MODELING SERVICE-TIME DISTRIBUTIONS WITH NON-EXPONENTIAL TAILS: BETA MIXTURES OF EXPONENTIALS

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ABSTRACT

Motivated by interest in probability density functions (pdf's) with nonexponential tails in queueing and related areas, we introduce and investigate two classes of beta mixtures of exponential pdf's. These classes include distributions introduced by Boxma and Cohen (1997) and Gaver and Jacobs (1998) to study queues with long-tail service-time distributions. When the standard beta pdf is used as the mixing pdf, we obtain pdf's with an exponentially damped power tail, i.e., $f(t) \sim \alpha t^{-q} e^{-\eta t}$ as $t \to \infty$. This pdf decays exponentially, but analysis is complicated by the power term. When the beta pdf of the second kind is used as the mixing pdf, we obtain pdf's with a power tail, i.e., $f(t) \sim \alpha t^{-q}$ as $t \to \infty$. We obtain explicit representations for the cumulative distributions functions, Laplace transforms, moments and asymptotics by exploiting connections to the Tricomi function. Properties of the power-tail class can be deduced directly from properties of the other class, because the power-tail pdf's are undamped versions of the other pdf's. The power-tail class can also be represented as gamma mixtures of Pareto pdf's. Both classes of pdf's have simple explicit Laguerre-series expansions.

1. Introduction

In this paper we introduce and investigate convenient classes of probability density functions (pdf's) on the nonnegative real line. These pdf's either have *power tails*, i.e., for which $f(t) \sim \alpha t^{-q}$ as $t \to \infty$ for positive parameters α and q (where $f(t) \sim g(t)$ as $t \to \infty$ means that $f(t)/g(t) \to 1$ as $t \to \infty$), or *exponentially damped power tails*, i.e., for which $f(t) \sim \alpha t^{-q} e^{-\eta t}$ as $t \to \infty$ for positive parameters α, q and η. By "convenient" we mean that the pdf and its associated cumulative distribution function (cdf), Laplace transform, moments and tail asymptotics should all be available explicitly or at least be readily computable, so that the pdf's can easily be used as components of stochastic models, e.g., as service-time pdf's in queueing models. We especially want computable Laplace transforms so that we can apply numerical transform inversion, e.g., to compute a waiting-time cdf when the service-time pdf is from one of these classes. The classes should also be sufficiently large that they cover an interesting range of cases.

As discussed in Feldmann and Whitt [18] and references therein, there currently is great interest in power-tail pdf's in the study of communication network performance because measurements indicate that many distributions have this property (e.g., file sizes), and that this property is largely responsible for observed traffic complexity (e.g., long-range dependence and self-similarity). The power-tail pdf's here are alternatives to the Pareto mixture of exponential (PME) pdf's introduced in Abate, Choudhury and Whitt [9].

This paper can be regarded as a continuation of the operational calculus for probability distributions via Laplace transforms in Abate and Whitt [4]. In particular, here we apply the stationary-excess operator $\mathcal{E}$, the stationary-lifetime operator $\mathcal{L}$, the unimodal operator $\mathcal{U}$ and the damping operator $\mathcal{D}$; see (1.12), (5.7), (5.12) and (1.21) below.

The assumed tail behavior makes these classes of pdf's special cases of class-III (long-tail) and class-II (semi-exponential tail) distributions in the terminology of Abate, Choudhury and Whitt [9] and Abate and Whitt [5]. Let $\hat{f}(s)$ be the Laplace transform of a pdf $f(t)$ and let $-s^*$ be the rightmost singularity of $\hat{f}(s)$. Then $f(t)$ and $\hat{f}(s)$ are classified as type I (exponential tail) if $s^* > 0$ and $\hat{g}(-s^*) = \infty$, type II (semi-exponential tail) if $s^* > 0$ and

$1 < g(-s^*) < \infty$, and type III (long or heavy tail) if $s^* = 0$. Among the class-I pdf's are pdf's with rational Laplace transforms. All phase-type distributions are class I. Many explicit results in queueing theory are available when the underlying distributions are class I. Our goal is to develop tractable classes of pdf's that are *not* class I.

The principal pdf's considered here are beta mixtures of exponential pdf's, but we consider both the *standard beta pdf*

$$b(p, q; y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1-y)^{q-1}, \quad 0 \le y \le 1, \tag{1.1}$$

and the *beta pdf of the second kind*

$$b_2(p, q; y) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} y^{p-1}(1+y)^{-(p+q)}, \quad y \ge 0, \tag{1.2}$$

for $p > 0$ and $q > 0$, where $\Gamma(x)$ is the gamma function; e.g., see p. 50 of Feller [19] and p. 51 of Johnson and Kotz [26]. The beta pdf of the second kind is obtained by considering the random variable $X/(1-X)$, where X has a standard beta pdf.

The (standard) *beta mixture of exponentials* (BME) pdf is defined as

$$v(p, q; t) \equiv \int_0^1 y^{-1} e^{-t/y} b(p, q; y) dy, \quad t \ge 0, \tag{1.3}$$

while the *second beta mixture of exponentials* (B_2ME) pdf is defined as

$$v_2(p, q; t) \equiv \int_0^\infty y^{-1} e^{-t/y} b_2(p, q; y) dy, \quad t \ge 0 \; . \tag{1.4}$$

The pdf's in (1.3) and (1.4) clearly are computable via numerical integration, but they are not especially convenient. We will show that some special cases have explicit formulas and others can be computed more rapidly by recursions.

We have used the beta pdf's in (1.3) and (1.4) to mix the *means* or times of the exponential pdf's. We could also mix the exponential *rates*. Since mixing rates is the form of the spectral representation, we call the associated mixing pdf the spectral density. (By making the change of variables $x = 1/y$, we see that a mixing density $w(y)$ is related to the associated spectral density $\phi(x)$ by $\phi(x) = x^{-2} w(x^{-1})$ and $w(y) = y^{-2}\phi(y^{-1})$.) The alternative spectral representations of (1.3) and (1.4) are

$$v(p, q; t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_1^\infty (xe^{-tx}) b_2(q, p; x-1) dx, \quad t \ge 0, \tag{1.5}$$

and

$$v_2(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\infty (xe^{-tx}) b_2(q,p;x)dx, \quad t \ge 0 \ . \tag{1.6}$$

Note that the spectral density for $v_2(p,q;t)$ is $b_2(q,p;y)$, the same as the mixing density but with the parameters p and q switched.

These two forms of mixtures can also be represented as products and quotients of independent random variables. Let X, $Y(p,q)$ and $Y_2(p,q)$ be independent random variables with pdf's e^{-y}, $b(p,q;y)$ and $b_2(p,q;y)$, respectively. Then the BME pdf $v(p,q;t)$ can be represented via the random variable $Z = Y(p,q)X$ or $Z = X/(1+Y_2(q,p))$, while the B_2ME pdf $v_2(p,q;t)$ can be represented via the random variable $Z_2 = Y_2(p,q)X$ or $Z_2 = X/Y_2(q,p)$. The random product representations are convenient for calculating moments, e.g., if $Z = YX$, then $EZ^n = EY^n EX^n$ for all n.

We could also consider a third beta mixture of exponentials (B_2ME), defined by

$$v_3(p,q;t) = \int_0^1 xe^{-tx} b(p,q;x)dx = \int_1^\infty y^{-1} e^{-t/y} b_2(q,p;y-1)dy \tag{1.7}$$

which is obtained by switching the BME mixing and spectral pdf's. The B_3ME pdf $v_3(p,q;t)$ can be represented by the random variable $Z_3 = X/Y(p,q)$ or $Z_3 = X(1+Y_2(q,p))$. The mixing representation shows that the B_2ME and B_3ME pdf's are similar, especially with regard to their tail behavior. We do not discuss the B_3ME pdf further here, but we do in Abate and Whitt [7].

Let $G^c(t) \equiv 1 - G(t)$ be the complementary cdf (ccdf) associated with a cdf G. By integrating in (1.3) and (1.4), we see that the associated ccdf's are simply related to the pdf's with different parameters, in particular,

$$V^c(p,q;t) = \frac{p}{p+q} v(p+1,q;t), \quad t \ge 0, \tag{1.8}$$

and

$$V_2^c(p,q;t) = \frac{p}{q-1} v_2(p+1,q-1,t), \quad t \ge 0 \ . \tag{1.9}$$

It turns out that BME and B_2ME have very nice structure. It may not be initially evident from (1.3)–(1.9), but these pdf's and ccdf's have the indicated tail behavior. The following are special cases of more general asymptotic expansions given in Section 3.

Theorem 1.1. *For all $p > 0$ and $q > 0$,*

$$V^c(p,q;t) \sim \frac{\Gamma(p+q)e^{-t}}{\Gamma(p)t^q} \quad as \quad t \to \infty \tag{1.10}$$

and

$$V_2^c(p,q;t) \sim \frac{\Gamma(p+q)}{\Gamma(p)t^q} \quad as \quad t \to \infty \ . \tag{1.11}$$

The BME and B_2ME classes have the two parameters $p > 0$ and $q > 0$, but we can also add an additional scale parameter, say a; e.g., if X has pdf $v(p,q;t)$, then X/a has pdf $av(p,q;at)$, $t \geq 0$. Hence BME and B_2ME are both three-parameter families, where the parameters run over the positive reals.

