This is an appendix to the main paper. Here is its abstract:

In order to understand queueing performance given only partial information about the model, we propose determining intervals of likely performance measures given that limited information. We illustrate this approach for the steady-state waiting time distribution in the $GI/GI/K$ queue given the first two moments of the interarrival-time and service-time distributions plus additional information about these underlying distributions, including support bounds, higher moments and Laplace transform values. As a theoretical basis, we apply the theory of Tchebycheff systems to determine extremal models (yielding tight upper and lower bounds) on the asymptotic decay rate of the steady-state waiting-time tail probability, as in the Kingman-Lundberg bound and large deviations asymptotics. We then can use these extremal models to indicate likely intervals of other performance measures. We illustrate by constructing such intervals of likely mean waiting times. Without extra information, the extremal models involve two-point distributions, which yield a wide range for the mean. Adding constraints on the third moment and a transform value produces three-point extremal distributions, which significantly reduce the range, yielding practical levels of accuracy.

Key words: performance approximations, queues, multi-server queues, bounds, mean waiting time, extremal queues, October 21, 2019

History: October 21, 2019

1. Overview of this Appendix

This appendix provides additional material expanding upon the main paper. It is divided into two parts. First, in §2-§5 we focus on the classic bounds based on the first two moments of the underlying cdf $F$ of an interarrival time $U$ and the cdf $G$ of a service times $V$, as in

\[(E[U], E[U^2], E[V], E[V^2]) \equiv (1, c_0^2, \rho, c_2^2)\]  

(1)
taken from (4) of the main paper. In these sections we consider the impact of the support bounds, but not yet the Laplace transform values, which play a prominent role in our main method for generating intervals of likely values for the mean in §4 of the main paper.

In §2 we discuss the wide range of possible values of the mean $E[W]$ given only the the first two moments of $F$ and $G$, without yet introducing the finite support bounds $M_a$ for $F$ and $M_s$ for $G$. Then in §3 we elaborate on §4.1 of the main paper on how to choose the support bounds. In §4 we supplement Table 1 of the main paper by presenting additional tables studying the direct application of Theorem 5 of the paper, which gives bounds based on adding only support bounds to the parameters in (1). In §5 we relate the support bound constraints to extra third-moment constraints. In particular, we show that for appropriate choices, there is an explicit one-to-one correspondence between the third moment and the support bound. Consequently, provided that we decide to use a support bound, we can specify either the third moment or the support bound, and the other will be determined. However, our approach in this paper is to introduce support bounds that should have only negligible impact on the mean waiting time. Thus, it should not be surprising that support bounds associated with natural third moments tend to take smaller values.

In §6-§7 we present additional results related to the Laplace transform constraints in Theorem 6 of the main paper, which forms the basis of our approach to the mean $E[W]$. In §6 we expand the study in §4.3 and §4.4 of the main paper, complementing Table 3 of the main paper. In §7 we present additional tables for the $M/M/1$ and $M/M/2$ model complementing Tables 2 and 4 of the main paper.


The standard way to evaluate approximations such as the heavy-traffic approximation (HTA)

$$E[W] \approx \frac{\rho^2(c_a^2 + c_s^2)}{2(1 - \rho)}.$$  

(2)

taken from (2) of the main paper is to compare it to simulation estimates for specific cases. This approach is of course excellent if we have a specific model we want to analyze. An alternative approach to obtain a broader understanding is to look at the set of all possible values, given the partial specification by the parameter 4-tuple in (1) when this can be done.
A principle conclusion of this line of work is that the range of possible values for $\mathbb{E}[W]$ given the partial information in (1) is remarkably wide. We now illustrate by providing simple approximation formulas for the absolute and relative errors, obtained by viewing the established bounds in a revealing way. In particular, it is helpful to look at the accuracy of the upper bound (UB) separately from the lower bound (LB), and it is helpful to use the simple HTA in (2) as a reference.

As an approximation for the UB, we use the (non-tight) Daley (1977) UB

$$\mathbb{E}[W] \leq \rho^2 \left( \frac{(2-\rho)c^2 + c^2_a}{2(1-\rho)} \right).$$

(3)

We could compute the exact tight UB using Chen and Whitt (2019), but we want to produce a simple formula. We use the tight LB

$$\mathbb{E}[W(LB)] = \frac{\rho((1+c^2s + \rho - 1)^+}{2(1-\rho)}.$$

(4)

The LB has long been known, see Stoyan and Stoyan (1974), §5.4 of Stoyan (1983), §V of Whitt (1984a), Ott (1987), Theorem 3.1 of Daley et al. (1992) and references there. It is significant that the LB is often 0 for smaller values of $\rho$; indeed it occurs whenever we can have $P(V-U \leq 0) = 1$, which cannot be effectively prevented by moment constraints alone. The main paper shows that a third moment and the transform constraints address this shortcoming.

Let the absolute upper error (AUE) and the relative upper error (RUE) of the heavy-traffic approximation (2) (HTA) be defined by the formulas

$$AUE \equiv UB - HTA \quad \text{and} \quad RUE \equiv \frac{UB - HTA}{HTA}.$$  

(5)

Similarly, let the absolute lower error (ALE) and the relative lower error (RLE) of the heavy-traffic approximation (2) be defined by the formulas

$$ALE \equiv HTA - LB \quad \text{and} \quad RUE \equiv \frac{HTA - LB}{HTA}.$$  

(6)

We subtract the smaller from the larger in each case, so that these measures of the possible errors are always positive. We use the HTA in the denominator because it produces more revealing simple formulas. (If we divided by the bound, then the RUE would decrease, but the RLE would increase.)

When $\rho$ and $\mathbb{E}[W]$ are not too small, it seems natural to focus on the relative error; otherwise it may be better to focus on the absolute error.
Proposition 1. (upper and lower errors for the mean) Suppose that we use the non-tight UB for the mean $E[W]$ in (3) and the tight LB in (4). For the LB, assume that $\rho > 1/(1+c^2_a)$; otherwise it must be 0. Then the upper and lower errors given the parameter four-tuple $(1, c^2_a, \rho, c^2_s)$ can be expressed as

\[
AUE = \rho c^2_a, \\
RUE = \left(\frac{2(1-\rho)}{\rho}\right) \left(\frac{c^2_a}{c^2_a+c^2_s}\right) \quad \text{or} \quad \frac{1-\rho}{\rho} \quad \text{if} \quad c^2_a = c^2_s, \\
ALE = \left(\frac{\rho^2}{2(1-\rho)}\right) \left(c^2_a + \frac{1-\rho}{\rho}\right) = \left(\frac{\rho^2 c^2_a}{2(1-\rho)}\right) + \frac{\rho}{2} \quad \text{and} \\
RLE = \frac{c^2_a + 1-\rho}{c^2_a + c^2_s} \quad \text{or} \quad \frac{1}{2} + \frac{1-\rho}{c^2_a} \quad \text{if} \quad c^2_a = c^2_s. 
\] (7)

Corollary 1. (monotonicity as functions of the parameters) The relative errors $RUE$ and $RLE$ in (7) are decreasing in $\rho$ and $c^2_s$ but are increasing in $c^2_a$.

Corollary 2. (heavy traffic and light traffic) The upper errors are asymptotically effective in the sense that $RUE(\rho) \to 0$ as $\rho \uparrow 1$, while $AUE(\rho) \to 0$ as $\rho \downarrow 0$. In contrast, $RLE(\rho) \to c^2_a/(c^2_a+c^2_s)$ as $\rho \uparrow 1$.

As in Corollary 1 of Whitt (1984a) for the $GI/M/1$ model, which shows that the overall relative error $(UB-LB)/LB$ for the mean queue length in the $GI/M/1$ model is $c^2_a$, Proposition 1 and Corollary 2 dramatically show the wide range of possible values. This suggests imposing further constraints on these distributions to concentrate on realistic “typical” cases, as was done in Kliincewicz and Whitt (1984) and Whitt (1984b) for the $GI/M/1$ model. This program was extended to phase-type distributions by Johnson and Taaffe (1991, 1993). The main paper carries out the same program for the more general $GI/GI/1$ and $GI/GI/K$ models in a new way (by initially focusing on the asymptotic decay rate).

