\sqrt{n}} \Rightarrow B(t) \quad \text{in} \quad D \quad \text{as} \quad n \to \infty. \quad (25)$$

We can thus complete the proof by applying the convergence-together theorem, Theorem 11.4.7 of [15], since

$$\frac{\Lambda_n(t) - nt}{\sqrt{n}} = \frac{\sqrt{n}(1 - \cos(\gamma_n t))}{\gamma_n} \Rightarrow 0 \quad \text{as} \quad n \to \infty \quad (26)$$

uniformly in t over compact intervals by virtue of the scaling in (21). ■

Observe that, with the scaling in (21), the variance satisfies

$$Var(\hat{Z}_n) = 1 + \frac{n\beta^2}{2(1+\gamma_n^2)} \to 1 \quad \text{as} \quad n \to \infty \quad (27)$$

We thus suggest combining the exact variance with Theorem 4.1 to generate the approximation for large n and γ:

$$\hat{Z}_n \approx N(0, \sigma_n^2) \quad \text{for} \quad \sigma_n^2 = 1 + \frac{n\beta^2}{2(1+\gamma_n^2)}. \quad (28)$$

Remark 4.1. (*an associated diffusion process limit*) A minor modification of the proof of Theorem 4.1 yields a periodic diffusion process limit for $Q_n(t)$ if we instead assume that $\beta_n\sqrt{n} \to \beta^* > 0$ with $\gamma_n = \gamma$ as $n \equiv \bar{\lambda}_n \to \infty$. Under that condition, $\hat{A}_n(t) \equiv n^{-1/2}[A_n(t) - nt] \Rightarrow B(\Lambda^*(t))$ in D, where B is again BM and $\Lambda^*(t) = \beta^*(1 - \cos(\gamma t))/\gamma$ and we can apply [14]. The resulting approximation for Z is a mixture of Gaussian distributions, which does not offer much advantage over the exact distribution in (5).

5. A Double Limit for Long Cycles

We can also establish a double limit for long cycles by letting $\gamma_n \downarrow 0$ as $n \to \infty$. We then obtain simple explicit expressions for the cdf and pdf, which help explain the figures. These are closely related to the classical arcsine law in probability theory, which in turn is a special beta distribution; see p. 50 of [16].

Theorem 5.1. (*double limit for the steady-state distribution with long cycles*) *If $\gamma_n \to 0$ as $n \to \infty$ for the sequence of $M_t/M/\infty$ models indexed by n, then*

$$\bar{Z}_n \equiv \frac{Z_n}{n} \Rightarrow Z \quad in \quad \mathbb{R} \quad as \quad n \to \infty, \tag{29}$$

where Z has cdf

$$P(Z \le 1 + \beta x) = \frac{1}{2} + \frac{\arcsin(x)}{\pi}, \quad -1 \le x \le 1, \tag{30}$$

and pdf

$$f_{(Z-1)/\beta}(x) = f_{(Z-1)/\beta}(-x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad 0 \le x \le 1. \tag{31}$$

or, equivalently,

$$f_Z(x) = f_Z(-x) = \frac{1}{\beta\pi\sqrt{1-[(x-1)/\beta]^2}}, \quad 1 \le x \le 1+\beta. \tag{32}$$

with

$$E[Z] = 1 \quad and \quad Var(Z) = \frac{\beta^2}{2}. \tag{33}$$

Proof. We use a minor modification of the proof of Theorem 3.1. For any fixed γ, perform the change of variables $u = \gamma t$ in (14) to obtain

$$\begin{aligned} P(\bar{Z}_n(\gamma) \le 1 + \beta x) &= P(Z_n(\gamma) \le n(1+\beta x)) = \frac{1}{2\pi}\int_0^{2\pi} \Pi_{n(1+\beta x)}(m_1(u/\gamma))\, du, \\ &\Rightarrow \frac{0}{2\pi}\int_0^{2\pi} 1_{(-\infty,\beta x]}(s(u/\gamma))\, du \equiv P(Z(\gamma) \le 1+\beta x) \quad \text{as} \quad n \to \infty, \end{aligned} \tag{34}$$