Being mixtures of exponential pdf's, the BME and B_2ME pdf's are all *completely monotone* (CM); e.g., see p. 439 of Feller [19]. Consequently, they can be approximated arbitrarily closely by hyperexponential (H_k) pdf's, which are *finite* mixtures of exponential pdf's; see Feldmann and Whitt [18]. The BME and B_2ME pdf's often might be preferred to H_k pdf's, however, because they have fewer parameters. On the other hand, since the standard beta cdf approaches the cdf of a point mass at m as $p \to \infty$, $q \to \infty$ with $p/(p+q) \to m$, we have the following result. (This proof and other omitted proofs are given in the final section.)

Theorem 1.2. *Finite mixtures of BMEs constitute a dense (in the sense of convergence in distribution) subset of the family of all CM pdf's.*

Since mixtures of beta pdf's can be quite different from single beta pdf's, it may actually be useful to consider mixtures of BMEs and B_2MEs. A mixture of two BMEs has seven parameters, the parameter triple (p,q,a) for each BME and the mixing probability. To get additional pdf's, we can also consider convolutions and mixtures of such convolutions. Mixtures of (m-fold) convolutions of BME pdf's are dense in the family of *all* probability distributions. However, BME and B_2ME pdf's are probably of most interest used directly or with only a small number of mixture and convolution operations.

As discussed in Abate and Whitt [4] and previous papers, a fundamental operation on probability distributions is the stationary-excess operation. The stationary-excess ccdf associated with a cdf G with mean $m_1(G)$ is

$$G_e^c(t) \equiv \mathcal{E}(G^c)(t) \equiv \frac{1}{m_1(G)} \int_t^\infty G^c(u)du, \quad t \geq 0, \tag{1.12}$$

while the associated stationary-excess pdf is

$$g_e(t) \equiv \mathcal{E}(g)(t) \equiv \frac{G^c(t)}{m_1(G)}, \quad t \geq 0. \tag{1.13}$$

It is significant that $\mathcal{E}$ maps the classes BME and B_2ME into themselves, in particular, the next result follows easily from (1.8) and (1.9). We use the fact that the mean of a BME (B_2ME) with parameters p and q is $p/(q+q)$ $(p/q-1)$; see Section 2.

Theorem 1.3. *For all $p > 0$ and $q > 0$,*

$$v_e(p, q; t) = v(p+1, q; t), \quad t \geq 0 \ , \tag{1.14}$$

and, for all $p > 0$ and $q > 1$,

$$v_{2e}(p, q; t) = v_2(p+1, q-1; t), \quad t \geq 0 \ . \tag{1.15}$$

Theorem 1.3 is important for constructing new members of the BME and B_2ME classes from given ones. For any function f, let $\hat{f}$ be its Laplace transform. Since $\hat{g}_e(s) = (1 - \hat{g}(s))/m_1(G)s$ for any pdf g, Theorem 1.3 implies the following.

Corollary. *For all $p > 0$ and $q > 0$,*

$$\hat{v}(p+1, q, s) = \frac{p+q}{sp}(1 - \hat{v}(p, q; s)) \tag{1.16}$$

and, for all $p > 0$ and $q > 1$,

$$\hat{v}_2(p+1, q-1, s) = \frac{q-1}{ps}(1 - \hat{v}_2(p, q; s)) \ . \tag{1.17}$$

It turns out that pdf's associated with the time-dependent behavior of reflected Brownian motion (RBM) previously studied in Abate and Whitt [1, 2, 3, 4] and elsewhere are BME pdf's (with the scale parameter chosen so that the beta pdf is on $[0, 2]$; e.g., see Theorem 4.2 of Abate and Whitt [3]. The previously exploited stationary-excess relations among these pdf's is largely explained by Theorem 1.3.

It is straightforward to calculate the Laplace transform of $v(p, q; t)$,

$$\hat{v}(p, q; s) \equiv \int_0^\infty e^{-st} v(p, q; t) dt \ , \tag{1.18}$$

obtaining an integral representation, which can be identified with an instance of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$; see (5.80) on p. 207 of Graham, Knuth and Patashnik [22] or 15.3.1 of Abramowitz and Stegun [12]; hereafter referred to as AS.

Theorem 1.4. *For all $p > 0$ and $q > 0$,*

$$\hat{v}(p,q;s) = \int_0^1 (1+sy)^{-1} b(p,q;y)dy = {}_2F_1(1,p;p+q;-s) \ , \tag{1.19}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function.

It turns out that Theorem 1.4 is very useful for computing BME Laplace transforms via continued fractions; see [8].

We can then apply Theorem 1.4 to obtain the following symmetry result for BME pdf's. Exploiting the Gauss hypergeometric function, we can also obtain this next result by an application of the Pfaff reflection law; see (5.101) on p. 217 of Graham, Knuth and Patashnik [22] or 15.3.4 of AS.

Theorem 1.5. *For all $p > 0$ and $q > 0$,*

$$\hat{v}(q,p;s) = \frac{1}{1+s}\hat{v}\left(p,q;-s/(1+s)\right) \ . \tag{1.20}$$

As noted in Section 5 of Abate, Choudhury and Whitt [9], there is a one-to-one correspondence between class II and class III pdf's; one type of pdf can be converted into the other by damping or undamping, i.e., by multiplying by an appropriate exponential and rescaling. It turns out the BME and B_2ME classes are related by such transformations. From (1.19) it is evident that the rightmost singularity of $\hat{v}(p,q;s)$ is $-s^* = -1$. The value of $\hat{v}(p,q;-s^*)$ can be obtained from the integral representation (1.19). It yields $\hat{v}(p,q;-1) = \infty$ for $q \le 1$ and $\hat{v}(p,q;-1) = 1 + p(q-1)^{-1}$ for $q > 1$, so that the BME pdf is class I for $q \le 1$ and class II for $q > 1$. Hence, BME pdf's can be mapped into B_2ME pdf's for all $q > 1$. Instead, we now map BME ccdf's into B_2ME ccdf's, which applies for all $q > 0$.

Theorem 1.6. *For all $p > 0$ and $q > 0$, the* B_2ME *ccdf and pdf can be represented as*

$$V_2^c(p,q;t) = e^t V^c(p,q;t), \quad t \ge 0 \ , \tag{1.21}$$

and

$$v_2(p,q;t) = e^t[v(p,q;t) - V^c(p,q;t)] = \frac{e^t v(p,q+1;t)}{\int_0^\infty e^u v(p,q+1;u)du}, \quad t \ge 0 \ ,$$
$$= \frac{q}{p+q} e^t v(p,q+1;t), \quad t \ge 0 \ . \tag{1.22}$$

Theorem 1.6 is useful because it enables us to obtain many properties of the B_2ME class directly from properties of the BME class. As indicated above, we could have damped the pdf's instead of the ccdf's as in the second formula in (1.22), but then the normalizing integral is finite only for $q > 1$. That alternative approach just shifts q by 1.

We obtain many structural results by making connections to appropriate special functions. The key special function is the *Tricomi function* $U(a,b,t)$, i.e., the second of the confluent hypergeometric functions; see Chapter 48 of Spanier and Oldham [32] or Chapter 13 of AS. In particular, the representation here follows form 13.2.6 of AS and (1.5) here after making the change of variables $y = x - 1$ in (1.5).

Theorem 1.7. *For all* $p > 0$ *and* $q > 0$*,*

$$v(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)} e^{-t} U(q, 2-p, t), \quad t \ge 0 \ , \tag{1.23}$$

where $U(a,b,t)$ *is the Tricomi function.*

From the perspective of queueing theory, this link to special functions in Theorems 1.4 and 1.7 is reminiscent of Srivastava and Kashyap [33], but BME and B_2ME pdf's are not discussed there. The function ${}_2F_1$ was also used in a different way in Example 4 of Duffield and Whitt [16].

We now apply Theorem 1.7 to obtain an interesting alternative characterization of B_2ME ccdf's as gamma mixtures of Pareto distributions. (This is not a scale mixture.) Boxma and Cohen [15] introduce the subclass of B_2ME distributions with $p = 2 - q$ in this form to study queues with long-tail service-time distributions. (They also include an atom at the origin.)

Theorem 1.8. *The* B_2*ME ccdf is a gamma mixture of Pareto ccdf's, i.e.,*

$$V_2^c(p,q;t) = \int_0^\infty \left(\frac{x}{x+t}\right)^q \frac{x^{p-1}e^{-x}}{\Gamma(p)} dx, \quad t \ge 0 \ . \tag{1.24}$$

Here is how the rest of this paper is organized. In Section 2 we give formulas for all BME and B_2ME moments. We then use the BME moments to obtain series representations for the Laplace transform. We then show that the BME and B_2ME pdf's and ccdf's admit explicit Laguerre-series expansions. These series representations can serve as an effective means of computation for any p and q, using the algorithm in Abate, Choudhury and Whitt [10].

In Section 3 we give asymptotic expansions as $t \to \infty$ and as $t \to 0$ for the pdf's and cdf's, and asymptotic expansions for the moments as $n \to \infty$. In Section 4 we observe that gamma pdf's with shape parameter $p < 1$ are BME pdf's, in particular, $v(p, 1-p; t)$. In Section 5 we given recurrence relations that facilitate determining new BME and B_2ME pdf's given established ones. In Section 5 we also establish connections to pdf's introduced by Gaver and Jacobs [20] and Boxma and Cohen [15] to study the M/G/1 queue with a long-tail service-time pdf. In particular, we analytically invert a transform in Gaver and Jacobs [20], solving a problem they pose.