3. Elaboration on Specifying Appropriate Support Bounds

In this section we elaborate on §4.1 of the main paper, expanding upon the discussion there. As we wrote there, most applications of the $GI/GI/1$ queueing model do not have interarrival-times and service-time distributions with finite support. We introduce the support bounds $M_a$ and $M_s$ as a device to help expose the typical range of possible values of the simple approximations for decay rate $\theta_W$ in equation (5) and §3.1 of the main paper.

We propose using values of $M_a$ and $M_s$ that should have negligible impact on the mean
waiting time in typical cases of interest, so that the bounds with \( M_a \) and \( M_s \) give a good indication of the likely set of possible values given the partial information. (§II.5.9 of Cohen (1982) provides theoretical support for this step.) Assumption 1 of the main paper about the critical singularity \( s^* \) of the moment generating function \( \hat{g}(-s) \) is critical. We show how to construct support bounds that are typical as well ones that are conservative.

3.1. Starting from a Model or Data

Starting from a specific model with unbounded \( U \) and \( V \), we suggest choosing the support bounds \( M_a \) and \( M_s \) so that

\[
P(U > M_a \mathbb{E}[U]) = P(U > M_a) = P(V > M_s \mathbb{E}[V]) = P(V > \rho M_s) = \epsilon
\]

for a suitably small \( \epsilon \) such as 0.001. We might take \( \epsilon = 0.0001 \) to be more conservative or \( \epsilon = 0.01 \) to narrow the range (but losing confidence in the reliability). With ample data, we would estimate the corresponding empirical complementary cdf (ccdf) of the service time, and use the same criterion in (8).

**Example 1.** (the \( M/M/1 \) case)

As a helpful orientation, we first consider the \( M/M/1 \) queue with arrival rate \( \lambda = 1 \) and mean service time \( \rho < 1 \). Notice that the service-time complementary cdf (ccdf) is

\[
P(V > x) = e^{-x/\rho}, \quad x \geq 0,
\]

so that its decay rate \( \theta_V \equiv \lim_{x \to \infty} \{-\log(P(V > x))/x\} \) is independent of \( x \), i.e.,

\[
\theta_V = -\log(P(V > x))/x = 1/\rho \quad \text{for all} \quad x.
\]

while the associated waiting time ccdf is

\[
P(W > x) = \rho e^{-(1-\rho)x/\rho}, \quad x \geq 0,
\]

so that its decay rate is

\[
\theta_W \equiv \lim_{x \to \infty} -\log(P(W > x)/x = (1 - \rho)/\rho.
\]

Hence, for the \( M/M/1 \) model we see that \( \theta_W/\theta_V = 1 - \rho < 1 \) which quantifies the well-known property that large waiting times are more likely than large service times, becoming ever more so as the traffic intensity approaches 1.
Moreover, in the present light-tail case, provided that $\rho$ is not too small, large waiting times are likely to be the result of several service times associated with a cluster of arrivals rather than one especially large service time.

For the $M/M/1$ model where $U$ and $V$ have exponential distributions, the target in (8) becomes $e^{-M} = \epsilon$. For $M_a = M_s = M = (4, 5, 6, 7, 8, 9, 10)$, the corresponding values are $\epsilon(M) = (0.0183, 0.0067, 0.0025, 0.0091, 0.0033, 0.01123, 0.00045)$. We use 7 and 9 in our experiments later.

Based on this analysis, we conduct a simulation comparison to show how the support bounds affect the decay rate $\theta_W$ of the extremal queues for the case $\rho = 0.7$ and $c_a^2 = c_s^2 = 1$ in Table 1. We implement Monte-Carlo Simulation with $N = 10^8$ and $R = 20$ to simulate the tail probability with different quantities and report 95% confidence interval length (CIL).

Table 1: Simulation comparison of the waiting time ccdf and delay rate $\theta_W$ for two-point extremal models with $M_a = \{1 + c_a^2, 5, 10\}$ and $M_s = \{1 + c_s^2, 5, 10\}$ under $c_a^2 = c_s^2 = 1, \rho = 0.7$

<table>
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<tr>
<th>$M_a = 2, M_s = 10$</th>
<th>$x = 10$</th>
<th>CIL</th>
<th>$x = 12$</th>
<th>CIL</th>
<th>$x = 14$</th>
<th>CIL</th>
<th>$x = 16$</th>
<th>CIL</th>
<th>$x = 18$</th>
<th>CIL</th>
</tr>
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<tbody>
<tr>
<td>$P(W &gt; x)$</td>
<td>2.99E-02</td>
<td>3.65E-05</td>
<td>3.54E-01</td>
<td>3.63E-05</td>
<td>8.85E-03</td>
<td>2.82E-05</td>
<td>4.83E-03</td>
<td>1.49E-05</td>
<td>2.64E-03</td>
<td>1.16E-05</td>
</tr>
<tr>
<td>$- \log(P(W &gt; x))/x$</td>
<td>3.54E-01</td>
<td>3.44E-01</td>
<td>3.38E-01</td>
<td>3.33E-01</td>
<td>3.30E-01</td>
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<td>3.30E-01</td>
<td>3.30E-01</td>
<td>3.30E-01</td>
</tr>
</tbody>
</table>

Table 1 shows that there is rapid convergence of $- \log(P(W > x))/x$ to the decay rate $\theta_W$ as $x$ increases; it is not necessary to make $x$ extraordinarily large. Notice that the estimated decay rate is monotone in Table 1, with the $M/M/1$ exact value $(1 - \rho)/\rho = 0.3/0.7 = 0.4285$ bounded below and above by the values for $(M_a, M_s) = (2, 5)$ and $(5, 2)$ taken from the last column of Table 1.
4. Direct Application of Theorem 5 to the Mean $\mathbb{E}[W]$

We now elaborate on §4.1 of the main paper by providing additional results about how the extremal UB model $F_0/G_u/1$ and LB model $F_u/G_0/1$ for the decay rate from Theorem 5 of the main paper apply to the mean $\mathbb{E}[W]$ with $K = 1$ when we introduce the support bounds $M_a$ and $M_s$ following the prescription in §4.1 of the main paper.

This issue relates strongly to Chen and Whitt (2018), which studied the extremal models for $E[W]$. For the mean, there is strong evidence (but not yet a mathematical proof) that the model $F_0/G_u/1$ directly yields the UB for the mean, for both bounded and unbounded support, just as it does for the decay rate. However, the situation is different for the LB, as discussed in §7 and EC.6 of Chen and Whitt (2018). For unbounded support, the tight LB is given here in (4). It is attained by the $D/A_3/1$ model, where $A_3$ denotes a three-point distribution, which has all mass on multiples of the deterministic interarrival time. The $D$ interarrival time violates the moment condition, but nevertheless is attained asymptotically. In Chen and Whitt (2018) we present evidence that the $F_u/A_3(u)/1$ model attains the LB, where $A_3(u)$ is a natural analog of $A_3$.

4.1. Elaborating on Table 1 of the Main Paper

Table 1 of the main paper shows how the support bounds reduce the range of the possible values of $\mathbb{E}[W]$. It reports results for the cases $(c_a,c_s^2) = (1.0,1.0)$, $(4.0,4.0)$, $(0.5,0.5)$, $(4.0,0.5)$, $(0.5,4.0)$.

We now supplement Table 1 of the main paper by showing the UB and LB for the mean $\mathbb{E}[W]$ with the support bounds chosen to satisfy (8) with targets $\epsilon = 0.001$ and $0.0001$ for all four cases of $c_a^2,c_s^2 \in \{0.5,4.0\}$ for 10 values of $\rho$. 