Next observe that

$$m_1(t/\gamma) = \left(1 + \frac{\beta}{1+\gamma^2}\right)(\sin(t) - \gamma\cos(t)) \to 1 + \beta\sin(t) \quad \text{as} \quad \gamma \downarrow 0$$

uniformly in t. Hence the convergence in the second line of (34) is uniform in γ with the limit being the expression for $\gamma = 0$. We thus have the limit

$$P(Z \equiv Z(0) \le 1 + \beta \sin(\pi(t - 0.5)) = t, \quad -1 \le t \le 1,$$

which directly implies (30) and (31). For the variance, start with a direct expression and then perform the change of variables $y = x^2$ to get

$$E[((Z-1)/\beta)^2] = \frac{2}{\pi}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{1}{\pi}\int_0^1 \frac{y}{\sqrt{y(1-y)}}\, dy = \frac{1}{2}, \qquad (35)$$

because the final integral is the expression for the mean of the arcsine or $Beta(1/2, 1/2)$ distribution on $[0, 1]$, which of course is $1/2$. ■

From Theorem 5.1, we have the approximations

$$P(Z_n \le nx) \approx P(Z \le x) \quad \text{and} \quad P(Z_n \le y) \approx P(Z \le y/n), \qquad (36)$$

so that

$$f_{Z_n}(y) \approx \frac{f_Z(y/n)}{n} \qquad (37)$$

for f_Z in (32).

The cdf's and pdf's are compared in Figure 4.

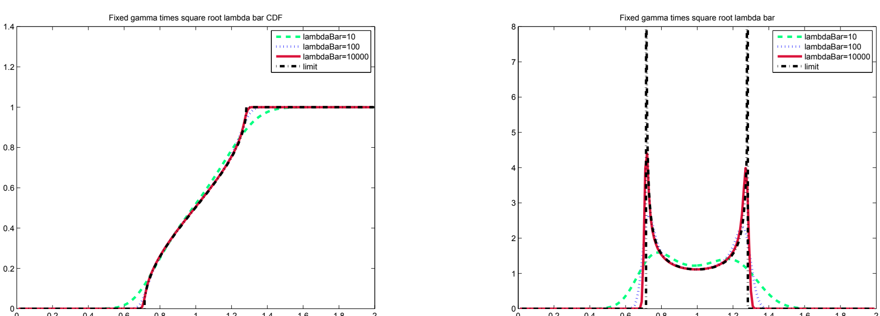

Figure 4: A comparison of the cdf (left) and pdf (right) of the limit Z in Theorem 5.1 with the exact values of the cdf and pdf of the scaled steady-state random variable $\bar{Z}_n$ in the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) for $\beta = 10/35 = 0.286$, $\gamma_n = 1/\sqrt{n}$ and three values of $n = \bar{\lambda}$: 10, 100 and 10,000.

6. Extensions

Extension of the results here for other related models follow by essentially the same arguments. First, extensions to the $M_t/GI/\infty$ model with the sinusoidal arrival rate function in (1) and a non-exponential service-time distribution follow from §4 of [3]. The same reasoning applies, but the formulas are more complicated. Second, extensions also hold for $M_t/GI/\infty$ models with other periodic arrival rate functions, but the expressions become even more complicated. The cdf of S remains easy to compute in the same way if (i) cycles can be defined so that the mean function is first increasing in the first part of the cycle and then decreasing thereafter, and (ii) the mean function is symmetric when it is reflected about its peak, so that the downward part in reverse time starting at the end of the cycle coincides with the upward part in forward time over its half cycle. Without condition (ii), we can treat the upward and downward portions separately and combine the results.

Finally, the heavy-traffic limit supporting the fluid approximation in §3 can also be established for more general models with periodic arrival rate functions, including the $G_t/G/\infty$ model with dependent service times studied in [17] and the non-Markovian $G_t/GI/s_t + GI$ model with time-varying finite capacity and customer abandonment from queue as in [13], exploiting the FWLLN's established in those papers. Just as here, the approximating steady-state distribution will be non-degenerate even though the fluid limits for the number of customers in the system are deterministic.

Acknowledgement. I thank Columbia undergraduate Ethan Kochav for assistance with the numerical examples and NSF for research support (grants CMMI 1066372 and and 1265070).