In Section 6 we discuss concrete examples. In Section 7 we show that all gamma mixtures of exponential pdf's can be represented as limits of BME pdf's and thus inherit BME properties. We discuss other related mixtures in Section 8, e.g., showing that beta mixtures of betas are again betas in certain situations, so that beta mixtures of BMEs are again BMEs. We give previously omitted proofs in Section 9. Finally, we make a few concluding remarks in Section 10.

2. Moments and Series Representations

Let $m_n(p,q)$ be the n^{th} moment of the BME pdf $v(p,q;t)$. Since the n^{th} moment of a mixture is just the mixture of the component n^{th} moments, $m_n(p,q)$ is just the n^{th} moment of the beta pdf $b(p,q;y)$ multiplied by $n!$, i.e.,

$$m_n(p,q) = \frac{(p)_n}{(p+q)_n} n! \tag{2.1}$$

where $(x)_n$ is the Pochhammer symbol; i.e., $(x)_0 = 1$ and $(x)_n = x(x+1)\ldots(x+n-1) = \Gamma(x+n)/\Gamma(x)$; see 6.1.22 of AS. Hence, the first two moments are

$$m_1(p,q) = \frac{p}{p+q} \quad \text{and} \quad m_2(p,q) = \frac{2p(p+1)}{(p+q)(p+q+1)}\ , \tag{2.2}$$

and the squared coefficient of variation (SCV, variance divided by the square of the mean) is

$$c^2(p,q) \equiv \frac{m_2(p,q)}{m_1(p,q)^2} - 1 = 1 + \frac{2q}{p(p+q+1)} \ . \tag{2.3}$$

Formula (2.3) is consistent with the fact that any mixture of exponentials must have $c^2 \geq 1$.

The normalized third moment is

$$\frac{m_3(p,q)m_1(p,q)}{m_2(p,q)^2} = \frac{1+q/(p+1)}{1+q/(p+2)} \ , \tag{2.4}$$

which is increasing in q for all p.

We can deduce from (1.10) that the tail of the BME distribution for fixed mean, gets heavier as q increases. We now establish related properties of the moments.

Theorem 2.1. *For all $p > 0$ and $q > 0$,*

$$\frac{m_{n+1}(p,q)}{m_1(p,q)m_n(p,q)} = \left(\frac{p+q}{p}\right)\left(\frac{p+n}{p+q+n}\right) \ , \tag{2.5}$$

which is increasing in q and n, but decreasing in p.

Corollary. *For fixed mean and variance, $m_n(p,q)$ is increasing in q and decreasing in p for all $n \geq 3$.*

Formula (2.1) directly gives a series representation of the Laplace transform, namely,

$$\hat{v}(p,q;s) \equiv \int_0^\infty e^{-st}v(p,q;t)dt = \sum_{n=0}^{\infty}(-s)^n \frac{(p)_n}{(p+q)_n} \ . \tag{2.6}$$

By the ratio formula, the radius of convergence of the series in (2.6) is

$$s^* = \lim_{n\to\infty} \frac{(p)_{n+1}(p+q)_n}{(p)_n(p+q)_{n+1}} = \lim_{n\to\infty} \frac{(p+n)(p+q+n-1)}{(p+n-1)(p+q+n)} = 1 \ . \tag{2.7}$$

This is consistent with remarks following Theorem 1.5.

Formula (2.6) looks promising for computing when $|s| < 1$. Otherwise, (2.6) might be not so convenient. Hence, we can consider an alternative series expansion based on the symmetric representation in Theorem 1.5. It provides a nice connection to Laguerre functions, because $s^n/(1+s)^{n+1}$ is the Laplace transform of the *Laguerre function*

$$l_n(t) = e^{-t} L_n(t), \quad t \geq 0 \ , \tag{2.8}$$

where

$$L_n(t) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-t)^k}{k!} \tag{2.9}$$

is the *Laguerre polynomial*; e.g., perform a change of variable upon 29.3.34 of AS to account for the e^{-t} factor, as in 29.2.12 of AS. (Note that we use the prefactor e^{-t} in (2.8) instead of $e^{-t/2}$, as is often done in order to have $\{l_n(t), n \geq 0\}$ be an orthonormal basis.) Hence, we can invert term by term to obtain an explicit Laguerre-series representation for $v(p, q; t)$. We can apply the Corollary 1 to Theorem 1.3 and Theorem 1.5 to obtain corresponding results for the BME and B_2ME ccdf's.

Theorem 2.2. *For each $p > 0$ and $q > 0$,*

$$\hat{v}(p, q; s) = \sum_{n=0}^{\infty} \frac{s^n}{(1+s)^{n+1}} \frac{(q)_n}{(p+q)_n} \ , \tag{2.10}$$

so that

$$v(p, q; t) = \sum_{n=0}^{\infty} l_n(t) \frac{(q)_n}{(p+q)_n} \ , \tag{2.11}$$

$$V^c(p, q; t) = \sum_{n=0}^{\infty} l_n(t) \frac{p}{p+q} \frac{(q)_n}{(p+q+1)_n} \ , \tag{2.12}$$

$$V_2^c(p, q; t) = \sum_{n=0}^{\infty} L_n(t) \frac{p}{(p+q)} \frac{(q)_n}{(p+q+1)_n} \ , \tag{2.13}$$

where $l_n(t)$ is the n^{th} Laguerre function in (2.8) and $L_n(t)$ is the Laguerre polynomial in (2.9).

Combining Theorems 1.7 and 2.2, we obtain a Laguerre-series representation for the Tricomi function, namely,

$$\frac{\Gamma(p+q)}{\Gamma(p)} U(q, 2-p, t) = \sum_{n=0}^{\infty} \frac{(q)_n}{(p+q)_n} L_n(t) \ . \tag{2.14}$$

It would seem that formula (2.14) should be well known, but it evidently is not. After much search, we found a source with formulas close to (2.14), from which (2.14) can be derived, in particular, formulas (48.3.3) and (48.3.16) in Hansen [23]. Even though extensive references are given to sources in Hansen [23], none is given for these two formulas.

We can also approach Theorems 1.5 and 2.2 in another way. Recognizing that the series in (2.6) does not always converge rapidly, we might make a transformation to obtain a more rapidly convergent series. If we use the Euler $(E, 1)$ transformation for this purpose, e.g., see p. 7 of Hardy [24], we obtain the same result. In fact, the Euler transformation in Theorem 2.3 provides a general way to construct Laguerre-series representations for pdf's and their Laplace transforms. We investigate this general approach in Abate and Whitt [7].

Theorem 2.3. *The Euler transformation of the series in* (*2.6*) *yields*

$$\hat{v}(p, q; s) = \frac{1}{1+s} \sum_{n=0}^{\infty} \left(\frac{s}{1+s} \right)^n \frac{(q)_n}{(p+q)_n} = \frac{1}{1+s} \hat{v} \left(q, p; \frac{-s}{1+s} \right) . \qquad (2.15)$$

Note that, for any s with $\text{Re}(s) > 0$, $|s/(1+s)| < 1$, so that the series in (2.15) converges geometrically fast. See [7] and [8] for further discussion about how to compute the Laplace transforms.

Theorem 2.2 provides an effective way to compute the pdf's and ccdf's for any p and q. As shown by Abate, Choudhury and Whitt [10], these Laguerre series can be difficult to compute directly, but there are effective ways to enhance the computation. Interestingly, Examples 2.1, 2.2 and 2.5 there are for BME pdf's (see Section 4 here), so we already have done considerable numerics for this class.

We now turn to the B_2ME moments. Let $\mu_n(p, q)$ be the n^{th} moment of $v_2(p, q; t)$. From (1.2),

$$\mu_n(p, q) = \frac{\Gamma(p+n)\Gamma(q-n)n!}{\Gamma(p)\Gamma(q)}, \quad \text{if} \quad q > n \ , \qquad (2.16)$$

with $\mu_n(p, q) = \infty$ if $n \leq q$. Hence, the first two moments, when finite, are

$$\mu_1(p, q) = \frac{p}{q-1} \quad \text{and} \quad \mu_2(p, q) = \frac{2p(p+1)}{(q-1)(q-2)} \ . \qquad (2.17)$$

The associated SCV is

$$c_2^2(p, q) \equiv \frac{\mu_2(p, q)}{\mu_1(p, q)^2} - 1 = 2\frac{(p+1)(q-1)}{p(q-2)} - 1 > 1 \ . \qquad (2.18)$$

It follows immediately from (1.21) that $V(p,q;t)$ is stochastically less than or equal to $V_2(p,q;t)$; i.e., $V^c(p,q;t) \leq V_2^c(p,q;t)$ for all t. Consequently, $m_n(p,q) \leq \mu_n(p,q)$ for all p,q and n. Moreover, from (2.3) and (2.16), we see that $c_2^2(p,q) > c^2(p,q)$ for all p,q (with $q > 2$ so that $\mu_2(p,q) < \infty$). In particular,

$$\frac{c_2^2(p,q)+1}{c^2(p,q)+1} = \frac{(q-1)(p+q+1)}{(q-2)(p+q)} > 1 \ . \tag{2.19}$$

Since the moments $\mu_n(p,q)$ in (2.16) are not finite for all n, we cannot obtain a power-series representation of the Laplace transform $\hat{v}_2(p,q;s)$ from (2.20). However, we can undamp the transform of $V^c(p,q;t)$ for this purpose. In particular, it follows from (1.8) that

$$\hat{V}^c(p,q;s) = \frac{p}{p+q}\hat{v}(p+1,q;s) \ . \tag{2.20}$$

Then it follows from Theorems 1.4–1.6 that

$$\begin{aligned} \hat{V}_2^c(p,q;s) &= \hat{V}^c(p,q;s-1) = \frac{p}{p+q}\hat{v}(p+1,q;s-1) \\ &= \left(\frac{p}{p+q}\right) {}_2F_1(1,p+1;p+q+1;1-s) \\ &= \frac{1}{s}\hat{v}\left(q,p+1,\frac{-(s-1)}{s}\right) \ . \end{aligned} \tag{2.21}$$

and

$$\hat{v}_2(p,q;s) = 1 - s\hat{V}_2^c(p,q;s) = 1 - \hat{v}\left(q,p+1,\frac{-(s-1)}{s}\right) \ . \tag{2.22}$$

3. Asymptotics

To obtain an asymptotic expansion for the pdf's as $t \to \infty$, which implies Theorem 1.1, we can simply apply 13.5.2 of AS with Theorem 1.7. (We can also obtain the asymptotic expansion directly by applying Watson's lemma to (1.5).)