## Table 2: Evaluation of $E[W]$ for $F_u/G_0/1$ and $F_0/G_u/1$ with $(M_a, M_s)$ for $c^2_a = c^2_s = 4$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Tight LB</th>
<th>$M_a = 39.9$</th>
<th>$M_a = 31.1$</th>
<th>HTA</th>
<th>$M_s = 31.1$</th>
<th>$M_s = 39.3$</th>
<th>Tight UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.044</td>
<td>0.401</td>
<td>0.402</td>
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<td>0.026</td>
<td>0.033</td>
<td>0.200</td>
<td>0.867</td>
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<td>0.30</td>
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<td>0.514</td>
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</tr>
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<td>0.40</td>
<td>0.333</td>
<td>0.458</td>
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<td>2.304</td>
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<td>0.50</td>
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<td>1.013</td>
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</tr>
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## Table 3: Evaluation of $E[W]$ for $F_u/G_0/1$ and $F_0/G_u/1$ with $(M_a, M_s)$ for $c^2_a = c^2_s = 0.5$

<table>
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<th>$M_a = 4.5$</th>
<th>$M_a = 3.5$</th>
<th>HTA</th>
<th>$M_s = 3.5$</th>
<th>$M_s = 3.5$</th>
<th>Tight UB</th>
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Table 4 Evaluation of $E[W]$ for $F_u/G_0/1$ and $F_0/G_a/1$ with $(M_a, M_s)$ for $c^2_a = 4, c^2_s = 0.5$

<table>
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<tr>
<th>$\rho$</th>
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<th>$M_a = 39.9$</th>
<th>$M_a = 31.1$</th>
<th>HTA</th>
<th>$M_s = 3.5$</th>
<th>$M_s = 4.5$</th>
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<td>20.546</td>
</tr>
<tr>
<td>0.95</td>
<td>4.037</td>
<td>32.083</td>
<td>34.934</td>
<td>40.613</td>
<td>42.955</td>
<td>42.970</td>
<td>43.033</td>
</tr>
</tbody>
</table>

Table 5 Evaluation of $E[W]$ for $F_u/G_0/1$ and $F_0/G_a/1$ with $(M_a, M_s)$ for $c^2_a = 0.5, c^2_s = 4.0$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Tight LB</th>
<th>$M_a = 4.5$</th>
<th>$M_a = 3.5$</th>
<th>HTA</th>
<th>$M_s = 31.1$</th>
<th>$M_s = 39.9$</th>
<th>Tight UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.025</td>
<td>0.064</td>
<td>0.065</td>
<td>0.072</td>
</tr>
<tr>
<td>0.20</td>
<td>0.000</td>
<td>0.034</td>
<td>0.046</td>
<td>0.113</td>
<td>0.184</td>
<td>0.188</td>
<td>0.200</td>
</tr>
<tr>
<td>0.30</td>
<td>0.107</td>
<td>0.179</td>
<td>0.192</td>
<td>0.289</td>
<td>0.388</td>
<td>0.393</td>
<td>0.409</td>
</tr>
<tr>
<td>0.40</td>
<td>0.333</td>
<td>0.462</td>
<td>0.487</td>
<td>0.600</td>
<td>0.720</td>
<td>0.726</td>
<td>0.746</td>
</tr>
<tr>
<td>0.50</td>
<td>0.750</td>
<td>0.957</td>
<td>0.988</td>
<td>1.125</td>
<td>1.263</td>
<td>1.270</td>
<td>1.289</td>
</tr>
<tr>
<td>0.60</td>
<td>1.500</td>
<td>1.841</td>
<td>1.869</td>
<td>2.025</td>
<td>2.176</td>
<td>2.186</td>
<td>2.212</td>
</tr>
<tr>
<td>0.70</td>
<td>2.917</td>
<td>3.464</td>
<td>3.494</td>
<td>3.675</td>
<td>3.841</td>
<td>3.851</td>
<td>3.875</td>
</tr>
<tr>
<td>0.80</td>
<td>6.000</td>
<td>6.973</td>
<td>6.985</td>
<td>7.200</td>
<td>7.374</td>
<td>7.379</td>
<td>7.422</td>
</tr>
<tr>
<td>0.90</td>
<td>15.750</td>
<td>17.973</td>
<td>17.993</td>
<td>18.225</td>
<td>18.408</td>
<td>18.427</td>
<td>18.470</td>
</tr>
<tr>
<td>0.95</td>
<td>35.625</td>
<td>40.183</td>
<td>40.322</td>
<td>40.613</td>
<td>40.811</td>
<td>40.826</td>
<td>40.871</td>
</tr>
</tbody>
</table>

Table 1 of the main paper and Tables 2-5 above show that the support bounds reduce the range of possible value in all cases. The tables also show that the cases differ dramatically. Just as in Corollaries 1 and 2, we see that the relative errors are remarkably small for
($c_a^2, c_s^2) = (0.5, 4.0)$, but remarkably large for $(c_a^2, c_s^2) = (4.0, 0.5)$, even with the support bounds.

4.2. Using Heavy-Traffic Approximations

To show that we could also start from the HT approximations for the mean $E[W]$ in (2) and for the decay rate $\theta_W$ in (8) of the main paper instead of the exact models based on $E_2$ and $H_2$ distributions, Table 6 compares the exact values of $\theta_W$ and $E[W]$ to these heavy-traffic approximations. Table 6 shows that the HTA in (2) overestimates the exact value when $c_a^2 = 0.5$, which is consistent with the refinement in (44) and (45) of Whitt (1983a).

<table>
<thead>
<tr>
<th>$c_a^2 = c_s^2 = 0.5$</th>
<th>$c_a^2 = c_s^2 = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Exact $\theta_W$</td>
</tr>
<tr>
<td>0.5</td>
<td>2.00</td>
</tr>
<tr>
<td>0.7</td>
<td>0.857</td>
</tr>
<tr>
<td>0.9</td>
<td>0.222</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c_a^2 = 4, c_s^2 = 0.5$</th>
<th>$c_a^2 = 0.5, c_s^2 = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Exact $\theta_W$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.826</td>
</tr>
<tr>
<td>0.7</td>
<td>0.260</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0537</td>
</tr>
</tbody>
</table>

These tables show that the support bounds reduce the range of possible value in all cases. The tables also show that the cases differ dramatically. Just as in Corollaries 1 and 2, we see that the relative errors are remarkably small for $(c_a^2, c_s^2) = (0.5, 4.0)$, but remarkably large for $(c_a^2, c_s^2) = (4.0, 0.5)$, even with the support bounds.

5. Relating Third Moments to Support Bounds

So far, we have obtained a reduced range of possible values of $E[W]$, by introducing the support bounds $M_a$ and $M_s$ in addition to the model parameters $(1, c_a^2, \rho, c_s^2)$. We can then apply Theorem 5 of the main paper. In §3 we chose $M_a$ and $M_s$ so that the approximate tail probability was suitably small, as in (8). To cover typical distributions, we used the approximate tail probabilities based on the decay rates of typical distributions.
An alternative way is to exploit third moments. For third moments, we might also specify candidate values by looking at candidate distributions with the given parameters \((1, c_a^2, \rho, c_s^2)\). Indeed that was done in §5.1 of Whitt (1983b), and we use the same prescription here. For \(c_a^2 \geq 1\), based on the \(H_2\) distribution with balanced means as before, \(m_{3,a} = 3c_a^2(1 + c_a^2)\). For \(c_a^2 < 1\), based on the \(E_k\) distribution, let \(m_{3,a} = (2c_a^2 + 1)(c_a^2 + 1)\).

We apply these “typical” third moments to go with \((1, c_a^2, \rho, c_s^2)\) by relating the third moments to the support bounds \(M\) associated with \(F_u\) and \(G_u\). We observe that the third moment of \(F_u\) is

\[
m_U^3 = \frac{c_a^2 M_a^3}{c_a^2 + (M_a - 1)^2} + \frac{(M_a - 1 - c_a^2)^3}{(M_a - 1)(c_a^2 + (M_a - 1)^2)},
\]

while the third moment of \(F_0\) is

\[
m_L^3 = \frac{c_a^2(1 + c_a^2)^3}{c_a^2 + c_a^4}.
\]

Now observe that the third moment in (13) is a strictly increasing function of \(M_a\), so that we can invert it to obtain \(M_a\) as a function of \(m_3\), getting

\[
M_a = \frac{-1 - c_a^2 + m_U^3 + \sqrt{1 + 6c_a^2 + 9c_a^2 + 4c_a^3 - 2m_U^3 - 6c_a^2m_U^3 + (m_U^3)^2}}{2c_a^2}.
\]

Hence, given typical values of \(m_3\) associated with any parameter 4-tuple \((1, c_a^2, \rho, c_s^2)\), we can construct a corresponding support bound \(M_a^*\) for which we can determine the range of possible mean values. With this approach, we obtain \(M_a^* = 13.081\) for \(c_a^2 = 4\), \(M_a^* = 3.414\) for \(c_a^2 = 1\) and \(M_a^* = 2.366\) for \(c_a^2 = 0.5\). We note that these values are substantially smaller than the values determined in §3. Table 7 presents the numerical ranges of third moment as a function of \(c^2\) and \(M\).