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7. Appendix

In this appendix we first give a full proof of Theorem 2.1 and then we establish limits for the moments displayed there as $\bar{\lambda} \to \infty$, $\gamma \to \infty$ and $\gamma \to 0$. We conclude with a figure related to Figure 1, showing the same example for $\bar{\lambda} = 35$ and 100.

7.1. Proof of Theorem 2.1

Recall that the first four moments of a Poisson distribution with mean m are $m_1 = m$, $m_2 = m+m^2$, $m_3 = m+3m^2+m^3$ and $m_4 = m+7m^2+6m^3+m^4$. Hence, from (5), we have the following formula for the first four moments of of the steady-state variable Z:

$$\begin{aligned}
E[Z] &= \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} m(t)\,dt = \bar{\lambda}, \\
E[Z^2] &= \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} (m(t) + m(t)^2)\,dt, \\
E[Z^3] &= \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} (m(t) + 3m(t)^2 + m(t)^3)\,dt, \\
E[Z^4] &= \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} (m(t) + 7m(t)^2 + 6m(t)^3 + m(t)^4)\,dt.
\end{aligned} \tag{38}$$

Recall from (2) that $s(t) = m(t) - 1$. To evaluate the integrals in (38), let

$$S_k = \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} s(t)^k\,dt = \frac{1}{2\pi}\int_0^{2\pi} s(u/\gamma)^k\,du, \quad k \geq 1, \tag{39}$$

where $s(u/\gamma) = (\beta/(1+\gamma^2))(\sin u - \gamma\cos u)$, with the last expression following from the change of variables $u = \gamma t$. Now recall the power-reduction formulas that follows from the double angle formula:

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos 2\theta}{2}, & \cos^2\theta &= \frac{1+\cos 2\theta}{2}, \\
\sin^3\theta &= \frac{3\sin\theta - \sin 3\theta}{4}, & \cos^3\theta &= \frac{3\cos\theta + \cos 3\theta}{4}, \\
\sin^4\theta &= \frac{3 - 4\cos 2\theta + 4\cos 4\theta}{8}, & \cos^4\theta &= \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}, \\
\sin\theta\cos\theta &= \frac{\sin 2\theta}{2}, & \sin^2\theta\cos^2\theta &= \frac{1-\cos 4\theta}{8}.
\end{aligned} \tag{40}$$

As a consequence of (40), we have $S_1 = S_3 = 0$ (and $S_{2k+1} = 0$ for all $k \geq 0$), and

$$\begin{aligned} S_2 &= \left(\frac{\beta}{1+\gamma^2}\right)^2 \left(\frac{1+\gamma}{2}\right), \\ S_4 &= \left(\frac{\beta}{1+\gamma^2}\right)^4 \left(\frac{3+6\gamma^2+3\gamma^4}{8}\right). \end{aligned} \tag{41}$$

Hence,

$$\begin{aligned} \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} m(t)^2\,dt &= \bar{\lambda}^2(1+S_2) = \bar{\lambda}^2 + \frac{\bar{\lambda}^2\beta^2}{2(1+\gamma^2)}, \\ \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} m(t)^3\,dt &= \bar{\lambda}^3(1+3S_2) = \bar{\lambda}^3 + \frac{3\bar{\lambda}^3\beta^2}{2(1+\gamma^2)}, \\ \frac{\gamma}{2\pi}\int_0^{2\pi/\gamma} m(t)^4\,dt &= \bar{\lambda}^4(1+6S_2+S_4), \\ &= \bar{\lambda}^4 + \frac{6\bar{\lambda}^4\beta^2}{2(1+\gamma^2)} + \frac{\bar{\lambda}^4\beta^4(3+6\gamma^2+3\gamma^4)}{8(1+\gamma^2)^4} \end{aligned} \tag{42}$$

for S_k in (39) and (41). Finally, we combine (38) and (42) to obtain the desired moments in Theorem 2.1. ■

7.2. Limiting Forms of the Moments

In this section we show how the moments in Theorem 2.1 behave as various parameters approach limits: $\bar{\lambda} \to \infty$, $\gamma \to \infty$ and $\gamma \to 0$.

7.2.1. Heavy-Traffic Limits

We next describe the asymptotic behavior of these moments as $\bar{\lambda} \to \infty$. Of particular interest are the skewness and the kurtosis, which approach 0 and -1.5, respectively.