Theorem 3.1. *For all $p > 0$ and $q > 0$,*

$$v(p,q;t) \sim \frac{\Gamma(p+q)}{\Gamma(p)}\frac{e^{-t}}{t^q}\sum_{n=0}^{\infty}(-1)^n\frac{(q)_n(p+q-1)_n}{n!t^n} \quad as \quad t \to \infty \ . \tag{3.1}$$

We can apply (1.8) and (1.21) to obtain corresponding asymptotic expansions for $V^c(p,q;t)$ and $V_2^c(p,q;t)$.

Corollary 1. *For all $p>0$ and $q>0$,*

$$V^c(p,q;t) \sim \frac{\Gamma(p+q)}{\Gamma(p)} \frac{e^{-t}}{t^q} \sum_{n=0}^{\infty} \frac{(-1)^n (q)_n (p+q)_n}{n! t^n} \quad as \quad t \to \infty \tag{3.2}$$

and

$$V_2^c(p,q;t) \sim \frac{\Gamma(p+q)}{\Gamma(p) t^q} \sum_{n=0}^{\infty} \frac{(-1)^n (q)_n (p+q)_n}{n! t^n} \quad as \quad t \to \infty\ . \tag{3.3}$$

We can apply (3.2) and Theorem 5.3 of Abate, Choudhury, Lucantoni and Whitt [11] to obtain an asymptotic expansion for the moments. Alternatively, since the moments are available explicitly in (2.1), we can also apply 6.1.47 of AS for this purpose.

Corollary. *As $n \to \infty$,*

$$\frac{m_n(p,q)}{n!} \sim \frac{\Gamma(p+q)}{\Gamma(p) n^q} \left[1 - \frac{q(p+q+1)}{n} + \frac{q(q+1)(p+q-1)(p+q)}{2n^2} \right] . \tag{3.4}$$

The explicit formulas make it possible to fit BME and B_2ME distributions to data or other distributions quite directly. For example, we might fit a B_2ME ccdf with parameter triple (p,q,a), where a is the scale parameter, to a ccdf $G^c(t)$ with tail asymptotics $G^c(t) \sim At^{-\beta}$ as $t \to \infty$ for $\beta > 1$ and mean $m \equiv m_1(G)$ by setting $q = \beta$, and $p = m(q-1)a$, because of (2.17). By the Corollary to Theorem 3.1, $A = \Gamma(p+q)/\Gamma(p)a^q$. Hence, we must solve the equation

$$A = \frac{\Gamma(m(q-1)a+q)}{\Gamma(m(q-1)a)a^q}\ . \tag{3.5}$$

Since the right side of (3.5) is decreasing in a, going from ∞ to 0, there is a unique solution a^* to (3.5).

We can also apply 13.5.8–13.5.12 of AS to describe the asymptotic behavior for small t. Let $\gamma \equiv .5772...$ be Euler's constant and let $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$ be the digamma function; see 6.3.1 of AS. The behavior is somewhat complicated, so we only describe three cases.

Theorem 3.2. *(a) If $p < 1$, then*

$$V^c(p,q;t) \sim e^{-t}\left(1 + \frac{\Gamma(p+q)\Gamma(-p)}{\Gamma(q)\Gamma(p)} t^p\right) \quad \text{as} \quad t \to 0 \ .$$

(b) If $p > 1$ and p non-integer, then

$$V^c(p,q;t) \sim e^{-t}\left(1 - \frac{q}{p-1} t\right) \quad \text{as} \quad t \to 0 \ .$$

(c) If $p = 1$, then

$$V^c(p,q;t) \sim e^{-t}(1 + q[\psi(1+q) + 2\gamma - 1 + \log t]t) \quad \text{as} \quad t \to 0 \ .$$

Proof. Combine (1.8) and (1.23) to express $V^c(p,q;t)$ in terms of U. For p noninteger, use 13.1.2 and 13.1.3 of AS. For $p = 1$, use 13.1.6 and 13.1.7 of AS.

4. Gamma Distributions

In this section we show that gamma pdf's with shape parameter less than 1 are all BME pdf's. This gives us a convenient starting point to construct other pdf's (as well as associated ccdf's and transforms) via recurrence relations discussed in the next section.

Let the gamma pdf be

$$\gamma(p;t) = \frac{t^{p-1}e^{-t}}{\Gamma(p)}, \quad t \geq 0 \ . \tag{4.1}$$

There is only the one parameter p in (4.1) because the scale parameter has been omitted. The associated Laplace transform is

$$\hat{v}(p;s) = (1+s)^{-p} \ . \tag{4.2}$$

Theorem 4.1. *For $0 < p < 1$,*

$$\hat{v}(p, 1-p;t) = \gamma(p;t), \quad t \geq 0 \ . \tag{4.3}$$

Proof. Paralleling (1.19), we represent the Laplace transform as the Stieltjes transform of the spectral density; i.e., assuming that a pdf f is a mixture of exponential (rates), i.e.,

$$f(t) = \int_0^\infty x e^{-xt} \phi(x) dx \ ,$$

it Laplace transform is the Stieltjes transform of $x\phi(x)$, i.e., by changing the order of integration,

$$\hat{f}(s) = \int_0^\infty \frac{x}{s+x} \phi(x) dx \ . \tag{4.4}$$

We can then calculate $x\phi(x)$ by inverting its Stieltjes transform, e.g., see p. 126 of Widder [34]. Starting with (4.2), we obtain

$$\phi(x) = \frac{-1}{\pi x} \mathrm{Im}\, \hat{\gamma}(p; -x) = \frac{\sin \pi p}{\pi x (x-1)^p} = \frac{1}{\Gamma(p)\Gamma(1-p) x (x-1)^p}, \quad x \geq 1 \ , \tag{4.5}$$

applying 6.1.17 and 4.3.4 of AS.

We can also approach (and generalize) Theorem 4.1 another way, using the following lemma, e.g., see p. 329 of Moran [28].

Lemma 4.2. *If X and Y are independent random variables with densities $\gamma(p;t)$ and $\gamma(q;t)$, then the ratio $X/(X+Y)$ has the beta $b(p,q;y)$ pdf and this ratio is independent of the denominator $X+Y$.*

We can apply Lemma 4.2 to establish the following generalizations of Theorem 4.1, evidently first due to Sawkins [30]. (We obtain Theorem 4.1 by letting $p+q=1$.)

Theorem 4.3. *A beta $b(p,q;y)$ scale mixture of gamma $\gamma(p+q;t)$ pdf's is gamma $\gamma(p;t)$; i.e., for all $p>0$ and $q>0$,*

$$\gamma(p;t) = \int_0^1 y^{-1} \gamma(p+q; t/y) b(p,q;y) dy = \int_0^1 y^{-1} b(p,q;t/y) \gamma(p+q;y) dy \ .$$

Proof. Let X and Y have the gamma pdf's $\gamma(p;t)$ and $\gamma(q;t)$. We can write

$$X = (X+Y) \frac{X}{(X+Y)} \ ,$$

but $(X+Y)$ has pdf $\gamma(p+q;t)$ and, by Lemma 4.2, the ratio has pdf $b(p,q;y)$ and is independent of $X+Y$. The moment sequences also provide a short proof: Note that the n^{th} moment is

$$(p)_n = ((p+q)_n)\left(\frac{(p)_n}{(p+q)_n}\right) .$$

5. Recurrence Relations

In this section we give recurrence relations that enable us to calculate new BME pdf's from given ones. Simple modifications of these recurrences apply to BME ccdf's and B_2ME pdf's and ccdf's by virtue of (1.8), (1.21) and (1.22); we will not state them.

In formulas (1.8), (1.14) and (1.16), we have seen that we can increase p by 1, i.e., go from $v(p,q;t)$ to $v(p+1,q;t)$ by integrating (or dividing by s in the Laplace transform). Now we want alternative recurrence relations that do not require integration. For this purpose, we can apply Theorem 1.7 and recurrence relations derived for the Tricomi function. The following four recurrence relations follow directly from 3.4.17, 3.4.16, 3.4.15 and 3.4.20 of AS, respectively.