<table>
<thead>
<tr>
<th>(M)</th>
<th>5.25</th>
<th>6.75</th>
<th>7.00</th>
<th>9.00</th>
<th>17.5</th>
<th>22.5</th>
<th>(m_L^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c^2) = 0.5</td>
<td>4.566</td>
<td>5.332</td>
<td>5.458</td>
<td>6.469</td>
<td>10.735</td>
<td>13.238</td>
<td>2.25</td>
</tr>
<tr>
<td>(c^2) = 4</td>
<td>26.235</td>
<td>33.217</td>
<td>34.333</td>
<td>43.000</td>
<td>78.030</td>
<td>98.256</td>
<td>25.000</td>
</tr>
</tbody>
</table>

We again applied simulation to study the performance of the extremal queues based on the third moments in addition to the basic model parameters in (1). As a first step, we
use the support bounds that come from the third moments via (15). Based on Theorem 5 of the main paper, the candidate UB and LB models for the mean $E[W]$, based on the reverse order for $\theta_W$, are $F_0/G_u/1$ and $F_u/G_0/1$. As shown in Chen and Whitt (2018), there is strong evidence that $F_0/G_u/1$ actually yields the tight UB for the mean $E[W]$, but it is known that $F_u/G_0/1$ is not actually the tight LB for $E[W]$, although it is close; see §7 and §EC.6 of Chen and Whitt (2018). There it is conjectured that the LB for $E[W]$ is attained by a special three-point distribution, denoted by $A_3(u)$. Table 8 compares the resulting LB and UB extremal queues to the HTA in (2). We include results for both the $F_u/A_3(u)/1$ and $F_u/G_0/1$ candidate LB models. (We again refer to §§7 and EC.6 for background.)

<table>
<thead>
<tr>
<th>$c^2_a = 4, c^2_s = 0.5, M_a = 13.1, M_s = 2.37$</th>
<th>$c^2_a = 4, c^2_s = 4, M_a = 13.1, M_s = 13.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$E[W(F_u, A_3(u))]$</td>
</tr>
<tr>
<td>--------</td>
<td>------------------</td>
</tr>
<tr>
<td>0.50</td>
<td>0.040</td>
</tr>
<tr>
<td>0.60</td>
<td>0.174</td>
</tr>
<tr>
<td>0.70</td>
<td>1.213</td>
</tr>
<tr>
<td>0.90</td>
<td>17.447</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c^2_a = 0.5, c^2_s = 0.5, M_a = 2.37, M_s = 2.37$</th>
<th>$c^2_a = 0.5, c^2_s = 4, M_a = 2.37, M_s = 13.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$E[W(F_u, A_3(u))]$</td>
</tr>
<tr>
<td>--------</td>
<td>------------------</td>
</tr>
<tr>
<td>0.50</td>
<td>0.058</td>
</tr>
<tr>
<td>0.60</td>
<td>0.200</td>
</tr>
<tr>
<td>0.70</td>
<td>0.537</td>
</tr>
<tr>
<td>0.80</td>
<td>1.336</td>
</tr>
<tr>
<td>0.90</td>
<td>3.749</td>
</tr>
</tbody>
</table>

From Table 8, we see that

$$E[W(F_u, A_3(u))] \leq E[W(F_u, G_0)] \leq E[W(F_0, G_u)]$$

in all cases. In addition,

$$E[W(F_u, G_0)] \leq HTA \leq E[W(F_0, G_u)]$$

in all cases except $(c^2_a, c^2_s) = (4.0, 4.0)$. In that case, the smaller values of $M$ than produced by (8) makes it important to use the better LB model $(F_u/A_3(u))$. 


Most important, we see that our range of possible values of the mean $\mathbb{E}[W]$ is reduced substantially by adding the additional parameters $(M_a, M_s)$ obtained from $(m_{a,3}, m_{s,3})$. To illustrate, note that the range in the case $(\rho, c_{a}^2, c_{s}^2) = (0.8, 4, 4)$ is reduced from $[6.000, 14.917]$ to $[10.593, 14.758]$. The change is obviously much greater for $F$ than for $G$.

6. The Impact of the Laplace Transform Constraints

We now elaborate on §4 of the main paper, which investigates the application of Theorem 6 in the main paper to obtain shorter intervals of likely values for the mean $E[W]$ by exploiting values of the Laplace transform $\hat{f}(s)$ and the moment generating function (mgf) $\hat{g}(-s)$. Recall that the Laplace transform is defined as

$$\hat{f}(s) \equiv \int_0^\infty e^{-st} dF(t), \quad s \geq 0 \tag{16}$$

When we look at $\hat{g}(-s)$, it corresponds to the mgf, i.e.,

$$\hat{g}(-s) \equiv \int_0^\infty e^{st} dG(t) = E[e^{sV}], \quad s \geq 0. \tag{17}$$

We now show how a direct application of Theorem 6 in the main paper reduces the range. In this section we avoid issues involving the singularity $s^*$ in Assumption 1 of the main paper, by primarily considering case (ii) in (30) of Theorem 6 in the main paper, in which

$$\mu_s < \theta_W < \mu_a, \tag{18}$$

which we achieve by following (7) of the main paper, i.e.,

$$\mu_s \equiv \theta_W/R \quad \text{and} \quad \mu_a \equiv R\theta_W \tag{19}$$

for suitable $R$. We begin by considering a range of $R$.

6.1. The Impact of Truncation

We initially truncate the basic models by $M_a, M_s$ because Theorem 6 of the main paper only applies to models with bounded support. In the implementation, we do not want to $\mu_s > s^*$. Thus, if we are considering one of the cases with $\mu_s \geq \theta_W$, then we first check to see if $R\theta_W > s^*$ for our largest value of $R$, which we take to be $R = 20$. If it is, then we create alternative values of $\mu_s$ in the interval $(\theta_W, s^*)$. In particular, we use

$$\mu_s \equiv \theta_W + \left(\frac{R}{25}\right)(s^* - \theta_W), \quad 1 \leq k \leq 4, \tag{20}$$
so that the values of $R$ remain in $\{5, 10, 15, 20\}$, but all values are within $(\theta_W, s^*)$.

Table 9 shows a careful comparison between parameters under truncation or not.

<table>
<thead>
<tr>
<th>$E_2/H_2/1$</th>
<th>Truncated</th>
<th>Original</th>
<th>Truncated</th>
<th>Original</th>
<th>$M/M/1$</th>
<th>Truncated</th>
<th>Original</th>
<th>Truncated</th>
<th>Original</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_a = R_s$</td>
<td>$\hat{f}(s)$</td>
<td>$\hat{f}(s)$</td>
<td>$\hat{g}(-s)$</td>
<td>$\hat{g}(-s)$</td>
<td>$R_a = R_s$</td>
<td>$\hat{f}(s)$</td>
<td>$\hat{f}(s)$</td>
<td>$\hat{g}(-s)$</td>
<td>$\hat{g}(-s)$</td>
</tr>
<tr>
<td>1</td>
<td>0.8630</td>
<td>0.8632</td>
<td>1.1568</td>
<td>1.1585</td>
<td>1</td>
<td>0.6998</td>
<td>0.7000</td>
<td>1.4293</td>
<td>1.4286</td>
</tr>
<tr>
<td>5</td>
<td>0.5229</td>
<td>0.5238</td>
<td>1.4582</td>
<td>1.5498</td>
<td>5</td>
<td>0.3180</td>
<td>0.3182</td>
<td>4.9983</td>
<td>4.9779</td>
</tr>
<tr>
<td>10</td>
<td>0.3205</td>
<td>0.3216</td>
<td>1.4582</td>
<td>1.5498</td>
<td>10</td>
<td>0.1890</td>
<td>0.1892</td>
<td>4.9983</td>
<td>4.9779</td>
</tr>
<tr>
<td>20</td>
<td>0.1558</td>
<td>0.1566</td>
<td>1.4582</td>
<td>1.5498</td>
<td>20</td>
<td>0.1044</td>
<td>0.1045</td>
<td>4.9983</td>
<td>4.9779</td>
</tr>
</tbody>
</table>

Additionally, the first three moments with and without truncation are close, i.e, $s_2 = 2.44, s_3 = 20.191$ for the truncated model and $s_2 = 2.45, 20.58$ for the original model for $E_2/H_2/1$. Since difference between parameters are negligible, it may suffice to apply the original $m_{a,2}, m_{a,3}$ instead of $m''_{a,2}, m''_{a,3}$ for reducing computation complexity. In other word, we could ignore the truncation effect and simply apply Theorem 6 of the main paper using parameters of the basic models without truncation.