Corollary 7.1. (*heavy traffic limits*) *If* $\bar{\lambda} \to \infty$, *then*

$$\begin{aligned} \frac{E[Z]}{\bar{\lambda}} &\to 1, \quad \frac{E[Z^2]}{\bar{\lambda}^2} \to 1 + \frac{\beta^2}{2(1+\gamma^2)}, \quad \frac{E[Z^3]}{\bar{\lambda}^3} \to 1 + \frac{3\beta^2}{2(1+\gamma^2)}, \\ \frac{E[Z^4]}{\bar{\lambda}^4} &\to 1 + \frac{6\beta^2}{2(1+\gamma^2)} + \frac{\beta^4(3+6\gamma^2+3\gamma^4)}{8(1+\gamma^2)^4}, \end{aligned} \tag{43}$$

so that the central moments satisfy

$$\begin{aligned}
\frac{Var(Z)}{\bar{\lambda}^2} &\equiv \frac{E[(Z-E[Z])^2]}{\bar{\lambda}^2} \to \frac{\beta^2}{2(1+\gamma^2)}, \\
\frac{E[(Z-E[Z])^3]}{\bar{\lambda}^3} &\to 0, \quad \frac{E[(Z-E[Z])^4]}{\bar{\lambda}^4} \to \frac{3\beta^4}{8(1+\gamma^2)^2},
\end{aligned} \tag{44}$$

and the skewness and kurtosis have the simple limits

$$\begin{aligned}
\gamma_1(Z) &\equiv \frac{E[(Z-E[Z])^3]}{E[(Z-E[Z])^2]^{3/2}} \to 0, \\
\gamma_2(Z) &\equiv \frac{E[(Z-E[Z])^4]}{E[(Z-E[Z])^2]^2} - 3 \to -1.5.
\end{aligned} \tag{45}$$

7.2.2. Short Cycles

We now consider the limits of the moments in (38) as $\gamma \to \infty$ and as $\gamma \to 0$. For the kurtosis, we see that the two iterated limits $\lim_{\bar{\lambda}\to\infty} \lim_{\gamma\to\infty}$ and $\lim_{\gamma\to\infty} \lim_{\bar{\lambda}\to\infty}$ do not agree.

Corollary 7.2. (*short cycles*) *As $\gamma \to \infty$, the moments in* (38) *approach the moments of a random variable Z_∞ having a Poisson distribution with mean $\bar{\lambda}$:*

$$\begin{aligned}
E[Z_\infty] &= \bar{\lambda}, \quad E[Z_\infty^2] = \bar{\lambda} + \bar{\lambda}^2, \quad E[Z_\infty^3] = \bar{\lambda} + 3\bar{\lambda}^2 + \bar{\lambda}^3, \\
E[Z_\infty^4] &= \bar{\lambda} + 7\bar{\lambda}^2 + 6\bar{\lambda}^3 + \bar{\lambda}^4.
\end{aligned}$$

The associated second, third and fourth central moments of Z_∞ are

$$\begin{aligned}
Var(Z_\infty) \equiv E[(Z_\infty - E[Z_\infty])^2] &= \bar{\lambda}, \quad E[(Z_\infty - E[Z_\infty])^3] = \bar{\lambda} \quad \textit{and} \\
E[(Z_\infty - E[Z_\infty])^4] &= \bar{\lambda} + 3\bar{\lambda}^2,
\end{aligned} \tag{46}$$

so that the skewness and kurtosis are

$$\begin{aligned}
\gamma_1(Z_\infty) &\equiv \frac{E[(Z_\infty - E[Z_\infty])^3]}{E[(Z_\infty - E[Z_\infty])^2]^{3/2}} = \frac{1}{\sqrt{\bar{\lambda}}}, \\
\gamma_2(Z_\infty) &\equiv \frac{E[(Z_\infty - E[Z_\infty])^4]}{E[(Z_\infty - E[Z_\infty])^2]^2} - 3 = \frac{1}{\bar{\lambda}},
\end{aligned} \tag{47}$$

both of which converge to 0 *as $\bar{\lambda} \to \infty$.*

7.2.3. Long Cycles

In contrast to Corollary 7.2, we see that the two iterated limits $\lim_{\bar{\lambda}\to\infty}\lim_{\gamma\to 0}$ and $\lim_{\gamma\to 0}\lim_{\bar{\lambda}\to\infty}$ do agree.