Theorem 5.1. *For all $p > 0$ and $q > 0$,*

$$(a)\quad v(p,q+1;t) = \left(\frac{p+q}{q}\right) v(p,q;t) - \frac{p}{q} v(p+1,q;t) \tag{5.1}$$

$$(b)\quad v(p+1,q;t) = \frac{p+q}{p(p-1)} t v(p-1,q;t) - \frac{(p+q)}{p(p+q-1)}(t+1-p)v(p,q;t) \tag{5.2}$$

$$(c)\quad v(p,q+1;t) = \left(\frac{p+q}{q}\right)\left(1+\frac{t+q-1}{p+q-1}\right) v(p,q;t) - \left(\frac{p+q}{q}\right) v(p,q-1;t) \tag{5.3}$$

$$(d)\quad v(p+1,q;t) = \left(\frac{p+q}{p}\right)\left(v(p,q-1;t) - \left(\frac{t+q-1}{p+q-1}\right) v(p,q;t)\right) . \tag{5.4}$$

Recurrence (5.1) can also be derived from a recurrence for the beta pdf

$$b(p,q;y) = \left(\frac{p+q}{p}\right) b(p,q;y) - \frac{p}{q} b(p+1,q;y) . \tag{5.5}$$

Combining (5.1) with (1.16), we obtain

$$\hat{v}(p, q+1; s) = \frac{p+q}{qs}[(1+s)\hat{v}(p, q; s) - 1] \ . \tag{5.6}$$

Formula (5.6) can also be obtained by combining (1.16) and (1.20).

As in (3.15) of Abate and Whitt [4], let $\mathcal{L}$ be the stationary-lifetime operator that maps a pdf f with finite mean $m_1(F)$ into the pdf

$$\mathcal{L}(f(t)) = \frac{tf(t)}{m_1(F)}, \quad t \geq 0 \ . \tag{5.7}$$

which has inverse (valid for unimodal pdf's)

$$f(t) \equiv \mathcal{L}^{-1}(g(t)) = -tg'(t), \quad t \geq 0 \ . \tag{5.8}$$

Paralleling Theorem 1.3, we can give the stationary-lifetime pdf associated with any BME pdf. (Integrate by parts in (1.3).)

Theorem 5.2. *For all $p > 0$ and $q > 0$,*

$$\mathcal{L}(v(p, q; t)) = (p+1)v(p+2, q-1; t) - pv(p+1, q; t) \ . \tag{5.9}$$

Recurrence (5.2) can also be derived from (5.9) and restated in that form, which is appealing because all component functions are pdf's.

Corollary. *For all $p > 0$ and $q > 0$,*

$$\begin{aligned} v(p+1, q; t) &= \frac{p+q}{p(p+q-1)}\mathcal{L}(v(p-1, q; t)) - \frac{1}{p+q-1}\mathcal{L}(v(p, q; t)) \\ &+ \left(1 - \frac{q}{p(p+q-1)}\right) v(p, q; t) \ , \end{aligned} \tag{5.10}$$

which is equivalent to (5.2).

Formula (2.1) implies that

$$m_n(p, q) = \left(1 + \frac{n}{p+q}\right) m_n(p, q+1) = \left(1 + \frac{n}{p-1}\right) m_n(p-1, q+1) \ . \tag{5.11}$$

As in Section 2 of Abate and Whitt [4], let $\mathcal{U}$ be the unimodal operator mapping any pdf into a unimodal (decreasing) pdf, which can be expressed via transforms as

$$\mathcal{U}(\hat{f}(s)) = \frac{1}{s}\int_0^s \hat{f}(z)dz \tag{5.12}$$

or directly as

$$\mathcal{U}(f(t)) = \int_t^\infty x^{-1} f(x) dx \ . \tag{5.13}$$

As noted there, $g = \mathcal{U}(f)$ if and only if $m_n(f) = (n+1)m_n(g)$ for all n. The moment characterization follows from the moment generating function representation. We see that this structure holds in the two formulas in (5.11) when $p+q = 1$ and $p = 2$, respectively. Hence, we can use the unimodal operator to generate new BME pdf's.

For this purpose, note that $\mathcal{U}$ maps gamma pdf's into associated incomplete gamma pdf's, i.e.,

$$\mathcal{U}(\gamma(p;t)) = \frac{1}{\Gamma(p)} \int_t^\infty e^{-y} y^{p-2} dy \equiv \frac{\Gamma(p-1,t)}{\Gamma(p)} \ , \quad t \geq 0 \ . \tag{5.14}$$

Hence, we have the following result.

Theorem 5.3. *For* $0 < p < 1$*,*

$$v(p, 2-p; t) = \mathcal{U}(v(p, 1-p; t)) = \frac{\Gamma(p-1,t)}{\Gamma(p)}, \quad t \geq 0, \tag{5.15}$$

and

$$v(1, q+1, t) = \mathcal{U}(v(2, q; t)), \quad t \geq 0 \ . \tag{5.16}$$

It turns out that the incomplete gamma pdf's $v(p, 2-p; t)$ coincide with a class of pdf's introduced by Gaver and Jacobs [20]. They consider the transform

$$\hat{g}(\alpha; s) = (\alpha s)^{-1}((1+s)^\alpha - 1), \quad 0 \leq \alpha < 1 \ , \tag{5.17}$$

obtained through manipulations of stable laws. It follows from (5.6) and Theorem 5.3 that

$$\hat{g}(1-p; t) = v(p, 2-p; t), \quad 0 < p < 2 \ ; \tag{5.18}$$

can be extended to $-1 < \alpha < 1$ and we give an alternative characterization of the distribution via Theorem 5.3.

Notable examples of the subclass $v(p; 2-p; t)$ are: for $p = 1/2$, the RBM first moment pdf $h_1(t)$ in Abate and Whitt [1]; for $p = 1$, the exponential-integral pdf $E_1(t)$, and for $p = 3/2$, the stationary-excess of $\gamma(1/2; t)$, denoted by $\gamma_e(1/2; t)$; see Tables 1 and 2. These cases can be determined directly from the incomplete gamma function representation in 6.53 of AS. Examples of pdf pairs (f, g) for which $g = U(f)$ are given in Table 1.

$f(t) = U^{-1}(g(t))$	$g(t) = U(f(t))$
$v(p, 1-p; t)$	$v(p, 2-p; t)$
$v(1/2, 1/2; t) = \gamma(t)$	$v(1/2, 3/2; t) = h_1(t)$
$2t\gamma(t)$	$v(3/2, 1/2; t) = \gamma_e(t)$
$4th_1(t)$	$v(3/2, 3/2; t) = h_{1e}(t)$
$2/3v(1/2, 3/2; t) + 1/3v(3/2, 3/2; t)$	$v(1/2, 5/2; t)$
e^{-t}	$v(1, 1; t) = E_1(t)$
$v(2, q; t)$	$v(1, q+1; t)$
$v(2, 1/2; t)$	$v(1, 3/2; t)$
$v(2, 1; t)$	$v(1, 2; t)$

Table 1: Examples of pdf's f and g satisfying $g = U(f)$.

$f(t)$	$g(t)$
$v(3/2, 1/2; t) = \gamma_e(t)$	$v(1/2, 3/2; t) = h_1(t)$
$v(3/2, 3/2; t) = h_{1e}(t)$	$v(1/2, 5/2; t) = \gamma_{ee}(t)$
$v(1/2, 1/2; t) = \gamma(t)$	$E_1(t)/(2\sqrt{\pi t})$
$2t\gamma(t)$	$v(1/2, 1/2; t) = \gamma(t)$
e^{-t}	$v(1/2, 1; t) = \sqrt{\pi}\,\mathrm{erfc}\,(\sqrt{t})/(2\sqrt{t})$
$v(3/2, 1; t)$	$v(1/2, 2; t)$
$v(3/2, q; t)$	$v(1/2, q+1; t)$

Table 2: Examples of pdf's f and g satisfying the moment relation in Theorem 5.4, i.e., for which $f(t) = -2tg'(t) - g(t)$.

It also turns out that the associated B$_2$ME pdf's $v_2(p, 2-p; t)$ correspond to pdf's introduced by Boxma and Cohen [15]. The ccdf is

$$V_2^c(p, 2-p; t) = e^t v(p, 1-p; t) - (t+1-p)e^t v(p, 2-p; t), \quad t \geq 0 \ . \tag{5.19}$$

We can apply the gamma relations established for $p + q = 1$ and $p + q = 2$ to obtain

$$V_2^c(p, 2-p; t) = \frac{e^t}{\Gamma(p)}(t^{p-1}e^{-t} - (t+1-p)\Gamma(p-1, t)) \ , \tag{5.20}$$

which can be shown to be equivalent to their formula (4.4).