6.2. The Parameter Pair $(R_a, R_s)$

However, we also report results exploring a more general two-parameter range, using $(R_a, R_s)$ with $R_a$ applying to $F$ and $R_s$ applying to $G$. In summary, we proceed as follows: Given an initially specified decay rate $\theta_W$, the range vector $(R_a, R_s)$ with $R_s \leq 1 \leq R_a$ and the specified parameters $(1, c^2_a, m_{a,3}, \mu_a, M_a)$ partially characterizing $F$ and $(1, c^2_s, m_{s,3}, \mu_s, M_s)$ partially characterizing $G$, where $\mu_s \equiv \theta_W/R_s \leq \theta_W < R_a \theta_W$, we identify the set of possible performance measures in two steps.

In the first step, we can determine the extremal distributions $F_L, G_L, F_U, G_U$ by solving $n$ equations in $n$ unknowns for the appropriate $n$. In the second step, we simulate $E[W(F_L, G_L)]$ and $E[W(F_U, G_U)]$ by Monte-Carlo simulation and obtain decay rates by solving equation (6) of the main paper for the LB and UB models $F_L/G_L$ and $F_U/G_U$.

We now illustrate the results.
6.3. The $H_2/H_2/1$ Model with $c_a^2 = c_s^2 = 4.0$

We use UB (LB) to refer to the minimum (maximum) decay rate, which yields our estimate of the UB (LB) for $E[W]$. Table 10 shows estimates of the UB and LB for the decay rate $\theta_W$ and the mean $E[W]$ in the case $c_a^2 = c_s^2 = 4$ and $\rho = 0.7$ for a range of $R_a$ and $R_s$ varying from 1 to 20, based on the first three moments and LT transforms from model $H_2/H_2/1$ with balanced means, which has exact mean $E[W(H_2, H_2)] = 6.608$ and exact decay rate $\theta_W = 0.1064$. (See Table 6.)

As indicated above, here we allow $R_a$ and $R_s$ to differ, but we still require that $\mu_s \equiv \theta_W/R_s$ and $\mu_a \equiv R_a \theta$ for $R_a \geq 1$ and $R_s \geq 1$.

<table>
<thead>
<tr>
<th>$R_s \backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s \backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (UB)</td>
<td>1</td>
<td>0.106</td>
<td>0.105</td>
<td>0.104</td>
<td>0.104</td>
<td>$E[W]$ (UB)</td>
<td>1</td>
<td>6.292</td>
<td>6.197</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.106</td>
<td>0.105</td>
<td>0.104</td>
<td>0.103</td>
<td></td>
<td>5</td>
<td>6.329</td>
<td>6.275</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.106</td>
<td>0.105</td>
<td>0.104</td>
<td>0.103</td>
<td></td>
<td>10</td>
<td>6.336</td>
<td>6.278</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.100</td>
<td>0.099</td>
<td>0.099</td>
<td>0.098</td>
<td></td>
<td>20</td>
<td>6.884</td>
<td>7.047</td>
</tr>
<tr>
<td>$\theta_W$ (LB)</td>
<td>1</td>
<td>0.106</td>
<td>0.107</td>
<td>0.108</td>
<td>0.108</td>
<td>$E[W]$ (LB)</td>
<td>1</td>
<td>6.667</td>
<td>6.516</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.107</td>
<td>0.108</td>
<td>0.108</td>
<td>0.108</td>
<td></td>
<td>5</td>
<td>6.692</td>
<td>6.485</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.107</td>
<td>0.108</td>
<td>0.108</td>
<td>0.108</td>
<td></td>
<td>10</td>
<td>6.693</td>
<td>6.474</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.109</td>
<td>0.110</td>
<td>0.110</td>
<td>0.110</td>
<td></td>
<td>20</td>
<td>6.866</td>
<td>6.664</td>
</tr>
</tbody>
</table>

Consistent with part (d) of Theorem 6 in the main paper, Table 10 show that the UB decay rate $\theta_W$ is monotone decreasing in $R_a$ and $R_s$, while the LB decay rate is monotone increasing. Moreover, recall we utilize information from the exact queueing models where $F$ and $G$ have unbounded support, so that we do not expect perfect consistency. On the other hand, there is less order in the corresponding values of $E[W]$. Nevertheless, from Table 10, we conclude that a reasonable range of $E[W]$ can be generated by $R_a = R_s = 20$.

To elaborate further, Table 11 shows the explicit numerical values of the three-point extremal distributions $F_L, G_L$ and $F_U, G_U$ obtained in the case $c_a^2 = c_s^2 = 4, \rho = 0.7$ with $R = R_a = R_s \in \{1, 5, 10, 20\}$, supporting Table 10.
Table 11  Numerical examples of extremal distributions

<table>
<thead>
<tr>
<th>$R_a = R_s = 1$</th>
<th>$F$</th>
<th>$G$</th>
<th>$R_a = R_s = 5$</th>
<th>$F$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_L/G_L/1$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$0.620$</td>
<td>$0.370$</td>
<td>$1.04E-02$</td>
<td>$0.677$</td>
<td>$0.317$</td>
<td>$6.08E-03$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$y_3$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$0$</td>
<td>$2.21$</td>
<td>$17.6$</td>
<td>$0$</td>
<td>$1.93$</td>
<td>$14.4$</td>
</tr>
<tr>
<td>$F_U/G_U/1$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$0.956$</td>
<td>$0.0433$</td>
<td>$2.88E-04$</td>
<td>$0.965$</td>
<td>$0.0345$</td>
<td>$1.73E-04$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$y_3$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$0.587$</td>
<td>$9.86$</td>
<td>$39.9$</td>
<td>$0.440$</td>
<td>$7.86$</td>
<td>$27.9$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$R_a = R_s = 10$</th>
<th>$F$</th>
<th>$G$</th>
<th>$R_a = R_s = 20$</th>
<th>$F$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_L/G_L/1$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$0.451$</td>
<td>$0.530$</td>
<td>$1.87E-02$</td>
<td>$0.654$</td>
<td>$0.338$</td>
<td>$7.88E-03$</td>
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<tr>
<td>$y_1$</td>
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<td>$y_3$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
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<td>$1.37$</td>
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<tr>
<td>$F_U/G_U/1$</td>
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<td>$q_2$</td>
<td>$q_3$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$0.917$</td>
<td>$0.0828$</td>
<td>$5.02E-04$</td>
<td>$0.962$</td>
<td>$0.0374$</td>
<td>$2.17E-04$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$y_2$</td>
<td>$y_3$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>$0.439$</td>
<td>$6.97$</td>
<td>$39.9$</td>
<td>$0.430$</td>
<td>$7.50$</td>
<td>$27.9$</td>
</tr>
</tbody>
</table>

Next, Figure 1 plots the extremal Laplace transforms $\hat{f}(s)$ and $1/\hat{g}(-s)$ for UB (LHS) and LB (RHS) for the case $c_a^2 = c_s^2 = 4$ and $\rho = 0.7$. The curves intersect at the decay rate $\theta_W$. The decay rate for $R_a = R_s = 1$ is 0.106, while for $R_a = R_s = 20$ it is 0.098 for the UB and 0.110 for the LB.