Corollary 7.3. (*long cycles*) *As $\gamma \to 0$, the moments in* (38) *approach*

$$\begin{aligned} E[Z_0] &= \bar{\lambda}, \quad E[Z_0^2] = (\bar{\lambda} + \bar{\lambda}^2) + \frac{\bar{\lambda}^2\beta^2}{2}, \\ E[Z_0^3] &= (\bar{\lambda} + 3\bar{\lambda}^2 + \bar{\lambda}^3) + \frac{(3\bar{\lambda}^2 + 3\bar{\lambda}^3)\beta^2}{2}, \\ E[Z_0^4] &= (\bar{\lambda} + 7\bar{\lambda}^2 + 6\bar{\lambda}^3 + \bar{\lambda}^4) + \frac{(7\bar{\lambda}^2 + 18\bar{\lambda}^3 + 6\bar{\lambda}^4)\beta^2}{2} + \frac{3\bar{\lambda}^4\beta^4}{8}. \end{aligned} \tag{48}$$

The associated second, third and fourth central moments of Z_0 are

$$\begin{aligned} Var(Z_0) &\equiv E[(Z_0 - E[Z_0])^2] = \bar{\lambda} + \frac{\bar{\lambda}^2\beta^2}{2}, \\ E[(Z_0 - E[Z_0])^3] &= \bar{\lambda} + \frac{3\bar{\lambda}^2\beta^2}{2}, \\ E[(Z_0 - E[Z_0])^4] &= \bar{\lambda} + 3\bar{\lambda}^2 + \frac{(7\bar{\lambda}^2 + 6\bar{\lambda}^3)\beta^2}{2} + \frac{3\bar{\lambda}^4\beta^4}{8}, \end{aligned} \tag{49}$$

so that the skewness and kurtosis satisfy

$$\begin{aligned} \gamma_1(Z_0) &\equiv \frac{E[(Z_0 - E[Z_0])^3]}{E[(Z_0 - E[Z_0])^2]^{3/2}} = \sqrt{2}\frac{(2\bar{\lambda} + 3\bar{\lambda}^2\beta^2)}{(2\bar{\lambda} + \bar{\lambda}^2\beta^2)^{3/2}} \sim \frac{3\sqrt{2}}{\bar{\lambda}\beta}, \\ \gamma_2(Z_0) &\equiv \frac{E[(Z_0 - E[Z_0])^4]}{E[(Z_0 - E[Z_0])^2]^2} - 3 \to \frac{3}{2} - 3 = -\frac{3}{2}. \end{aligned} \tag{50}$$

as $\bar{\lambda} \to \infty$.

7.3. Additional Plots

We conclude by giving plots of more cases related to Figure 1.

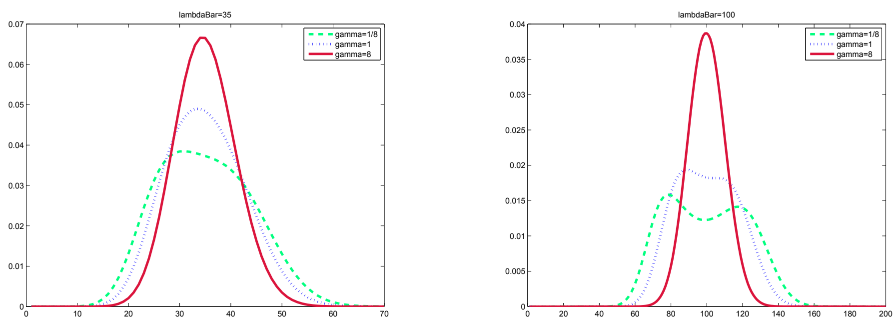

Figure 5: The steady-state pmf in the $M_t/M/\infty$ model with the sinusoidal arrival rate function in (1) for $\bar{\lambda} = 35$ (left) and $\bar{\lambda} = 100$ (right), $\beta = 10/35 = 0.286$ and three values of γ: 1/8, 1 and 8.