Especially tractable is the stationary-excess pdf in the $(1/2, 3/2)$ case. Then

$$V_2^c(1/2, 3/2; t) \quad = \quad (2t+1)e^t\,\mathrm{erfc}\,(\sqrt{t}) - 2\sqrt{t/\pi} \ , \tag{5.21}$$

$$v_2(1/2, 3/2; t) = e^t h_1(t) - \frac{1}{4} e^t h_{1e}(t) \quad (5.22)$$

$$\hat{v}_2(1/2, 3/2; s) = \frac{1}{s-1}\left(\frac{2s}{1+\sqrt{s}} - 1\right) = 1 - \frac{s}{(1+\sqrt{s})^2} \quad (5.23)$$

$$\hat{v}_{2e}(1/2, 3/2; s) = \frac{1}{(1+\sqrt{s})^2} , \quad (5.24)$$

where $h_1(t)$ is again the RBM first-moment pdf in Table 1 and erfc(t) is the complementary error function, which is related to the standard (mean 0, variance 1) normal ccdf $\Phi^c(t)$ by erfc$(t) = 2\Phi^c(\sqrt{2}t)$; see 7.1.1 and 26.2.29 of AS. Boxma and Cohen showed that the M/G/1 steady-state waiting-time distribution can be solved explicitly when the service-time pdf is the B_2ME pdf $v_2(1/2, 3/2; t)$. We extend this explicit representation to a larger class of service-time pdf's, all with the tail asymptotics $f(t) \sim \alpha t^{-3/2}$, in Abate and Whitt [6]. In Proposition 8.2 of Abate and Whitt [4] we had previously obtained explicit solutions for the M/G/1 waiting-time distribution for a class of service-time pdf's including the BME pdf $v(1/2, 3/2; t)$. Using formula (5.24), the argument for $v_2(1/2, 3/2; t)$ is a natural extension of the previous one.

Now consider the case $p = 3/2$ in the second formula in (5.11), which yields the relation

$$m_n(3/2, q) = (2n+1) m_n(1/2, q+1) . \quad (5.25)$$

It turns out that this moment relation induces a relation between cdf's similar to the unimodal operator in (5.12) and (5.13).

The following can be established by relating the coefficients of the moment generating functions.

Theorem 5.4. *If $f(t)$ and $g(t)$ are pdf's satisfying $m_n(F) = (2n+1)m_n(G)$ for all n, then $f(t) \equiv -2tg'(t) - g(t)$, $F^c(t) = G^c(t) + 2tg(t)$, $\hat{f}(s) = \hat{g}(s) + 2s\hat{g}'(s)$ and*

$$\hat{g}(s) = \frac{1}{2\sqrt{s}} \int_0^s \frac{\hat{f}(z)}{\sqrt{z}} dz . \quad (5.26)$$

Corollary. *For all $q > 0$,*

$$v(3/2, q; t) = -2tv'(1/2, q+1, t) - v(1/2, q+1; t) , \quad t \geq 0 . \quad (5.27)$$

Examples of pdf pairs (f, g) for which $f(t) = -2tg'(t) - g(t)$ are given in Table 2. Finally, note that (5.11) provides generalizations of the two moment relations that we have considered in detail.

6. Concrete Examples

In Section 2 we showed that the BME pdf's $v(p,q;t)$, cdf's $V^c(p,q;t)$ and Laplace transform $\hat{v}(p,q;s)$ can readily be computed for any p and q, exploiting the series representations (2.6), (2.10)–(2.13). Explicit formulas are also available in special cases, as is evident from Section 13.6 of AS. When $p+q$ is an integer, we can obtain explicit formulas starting from the gamma case in which $p+q=1$ and then applying the recurrence relation in Section 5.

Convenient explicit formulas can also be obtained when p and q are both integer multiples of 1/2. Indeed, many of these cases correspond to pdf's that we have considered in previous work. The results are summarized in Tables 3–5. Table 3 contains cases in which both p and q are odd multiples of 1/2.

p	q	$b(p,q;y)$	$m_n(p,q)/n!$	$\hat{v}(p,q;s)$	$v(p,q;t)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\pi\sqrt{y}\sqrt{1-y}}$	$\beta_n \equiv \frac{1}{4^n}\binom{2n}{n}$	$\hat{\gamma}(s) \equiv \hat{\gamma}(1/2;s) \equiv \frac{1}{\sqrt{1+s}}$	$\gamma(t) \equiv \gamma(1/2;t) \equiv \frac{e^{-t}}{\sqrt{\pi t}}$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{2\sqrt{y}}{\pi\sqrt{1-y}}$	$\frac{2n+1}{n+1}\beta_n$	$\hat{\gamma}_e(s) \equiv \frac{2}{s}[1-\hat{\gamma}(s)]$	$\gamma_e(t) \equiv 2\,\mathrm{erfc}\,(\sqrt{t})$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{2\sqrt{1-y}}{\pi\sqrt{y}}$	$\frac{\beta_n}{n+1}$	$\hat{h}_1(s) \equiv \frac{2}{1+\sqrt{1+s}}$	$h_1(t) \equiv 2\gamma(t) - \gamma_e(t)$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{8}{\pi}\sqrt{y}\sqrt{1-y}$	$\frac{(2n+1)2\beta_n}{(n+1)(n+2)}$	$\hat{h}_{1e}(s) \equiv \frac{4}{s}[1-\hat{h}_1(s)]$	$h_{1e}(t) \equiv 2(2t+1)\gamma_e(t) - 8t\gamma(t)$
$\frac{1}{2}$	$\frac{5}{2}$	$\frac{8}{\pi}\frac{(1-y)^{3/2}}{\sqrt{y}}$	$\frac{2\beta_n}{(n+1)(n+2)}$	$\frac{4}{3s}[(1+s)\hat{h}_1(s)-1]$	$\frac{4}{3}h_1(t) - \frac{1}{3}h_{1e}(t)$
$\frac{5}{2}$	$\frac{1}{2}$	$\frac{8}{3\pi}\frac{y^{3/2}}{\sqrt{1-y}}$	$\frac{(2n+1)(2n+3)2\beta_n}{3(n+1)(n+2)}$	$\hat{\gamma}_{ee}(s) \equiv \frac{4}{3s}[1-\hat{\gamma}_e(s)]$	$\gamma_{ee}(t) \equiv \frac{8}{3}t\gamma(t) - \frac{2}{3}(2t-1)\gamma_e(t)$

Table 3: Properties of BME pdf's when both p and q are odd multiples of 1/2.

The first entry in Table 3 is $v(1/2,1/2;t) = \gamma(t) \equiv \gamma(1/2;t)$, which follows from Section 4. The next entries are its stationary-excess $\gamma_e(t) \equiv v(3/2,1/2;t)$ and the RBM first-moment pdf $h_1(t) \equiv v(1/2,3/2;t)$. As noted in Corollaries 1.3.2, 1.5.1 and 1.5.2 of Abate and Whitt [1], $h_1(t)$ has the interesting property that $h_{1e}(t) = h_1(t) * h_1(t)$, where $*$ denotes convolution. This property can be seen from the h_1 moment sequence, i.e.,

$$\frac{m_n(H)}{n!} = \frac{1}{n+1}\binom{2n}{n} \equiv C_n\ , \tag{6.1}$$

where $\{C_n\} \equiv \{1,2,5,14,42,\ldots\}$ is the sequence of Catalan numbers, with defining property

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \tag{6.2}$$

where $C_0 = 1$. When rescaled, $h_{1e}(t) \equiv v(3/2, 3/2; t)$ coincides with the RBM correlation function; see p. 320 of Abate and Whitt [2].

Table 4 displays cases in which p and q are both integers. We start with

p	q	$m_n(p,q)/n!$	$\hat{v}(p,q;s)$	$v(p,q;t)$
1	1	$\frac{1}{n+1}$	$\hat{E}_1(s) \equiv \frac{1}{s}\log(1+s)$	$E_1(t)$
2	1	$\frac{2}{n+2}$	$\frac{2}{s}[1-\hat{E}_1(s)]$	$2E_2(t) \equiv 2e^{-t} - 2tE_1(t)$
1	2	$\frac{2}{(n+1)(n+2)}$	$\frac{2}{s}[(1+s)\hat{E}_1(s) - 1]$	$2E_1(t) - 2E_2(t)$
2	2	$\frac{6}{(n+2)(n+3)}$	$\frac{6}{s^2}\left[-(1+s)\hat{E}_1(s) + \frac{s}{2} + 1\right]$	$6E_2(t) - 6E_3(t)$
1	3	$\frac{6}{(n+1)(n+2)(n+3)}$	$\frac{3}{s^2}\left[(1+s)^2\hat{E}_1(s) - \frac{3s}{2} - 1\right]$	$3E_1(t) - 6E_2(t) + 3E_3(t)$
3	1	$\frac{3}{n+3}$	$\frac{3}{2s}[1-\hat{E}_{1e}(s)]$	$3E_3(t)$

Table 4: Properties of BME pdf's when both p and q are integers.

the exponential-integral pdf $v(1,1;t) \equiv E_1(t)$. Indeed, from (1.5) here and 5.1.4 of AS, we see that

$$v(n,1;t) = nE_n(t) = n\int_1^\infty x^{-n}e^{-tx}dx \ . \tag{6.3}$$

From 5.1.52 of AS, we see that

$$v(n,1;t) \to e^{-t} \quad \text{as} \quad n \to \infty \ . \tag{6.4}$$

Indeed, we see that $v(p,q;t) \to e^{-t}$ as $p \to \infty$ because the distribution of $b(p,q;y)$ approaches to a unit point mass at 1 as $p \to \infty$. Given Theorem 1.3, the limit (6.4) can also be deduced from limits of iterates of the stationary-excess operator; see Harkness and Shantaram [25], Shantaram and Harkness [31], and van Beek and Braat [13].