![Figure 1](image-url)  
Figure 1  Display of $\hat{f}(s)$ and $1/\hat{g}(-s)$ for UB (LHS) and LB (RHS) for the case $c_a^2 = c_s^2 = 4$ and $\rho = 0.7$: the decay rate for $R_a = R_s = 1$ is 0.106 and for $R_a = R_s = 20$ in UB is 0.098 and in LB is 0.110
Next Tables 12 and 13 show the estimated extremal values of $\theta_W$ and $E[W]$ as a function of $R_a, R_s \in \{1, 5, 10, 20\}$ based on simulation for $\rho = 0.5, 0.9$ for this same case $(c_a^2, c_s^2) = (4, 4)$.

### Table 12  The improved LB and UB based on information of $H_2/H_2/1$ with $\rho = 0.5$

<table>
<thead>
<tr>
<th>$R_s\backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s\backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (UB)</td>
<td>1</td>
<td>0.244</td>
<td>0.235</td>
<td>0.231</td>
<td>0.227</td>
<td>$E[W]$ (UB)</td>
<td>1</td>
<td>1.97</td>
<td>2.32</td>
</tr>
<tr>
<td>5</td>
<td>0.239</td>
<td>0.231</td>
<td>0.227</td>
<td>0.224</td>
<td>5</td>
<td>1.95</td>
<td>2.02</td>
<td>1.97</td>
<td>2.20</td>
</tr>
<tr>
<td>10</td>
<td>0.239</td>
<td>0.230</td>
<td>0.227</td>
<td>0.224</td>
<td>10</td>
<td>1.96</td>
<td>2.03</td>
<td>1.98</td>
<td>2.20</td>
</tr>
<tr>
<td>20</td>
<td>0.238</td>
<td>0.230</td>
<td>0.227</td>
<td>0.224</td>
<td>20</td>
<td>1.96</td>
<td>2.03</td>
<td>1.99</td>
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<table>
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<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (LB)</td>
<td>1</td>
<td>0.244</td>
<td>0.251</td>
<td>0.253</td>
<td>0.255</td>
<td>$E[W]$ (LB)</td>
<td>1</td>
<td>2.09</td>
<td>1.87</td>
</tr>
<tr>
<td>5</td>
<td>0.250</td>
<td>0.258</td>
<td>0.261</td>
<td>0.263</td>
<td>5</td>
<td>2.12</td>
<td>1.83</td>
<td>1.80</td>
<td>1.75</td>
</tr>
<tr>
<td>10</td>
<td>0.251</td>
<td>0.258</td>
<td>0.261</td>
<td>0.263</td>
<td>10</td>
<td>2.12</td>
<td>1.84</td>
<td>1.80</td>
<td>1.75</td>
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<tr>
<td>20</td>
<td>0.251</td>
<td>0.259</td>
<td>0.261</td>
<td>0.264</td>
<td>20</td>
<td>2.12</td>
<td>1.85</td>
<td>1.80</td>
<td>1.75</td>
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</table>

### Table 13  The improved LB and UB based on information of $H_2/H_2/1$ for $\rho = 0.9$

<table>
<thead>
<tr>
<th>$R_s\backslash R_a$</th>
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<th>$R_s\backslash R_a$</th>
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<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (UB)</td>
<td>1</td>
<td>0.0278</td>
<td>0.0277</td>
<td>0.0277</td>
<td>0.0277</td>
<td>$E[W]$ (UB)</td>
<td>1</td>
<td>32.9</td>
<td>31.4</td>
</tr>
<tr>
<td>5</td>
<td>0.0278</td>
<td>0.0277</td>
<td>0.0277</td>
<td>0.0277</td>
<td>5</td>
<td>32.8</td>
<td>31.8</td>
<td>31.5</td>
<td>32.1</td>
</tr>
<tr>
<td>10</td>
<td>0.0278</td>
<td>0.0277</td>
<td>0.0277</td>
<td>0.0277</td>
<td>10</td>
<td>32.8</td>
<td>31.7</td>
<td>31.6</td>
<td>32.2</td>
</tr>
<tr>
<td>20</td>
<td>0.0278</td>
<td>0.0277</td>
<td>0.0277</td>
<td>0.0277</td>
<td>20</td>
<td>33.0</td>
<td>31.6</td>
<td>31.5</td>
<td>32.2</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>$R_s\backslash R_a$</th>
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<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s\backslash R_a$</th>
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<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (LB)</td>
<td>1</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>$E[W]$ (LB)</td>
<td>1</td>
<td>32.8</td>
<td>32.6</td>
</tr>
<tr>
<td>5</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>5</td>
<td>32.8</td>
<td>32.7</td>
<td>32.9</td>
<td>33.4</td>
</tr>
<tr>
<td>10</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>10</td>
<td>32.7</td>
<td>32.8</td>
<td>32.7</td>
<td>33.4</td>
</tr>
<tr>
<td>20</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>0.0278</td>
<td>20</td>
<td>32.7</td>
<td>32.9</td>
<td>32.9</td>
<td>32.4</td>
</tr>
</tbody>
</table>

### 6.4. The $H_2/E_2/1$ Model with $c_a^2 = 4.0, c_s^2 = 0.5$

Next, Tables 14-16 show corresponding results for the case $c_a^2 = 4, c_s^2 = 0.5$, based on the first third moments and LT transform values from the model $H_2/E_2/1$, again using $H_2$ with balanced means. The exact values for the original $H_2/E_2/1$ model are given in Table 6. The exact values for $\rho = 0.7$ are $E[W(H_2, E_2)] = 3.368$ and exact decay rate $\theta_W = 0.2602$. 
Table 14  The improved LB and UB based on information of $H_2/E_2/1$

$c_2^a = 4, c_2^s = 0.5, \rho = 0.7, \theta_w = 0.2602$ and exact $E[W(H_2,E_2)] = 3.368$

<table>
<thead>
<tr>
<th>$R_s \setminus R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s \setminus R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (UB)</td>
<td>1</td>
<td>0.260</td>
<td>0.228</td>
<td>0.218</td>
<td>0.210</td>
<td>$E[W]$ (UB)</td>
<td>1</td>
<td>3.438</td>
<td>3.920</td>
</tr>
<tr>
<td>5</td>
<td>0.260</td>
<td>0.228</td>
<td>0.218</td>
<td>0.210</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.260</td>
<td>0.228</td>
<td>0.217</td>
<td>0.210</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.260</td>
<td>0.228</td>
<td>0.217</td>
<td>0.210</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_W$ (LB)</td>
<td>1</td>
<td>0.260</td>
<td>0.291</td>
<td>0.306</td>
<td>0.321</td>
<td>$E[W]$ (LB)</td>
<td>1</td>
<td>3.343</td>
<td>2.951</td>
</tr>
<tr>
<td>5</td>
<td>0.260</td>
<td>0.291</td>
<td>0.306</td>
<td>0.321</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.260</td>
<td>0.292</td>
<td>0.307</td>
<td>0.321</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.260</td>
<td>0.292</td>
<td>0.307</td>
<td>0.321</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 15  The improved LB and UB based on information of $H_2/E_2/1$ for $\rho = 0.5$

$c_2^a = 4, c_2^s = 0.5, \theta_w = 0.8260, \rho = 0.5$

<table>
<thead>
<tr>
<th>$R_s \setminus R_a$</th>
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<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s \setminus R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_W$ (UB)</td>
<td>1</td>
<td>0.826</td>
<td>0.531</td>
<td>0.484</td>
<td>0.456</td>
<td>$E[W]$ (UB)</td>
<td>1</td>
<td>0.93</td>
<td>1.53</td>
</tr>
<tr>
<td>5</td>
<td>0.826</td>
<td>0.531</td>
<td>0.484</td>
<td>0.456</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.826</td>
<td>0.531</td>
<td>0.484</td>
<td>0.456</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.814</td>
<td>0.530</td>
<td>0.483</td>
<td>0.456</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_W$ (LB)</td>
<td>1</td>
<td>0.826</td>
<td>1.21</td>
<td>1.44</td>
<td>1.64</td>
<td>$E[W]$ (LB)</td>
<td>1</td>
<td>0.860</td>
<td>0.498</td>
</tr>
<tr>
<td>5</td>
<td>0.827</td>
<td>1.22</td>
<td>1.45</td>
<td>1.66</td>
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</tr>
<tr>
<td>10</td>
<td>0.827</td>
<td>1.22</td>
<td>1.45</td>
<td>1.66</td>
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</tr>
<tr>
<td>20</td>
<td>0.831</td>
<td>1.24</td>
<td>1.49</td>
<td>1.74</td>
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</tr>
</tbody>
</table>
Again, consistent with part (d) of Theorem 6 in the main paper, these tables show that the UB decay rate $\theta_W$ is monotone decreasing in $R$, while the LB decay rate is monotone increasing. Recall that we utilize information from the exact queueing models where $F$ and $G$ have unbounded support, so that we do not expect perfect consistency.

Next, Table 17 presents the extremal decay rates that go with the associated mean values $E[W]$ in Table 3 of the main paper. We obtain the rates here by solving the key equation (6) of the main paper for the original $E_2$ and $H_2$ distributions, so there is good numerical precision, but there is a minor difference from the truncated model, which explains the lack of precise order in a few cases.
Table 17  The decay rates for all basic models under $\rho = 0.7$

<table>
<thead>
<tr>
<th>$E_2/H_2/1$</th>
<th>$R_s \setminus R_a$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$H_2/E_2/1$</th>
<th>$R_s \setminus R_a$</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_s \leq \theta_W \leq \mu_a$</td>
<td>UB</td>
<td>0.150</td>
<td>0.150</td>
<td>0.149</td>
<td>$\mu_s \leq \theta_W \leq \mu_a$</td>
<td>UB</td>
<td>0.228</td>
<td>0.217</td>
<td>0.210</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.156</td>
<td>0.156</td>
<td>0.164</td>
<td></td>
<td>LB</td>
<td>0.291</td>
<td>0.307</td>
<td>0.321</td>
</tr>
<tr>
<td>$R_s \setminus R_a$</td>
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<td>10</td>
<td>20</td>
<td>$R_s \setminus R_a$</td>
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<td>10</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_s \geq \theta_W \geq \mu_a$</td>
<td>UB</td>
<td>0.151</td>
<td>0.150</td>
<td>0.143</td>
<td>$\mu_s \geq \theta_W \geq \mu_a$</td>
<td>UB</td>
<td>0.246</td>
<td>0.243</td>
<td>0.242</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.153</td>
<td>0.153</td>
<td>0.150</td>
<td></td>
<td>LB</td>
<td>0.283</td>
<td>0.286</td>
<td>0.288</td>
</tr>
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<td>20</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_s, \mu_a \leq \theta_W$</td>
<td>UB</td>
<td>0.150</td>
<td>0.149</td>
<td>0.149</td>
<td>$\mu_s, \mu_a \leq \theta_W$</td>
<td>UB</td>
<td>0.246</td>
<td>0.243</td>
<td>0.242</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.156</td>
<td>0.156</td>
<td>0.164</td>
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<td>LB</td>
<td>0.283</td>
<td>0.287</td>
<td>0.289</td>
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<td>$R_s \setminus R_a$</td>
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<td>20</td>
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<td></td>
</tr>
<tr>
<td>$\mu_s, \mu_a \geq \theta_W$</td>
<td>UB</td>
<td>0.151</td>
<td>0.150</td>
<td>0.143</td>
<td>$\mu_s, \mu_a \geq \theta_W$</td>
<td>UB</td>
<td>0.228</td>
<td>0.218</td>
<td>0.210</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.153</td>
<td>0.154</td>
<td>0.150</td>
<td></td>
<td>LB</td>
<td>0.291</td>
<td>0.306</td>
<td>0.320</td>
</tr>
<tr>
<td>$E_2/E_2/1$</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>$H_2/H_2/1$</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>$\mu_s \leq \theta_W \leq \mu_a$</td>
<td>UB</td>
<td>0.842</td>
<td>0.833</td>
<td>0.825</td>
<td>$\mu_s \leq \theta_W \leq \mu_a$</td>
<td>UB</td>
<td>0.105</td>
<td>0.104</td>
<td>0.098</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.880</td>
<td>0.889</td>
<td>0.893</td>
<td></td>
<td>LB</td>
<td>0.108</td>
<td>0.108</td>
<td>0.110</td>
</tr>
<tr>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_s \geq \theta_W \geq \mu_a$</td>
<td>UB</td>
<td>0.847</td>
<td>0.841</td>
<td>0.825</td>
<td>$\mu_s \geq \theta_W \geq \mu_a$</td>
<td>UB</td>
<td>0.106</td>
<td>0.105</td>
<td>0.103</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.861</td>
<td>0.859</td>
<td>0.842</td>
<td></td>
<td>LB</td>
<td>0.107</td>
<td>0.107</td>
<td>0.108</td>
</tr>
<tr>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_s, \mu_a \leq \theta_W$</td>
<td>UB</td>
<td>0.849</td>
<td>0.848</td>
<td>0.848</td>
<td>$\mu_s, \mu_a \leq \theta_W$</td>
<td>UB</td>
<td>0.106</td>
<td>0.105</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.866</td>
<td>0.867</td>
<td>0.867</td>
<td></td>
<td>LB</td>
<td>0.107</td>
<td>0.107</td>
<td>0.111</td>
</tr>
<tr>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>$R_s \setminus R_a$</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_s, \mu_a \geq \theta_W$</td>
<td>UB</td>
<td>0.839</td>
<td>0.826</td>
<td>0.805</td>
<td>$\mu_s, \mu_a \geq \theta_W$</td>
<td>UB</td>
<td>0.105</td>
<td>0.104</td>
<td>0.101</td>
</tr>
<tr>
<td></td>
<td>LB</td>
<td>0.874</td>
<td>0.880</td>
<td>0.863</td>
<td></td>
<td>LB</td>
<td>0.107</td>
<td>0.108</td>
<td>0.108</td>
</tr>
</tbody>
</table>

Remark 1. In general, we cannot claim that the bounds for $\theta_W$ yield bounds for $\mathbb{E}[W]$, so the connection is heuristic. From equation (3.205) in §II.5.11 of Cohen (1982), it follows that for the $K_2/GI/1$ model that $\mathbb{E}[W] = A + \theta_W^{-1}$, where $A$ is a constant that depends on the parameters in (1) and $F$ within $K_2$, but not otherwise on $G$. As a consequence, for fixed $F$, $\mathbb{E}[W]$ is a strictly decreasing function of $\theta_W$ for given first two moments.
6.5. The Possibility of Using Heavy-Traffic Approximations

Tables 18 and 19 show the improved LB and UB for the mean $E[W]$ starting with the exact decay rates of the base models and the approximation in (8) of the main paper. These show that we could also work with the HT approximations.

<table>
<thead>
<tr>
<th>Table 18</th>
<th>The improved LB and UB for $GI/GI/1$ Queues under Exact Decay Rates</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho = 0.5$</td>
</tr>
<tr>
<td></td>
<td>UB</td>
</tr>
<tr>
<td></td>
<td>LB</td>
</tr>
<tr>
<td>$\rho = 0.5$</td>
<td>$c_a^2 = c_s^2 = 4$</td>
</tr>
<tr>
<td></td>
<td>UB</td>
</tr>
<tr>
<td></td>
<td>LB</td>
</tr>
<tr>
<td>$\rho = 0.5$</td>
<td>$c_a^2 = 4, c_s^2 = 0.5$</td>
</tr>
<tr>
<td></td>
<td>UB</td>
</tr>
<tr>
<td></td>
<td>LB</td>
</tr>
</tbody>
</table>
Table 19  The improved LB and UB for $GI/GI/1$ Queues under Approximate Decay Rates