We can obtain the transform $\hat{v}(1,1;s)$ as the limit of $\hat{v}(p,2-p;s)$ as $p \to 1$, using (5.17) and (5.18), i.e.,

$$\hat{v}(p,2-p;s) = \lim_{p\to 1}\frac{1}{(1-p)s}((1+s)^{1-p} - 1) = \frac{\log(1+s)}{s} \ . \tag{6.5}$$

We note that the cases $v(1/2,1/2;t) \equiv \gamma(t)$ and $v(1,1;t) \equiv E_1(t)$ were used to study the Laguerre-series algorithm in Abate, Choudhury and Whitt [10]; see Tables 1 and 2 there.

Table 5 contains cases in which one of p and q is integer while the other is an odd multiple of 1/2. The first entry $v(1/2,1;t)$ is a curious pdf. Noting

p	q	$b(p,q;y)$	$M_n(p,q)/n!$	$\hat{v}(p,q;s)$	$v(p,q;t)$
$\frac{1}{2}$	1	$\frac{1}{2\sqrt{y}}$	$\frac{1}{2n+1}$	$\hat{r}(s) \equiv \frac{\arctan\sqrt{s}}{\sqrt{s}}$	$r(t) \equiv \frac{\sqrt{\pi}}{2}\frac{\operatorname{erfc}\sqrt{t}}{\sqrt{t}}$
$\frac{3}{2}$	1	$\frac{3}{2}\sqrt{y}$	$\frac{3}{2n+3}$	$\hat{r}_e(s) \equiv \frac{3}{s}[1-\hat{r}(s)]$	$r_e(t) \equiv 3e^{-t} - 6tr(t)$
$\frac{1}{2}$	2	$\frac{3}{4}\frac{(1-y)}{\sqrt{y}}$	$\frac{3}{(2n+1)(2n+3)}$	$\frac{3}{2s}[(1+s)\hat{r}(s)-1]$	$\frac{3}{2}r(t) - \frac{1}{2}r_e(t)$
1	$\frac{1}{2}$	$\frac{1}{2\sqrt{1-y}}$	$\frac{1}{(2n+1)\beta_n}$	$\hat{\rho}(s) \equiv \frac{\log(\sqrt{s}+\sqrt{1+s})}{\sqrt{s}\sqrt{1+s}}$	$\rho(t) \equiv \frac{e^{-t/2}}{2}K_0\left(\frac{t}{2}\right)$
2	$\frac{1}{2}$	$\frac{3y}{4\sqrt{1-y}}$	$\frac{3(n+1)}{(2n+3)(2n+1)\beta_n}$	$\hat{\rho}_e(s) \equiv \frac{3}{2s}[1-\hat{\rho}(s)]$	$\rho_e(t)$
1	$\frac{3}{2}$	$\frac{3\sqrt{1-y}}{2}$	$\frac{3}{(2n+3)(2n+1)\beta_n}$	$\frac{3}{s}[(1+s)\hat{\rho}(s)-1]$	$3\rho(t) - 2\rho_e(t)$

Table 5: Properties of BME pdf's when one of p and q is an integer and the other is an odd multiple of 1/2.

that its moments are $n!/(2n+1)$ and recalling that the power series for the arctan function has coefficients $(2n+1)^{-1}$, see 2.5.9 on p. 53 of Wilf [35], we see that

$$\hat{v}(1/2,1;s) = \hat{r}(s) \equiv \frac{\arctan\sqrt{s}}{\sqrt{s}} \ . \tag{6.6}$$

We can also obtain (6.6) from the transform representation in Theorem 1.4, using 15.1.5 of AS. The $(1/2,2)$ and $(3/2,1)$ entries can be determined by applying (5.1) and Theorem 5.4.

The pdf $v(1,1/2;t)$ in Table 5 is determined from an integral representation of the Bessel function $K_0(t)$; see 9.6.23 and 13.6.21 of AS. For the transform, see p. 119 of Oberhettinger and Badii [29]. Note that $\hat{v}(1,1/2;s) = \hat{\gamma}(s)\hat{f}(s)$, where

$$\hat{f}(s) = \log(\sqrt{s}+\sqrt{1+s})/\sqrt{s} \tag{6.7}$$

and

$$f(t) = E_1(t)/2\sqrt{\pi t}, \quad t \geq 0 \ , \tag{6.8}$$

which has moments $\beta_n/(2n+1)$. Hence, $f(t)$ and $\gamma(t)$ are related as in Table 2 and we have an entry there.

We obtain

$$v(2,1/2;t) = 2\left(\frac{3}{8} + e^{-t/2}K_1(t/2)\right) - \frac{3}{2}tv(1,1/2;t)$$

from 11.3.15 of AS. We obtain $v(1,3/2;t)$ from (5.1).

From the concrete examples for BME distributions, we obtain corresponding concrete examples of B_2ME distributions. We give an illustrative example

Example 6.1. We consider the case $p = 1$ and $q = 3$. By (1.21) and (1.8), the ccdf is

$$V_2^c(1,3;t) = e^t V_2(1,3,t) = \frac{1}{4} e^t v(2,3;t) = \frac{3}{4} e^t (E_2(t) - 2E_3(t) + E_4(t)) \ . \quad (6.9)$$

After some algebra, we find that the transform is

$$\hat{v}_2(1,3;s) = \frac{\left(1 - \frac{7}{2}s + \frac{11}{2}s^2\right)}{(1-s)^3} + \frac{3s^3}{(1-s)^4} \log s \ . \quad (6.10)$$

From (6.10), we see that $m_1 = 1/2$, $m_2 = 2$ and $m_3 = \infty$.

7. Gamma Mixtures of Exponential Distributions

In this section we show that all gamma mixtures of exponential (GME) pdf's can be obtained as limits of BME pdf's. These GME pdf's can in turn be represented in terms of Bessel functions K_p.

Theorem 7.1. *For all $p > 0$,*

$$\lim_{q\to\infty} \frac{v(p,q;t/q)}{q} = \int_0^\infty \mu^{-1} e^{-t/\mu} \frac{\mu^{p-1} e^{-\mu}}{\Gamma(p)} d\mu = \frac{2}{\Gamma(p)} t^{(p-1)/2} K_{p-1}(2\sqrt{t}) \ . \quad (7.1)$$

Proof. Apply (1.3) and make the change of variables $\mu = qy$ to obtain

$$\frac{v(p,q;t/q)}{q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)q^p} \int_0^q \mu^{-1} e^{-t/\mu} \mu^{p-1} \left(1 - \frac{\mu}{q}\right)^{q-1} d\mu \ .$$

Since $\Gamma(p+q)/\Gamma(q)q^p \to 1$ and $(1-(\mu/q))^{q-1} \to e^{-\mu}$ as $q \to \infty$, we have the first limit. For the Bessel function limit, apply 13.3.3 of AS. ■

Let $g(p;t)$ represent the GME pdf with parameter p. Note that the stationary-excess property in Theorem 1.3 is inherited: $g_e(p;t) = g(p+1;t)$. From p. 155 of Oberhettinger and Badii [29], we obtain its Laplace transform

$$\hat{g}(p;s) = s^{-p} e^{1/s} \Gamma(1-p, 1/s) \ . \quad (7.2)$$

Two special cases of interest are $p = 1/2$ and $p = 1$, for which we get

$$\hat{g}(1/2;s) = \sqrt{\pi/s} e^{1/s} \operatorname{erfc}(1/\sqrt{s}) \quad (7.3)$$

and

$$\hat{g}(1;s) = s^{-1}e^{s^{-1}}E_1(s^{-1}) \tag{7.4}$$

from 6.5.17 and 6.5.15 of AS.

For purposes of numerical inversion, we can effectively compute these transforms by using the power series expansion for $\text{erfc}(z)$ and $E_1(z)$. For other cases where p is a multiple of $1/2$, we can use $\hat{g}(1+p;s) = (1/sm_1)(1-\hat{g}(p;s))$. For cases in which p is not a multiple of $1/2$, we can use the following series developed from 6.5.4 of AS

$$\hat{g}(p;s) = \frac{e^{1/s}}{s}\left(\Gamma(1-p)s^{1-p} - \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1-p)n!s^n}\right) . \tag{7.5}$$

We remark that the GME pdf with $p = 1/2$ coincides with the Weibull pdf with shape parameters $1/2$ and mean $1/2$; i.e., the ccdf is $e^{-2\sqrt{t}}$. This can be seen from its moment sequence. The moments of a GME pdf are $m_n(p) = n!(p)_n$. Hence,

$$m_n(1/2) = n!\left(\frac{n!}{4^n}\binom{2n}{n}\right) = \frac{(2n)!}{4^n} . \tag{7.6}$$

The exponential mixture of exponentials arising when $p = 1$ is the heavy-traffic limit for the waiting time in the M/G/1 queue with random order of service; see p. 89 of Kingman [27].

8. Other Scale Mixtures

As indicated in the introduction, scale mixtures are usefully viewed as products of independent random variables. The scale mixture can be represented as a random variable Z, where $Z = XY$ for independent random variables X and Y. It follows trivially that successive mixture operations are commutative and associative, i.e., $Y_1Y_2 = Y_2Y_1$ and $(Y_1Y_2)Y_3 = Y_1(Y_2Y_3)$ for independent random variables Y_1, Y_2 and Y_3.

The random product representation shows that mixtures can be conveniently characterized by their moment sequences provided that the moment sequences of the component random variables are known, as is the case for beta and exponential random variables. We now give some further results along these lines.

We first show that beta scale mixtures of betas are again betas in certain circumstances. Of particular note is the quick proof.

Theorem 8.1. *For all $p > 0$ and $q_2 > q_1$,*

$$\begin{aligned} b(p, q_2; t) &= \int_0^t y^{-1} b(p, q_1; t/y) b(p + q_1, q_2 - q_1, y) dy \\ &= \int_0^t y^{-1} b(p + q_1, q_2 - q_1; t/y) p(p, q_1; y) dy \ . \end{aligned}$$

Proof. In the random product XY, the n^{th} moments are

$$\left(\frac{(p)_n}{(p+q_2)_n} \right) = \left(\frac{(p)_n}{(p+q_1)_n} \right) \left(\frac{(p+q_1)_n}{(p+q_2)_n} \right) \ .$$

From the moment sequences, it is also easy to see that not all beta mixtures of betas are betas.

We apply Theorem 8.1 to show that beta mixtures of BMEs (BMBMEs) are again BMEs in certain circumstances.

Corollary. *For all $p > 0$ and $q_2 > q_1 > 0$,*

$$\begin{aligned} v(p, q_2; t) &= \int_0^1 y^{-1} v(p, q_1; t/y) b(p + q_1, q_2 - q_1; y) dy \\ &= \int_0^1 y^{-1} v(p + q_1, q_2 - q_1; t/y) b(p, q_1; y) dy \ . \end{aligned}$$

We now show that BME pdf's multiplied by a power of t and renormalized are beta mixtures of gamma pdf's. Such pdf's arise from applying the stationary-lifetime operator $\mathcal{L}$; see (5.7).

Theorem 8.2. *For all $p > 0$, $q > 0$ and $\alpha > p - 1$,*

$$f(t) \equiv \frac{\Gamma(\alpha+q)\Gamma(p)}{\Gamma(\alpha)\Gamma(1+\alpha-p)\Gamma(p+q)} t^{\alpha-p} v(p, q; t) \tag{8.1}$$

is a bonafide cdf with moments

$$m_n = \frac{(1+\alpha-p)_n (\alpha)_n}{(q+\alpha)_n} \ , \tag{8.2}$$

so that

$$\begin{aligned} f(t) &= \int_0^1 y^{-1} \gamma(1 + \alpha - p; t/y) b(\alpha, q; y) dy \\ &= \int_0^1 y^{-1} b(\alpha, q; t/y) \gamma(1 + \alpha - p; y) dy \ . \end{aligned} \tag{8.3}$$

Proof. First express $f(t)$ as

$$f(t) = \frac{\Gamma(\alpha+q)t^{\alpha-p}}{\Gamma(\alpha)\Gamma(1+\alpha-p)} e^{-t} U(q, 2-p; t) \ .$$

Then apply (7) on p. 270 of Erdelyi [17] plus 15.3.5 and 13.1.29 of AS to obtain

$$\begin{aligned} \hat{f}(s) &= {}_2F_1(1+\alpha-p, \alpha, q+\alpha; -s) \\ &= \sum_{n=0}^{\infty} \frac{(1+\alpha-p)_n(\alpha)_n}{(q+\alpha)_n} \frac{(-s)^n}{n!} \ . \end{aligned}$$

We now give some other examples from Theorem 8.2. First, if $\alpha = 1/2$, $p = 1$ and $q = 1$, then

$$f(t) = E_1(t)/2\sqrt{\pi t}, \quad t \geq 0 \ . \tag{8.4}$$

Next, if $\alpha = 1/2$, $p = 1$ and $q = 1/2$, then

$$f(t) = e^{-t/2} K_0(t/2)/\sqrt{\pi^3 t}, \quad t \geq 0 \ . \tag{8.5}$$

Finally, if $\alpha = 1/4$, $p = 1/2$ and $q = 5/4$, then

$$f(t) = \frac{4}{3\sqrt{2}\Gamma(3/4)} t^{-1/4} v(1/2, 5/4; t), \quad t \geq 0 \ , \tag{8.6}$$

and $f(t) * f(t) = h_1(t) = v(1/2, 3/2; t)$. None of these three examples are BME pdf's. Cases in which f is a BME are contained in Theorem 4.3; assume that $q = 1 - 2p$ and $p < 1/2$. We do not yet know all cases in which the BMBMEs in Theorem 8.2 are BME.

9. Omitted Proofs

We now give five omitted proofs.

Proof of Theorem 1.2. Recall that cdf's F_n converge weakly to a cdf F if and only if either (i) $F_n(x) \to F(x)$ for all x that are continuity points of F or (ii) $\int_0^\infty g dF_n \to \int_0^\infty g dF$ for all continuous bounded real-valued functions g; see Billingsley [14]. Apply this with the integral representation of the cdf paralleling (1.5).

Proof of Theorem 1.5. Observe that $b(q,p;y) = b(p,q;1-y)$, $0 \le y \le 1$, for all $p > 0$ and $q > 0$ and apply Theorem 1.4. Hence,

$$\begin{aligned}
\hat{v}(q,p;s) &= \int_0^1 \frac{1}{1+ys} b(p,q;1-y)dy \\
&= \int_0^1 \frac{1}{1+(1-z)s} b(p,q;z)dz \\
&= \frac{1}{1+s}\int_0^1 \frac{1}{1+z\left(\frac{-s}{1+s}\right)} b(p,q;z)dz \\
&= \frac{1}{1+s}\hat{v}(p,q;-s/(1+s)) \ .
\end{aligned}$$

Proof of Theorem 1.6. Perform the change of variables $y = x - 1$ in (1.5) to get

$$v(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} e^{-t} \int_0^\infty e^{-ty} y^{q-1}(1+y)^{1-p-q} dy \ . \tag{9.1}$$

Hence we can integrate in (9.1) to get the ccdf and then undamp the ccdf by multiplying by e^t to get

$$e^t V^c(p,q;t) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^\infty e^{-ty} \frac{y^{q-1}}{(1+y)^{p+q}} dy \ , \tag{9.2}$$

which has the corresponding mixing representation

$$e^t V^c(p,q;t) = \int_0^\infty e^{-t/y} b_2(p,q;y)dy \equiv V_2^c(p,q;t) \ . \tag{9.3}$$

From the right side of (9.2) or (9.3), we see that $e^t V^c(p,q;t)$ is a bonafide cdf (is decreasing).

Next,

$$\begin{aligned}
v_2(p,q;t) &= \frac{-d}{dt} e^t V^c(p,q;t) \quad = \quad e^t[v(p,q;t) - V^c(p,q;t)] \\
&= e^t\left[\int_0^1 \left(\frac{1}{y} - 1\right) e^{-t/y} b(p,q;y)dy\right] \\
&= \frac{q}{p+q} e^t v(p,q+1;t) \ ,
\end{aligned}$$

which must equal the normalized version of $e^t v(p,q+1;t)$.

Proof of Theorem 1.8. Combining (1.8) and Theorem 1.7,

$$V_2^c(t) = \frac{\Gamma(p+q)}{\Gamma(p)} V(q, 1-p, t) \tag{9.4}$$

but, from 13.1.29 of AS,

$$U(q, 1-p, t) = t^p U(p+q, p+1, t) \ , \tag{9.5}$$

so that

$$V_2^c(t) = \frac{t^p}{\Gamma(p)} \int_0^\infty e^{-t\mu} \frac{\mu^{p+q-1}}{(1+\mu)^q} d\mu \ . \tag{9.6}$$

The result follows from (9.6) by making the change of variables $x = t\mu$.

Proof of Theorem 2.3. From p. 7 of Hardy [24], the Euler transformation is

$$\sum_{n=0}^{\infty} a_n (-s)^n = \frac{1}{1+s} \sum_{n=0}^{\infty} b_n \left(\frac{s}{1+s} \right)^n$$

where

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k \ .$$

However,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(p)_k}{(p+q)_k} = \frac{(q)_n}{(p+q)_n}$$

by (7.1) on p. 58 of Gould [21].

10. Concluding Remarks

A major theme here, extending Abate and Whitt [4], has been the development of an operational calculus for probability distributions. For instance, the two classes BME and B_2ME were related by the exponential damping operator in Theorem 1.6. Here we made greater use of moment sequences to characterize probability distributions. This is illustrated most clearly by Theorems 4.3, 5.3, 5.4, 8.1, and 8.2. Since scale mixtures correspond to products of independent random variables, the n^{th} moment of a scale mixture is easily seen to be the product of the component n^{th} moments. For distributions such as beta and gamma (which includes exponential), where the moments have a convenient explicit form, it is possible to work effectively with the moment sequences.

A natural extension to consider is BME distributions where the beta pdf has support $[a, b]$ for $0 < a < b$. By introducing the scale parameter, we already have implicitly treated the case $[0, b]$ for all $b > 0$, but $a > 0$ is different. Examples of these more general BME pdf's appear in Abate and Whitt [3]. For the M/M/1 queue, the beta mixing pdf's have support $[a, b]$ for $a > 0$. In the heavy-traffic limit with $\rho \to 1$, $a \to 0$. Hence, the results here apply to the RBM distributions in Abate and Whitt [1, 2, 3].

We conclude with an observation about BME pdf's (a conjecture by us proved by a referee). In particular, $1/(1+s)\hat{v}(p, q; s)$ is the Laplace transform of a bonafide pdf with mean $q/(p+q)$ and SCV $c^2 = 1 + 2p(p+q)/q(p+q+1)$.

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