<table>
<thead>
<tr>
<th>$\rho = 0.5$</th>
<th>$c_a^2 = c_s^2 = 0.5$</th>
<th>$\rho = 0.7$</th>
<th>$c_a^2 = c_s^2 = 0.5$</th>
<th>$\rho = 0.9$</th>
<th>$c_a^2 = c_s^2 = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>20</td>
<td>5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>UB</td>
<td>0.220</td>
<td>0.238</td>
<td>0.247</td>
<td>UB</td>
<td>0.721</td>
</tr>
<tr>
<td>LB</td>
<td>0.153</td>
<td>0.145</td>
<td>0.143</td>
<td>LB</td>
<td>0.649</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\rho = 0.5$</th>
<th>$c_a^2 = c_s^2 = 4$</th>
<th>$\rho = 0.7$</th>
<th>$c_a^2 = c_s^2 = 4$</th>
<th>$\rho = 0.9$</th>
<th>$c_a^2 = c_s^2 = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>20</td>
<td>5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>UB</td>
<td>2.02</td>
<td>1.98</td>
<td>2.19</td>
<td>UB</td>
<td>6.27</td>
</tr>
<tr>
<td>LB</td>
<td>1.84</td>
<td>1.80</td>
<td>1.75</td>
<td>LB</td>
<td>6.48</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\rho = 0.5$</th>
<th>$c_a^2 = 4, c_s^2 = 0.5$</th>
<th>$\rho = 0.7$</th>
<th>$c_a^2 = 4, c_s^2 = 0.5$</th>
<th>$\rho = 0.9$</th>
<th>$c_a^2 = 4, c_s^2 = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>10</td>
<td>20</td>
<td>5</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>UB</td>
<td>1.35</td>
<td>1.62</td>
<td>1.78</td>
<td>UB</td>
<td>3.84</td>
</tr>
<tr>
<td>LB</td>
<td>0.629</td>
<td>0.494</td>
<td>0.390</td>
<td>LB</td>
<td>3.01</td>
</tr>
</tbody>
</table>

7. More on the $M/M/K$ Model

In this section we present results for $M/M/1$ and $M/M/2$ complementings Table 2 and 4 of the main paper. As above in this appendix, we use the parameter pair $(R_a, R_s)$.

7.1. Results for $M/M/1$

We start by presenting results for the $M/M/1$ model that complement Table 2 of the main paper. First, Table 20 shows results for $M/M/1$ model using case (ii) of (30) in Theorem 6.
Table 20  The Decay Rates and Set-valued Approximations of $M/M/1$ under Different $\mu_a, \mu_s$ in case (ii) of (30) in Theorem 6

\[ c_a^2 = c_s^2 = 1, \theta_w = 0.4286, \rho = 0.7, \mathbb{E}[W(M, M)] = 1.63, \mu_s \leq \theta_w \leq \mu_a \]

\[
\begin{array}{cccccc}
R_s \setminus R_a & 1 & 5 & 10 & 20 & R_s \setminus R_a & 1 & 5 & 10 & 20 \\
\hline
\theta_W \text{ (UB)} & & & & & \theta_W \text{ (LB)} & & & & \\
1 & 0.429 & 0.423 & 0.419 & 0.416 & 1 & 1.60 & 1.56 & 1.62 & 1.69 \\
5 & 0.427 & 0.421 & 0.418 & 0.415 & 5 & 1.61 & 1.59 & 1.61 & 1.68 \\
10 & 0.427 & 0.421 & 0.418 & 0.415 & 10 & 1.61 & 1.61 & 1.62 & 1.68 \\
20 & 0.427 & 0.421 & 0.418 & 0.415 & 20 & 1.61 & 1.58 & 1.60 & 1.68 \\
\end{array}
\]

\[ \mathbb{E}[W] \text{ (UB)} & \mathbb{E}[W] \text{ (LB)} \\
1 & 1.67 & 1.67 & 1.68 & 1.67 \\
5 & 1.67 & 1.67 & 1.68 & 1.67 \\
10 & 1.67 & 1.67 & 1.68 & 1.67 \\
20 & 1.67 & 1.67 & 1.67 & 1.67 \\
\]

Next, Table 21 shows results for $M/M/1$ model using case (i) of (30) in Theorem 6 with $\mu_a, \mu_s \leq \theta_w$.

Table 21  The Decay Rates and Set-valued Approximations of $M/M/1$ under Different $\mu_a, \mu_s$ in case (i) of (30) in Theorem 6

\[ c_a^2 = c_s^2 = 1, \theta_w = 0.4286, \rho = 0.7, \mathbb{E}[W(M, M)] = 1.63 \]

\[
\begin{array}{cccccc}
R_s \setminus R_a & 1 & 5 & 10 & 20 & R_s \setminus R_a & 1 & 5 & 10 & 20 \\
\hline
\theta_W(F_L, G_U) & & & & & \theta_W(F_U, G_L) & & & & \\
1 & 0.429 & 0.423 & 0.419 & 0.416 & 1 & 1.66 & 1.65 & 1.65 & 1.65 \\
5 & 0.427 & 0.421 & 0.418 & 0.415 & 5 & 1.66 & 1.65 & 1.65 & 1.65 \\
10 & 0.427 & 0.421 & 0.418 & 0.415 & 10 & 1.66 & 1.65 & 1.65 & 1.65 \\
20 & 0.427 & 0.421 & 0.418 & 0.415 & 20 & 1.66 & 1.65 & 1.65 & 1.65 \\
\end{array}
\]

\[ \mathbb{E}[W] \text{ (UB)} & \mathbb{E}[W] \text{ (LB)} \\
1 & 1.55 & 1.56 & 1.56 & 1.56 \\
5 & 1.55 & 1.56 & 1.56 & 1.56 \\
10 & 1.55 & 1.56 & 1.56 & 1.56 \\
20 & 1.55 & 1.56 & 1.56 & 1.56 \\
\]
Table 22 shows results for $M/M/1$ model using case (iii) of (30) in Theorem 6 with $\mu_a, \mu_s \geq \theta_W$.

### Table 22  The Decay Rates and Set-valued Approximations of $M/M/1$ under Different $\mu_a, \mu_s$ in case (iii) of (30) in Theorem 6

$$c_a^2 = c_s^2 = 1, \theta_w = 0.4286, \rho = 0.7, \mathbb{E}[W(M,M)] = 1.63$$

<table>
<thead>
<tr>
<th>$\theta_W(F_U, G_L)$</th>
<th>$R_s \setminus R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s \setminus R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.429</td>
<td>0.423</td>
<td>0.419</td>
<td>0.416</td>
<td>1</td>
<td>1.67</td>
<td>1.71</td>
<td>1.72</td>
<td>1.73</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.422</td>
<td>0.417</td>
<td>0.413</td>
<td>0.411</td>
<td>5</td>
<td>1.68</td>
<td>1.71</td>
<td>1.72</td>
<td>1.71</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.422</td>
<td>0.417</td>
<td>0.413</td>
<td>0.411</td>
<td>10</td>
<td>1.68</td>
<td>1.71</td>
<td>1.72</td>
<td>1.71</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.422</td>
<td>0.417</td>
<td>0.413</td>
<td>0.411</td>
<td>20</td>
<td>1.68</td>
<td>1.71</td>
<td>1.72</td>
<td>1.71</td>
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</tbody>
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<table>
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<tr>
<th>$\theta_W(F_L, G_U)$</th>
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<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>$R_s \setminus R_a$</th>
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<th>5</th>
<th>10</th>
<th>20</th>
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<td>1.63</td>
<td>1.62</td>
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</tr>
<tr>
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<td>0.429</td>
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<td>0.435</td>
<td>0.437</td>
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<td>1.67</td>
<td>1.65</td>
<td>1.64</td>
<td>1.62</td>
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<td>0.435</td>
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<td>1.67</td>
<td>1.65</td>
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<td>1.65</td>
<td>1.63</td>
<td>1.62</td>
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</table>

### 7.2. Corresponding Results for $M/M/2$

Tables 23 and 24 present corresponding results for the $M/M/2$ model in cases (ii) and (iii) of (30) in Theorem 6.
Table 23  The Set-valued Approximations for $M/M/2$ in case (ii): $\mu_s \leq \theta_W \leq \mu_a$

c_a^2 = c_s^2 = 1, \theta_w = 0.4286, \rho = 0.7, \mathbb{E}[W(M,M)] = 1.35, \mu_s \leq \theta_W \leq \mu_a

<table>
<thead>
<tr>
<th>$\theta_W$ (UB)</th>
<th>$R_s \backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
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</thead>
<tbody>
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<td>0.417</td>
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<td>0.422</td>
<td>0.418</td>
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<tr>
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<td>0.427</td>
<td>0.421</td>
<td>0.418</td>
<td>0.415</td>
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<td>0.427</td>
<td>0.421</td>
<td>0.418</td>
<td>0.415</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta_W$ (LB)</th>
<th>$R_s \backslash R_a$</th>
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<td>0.434</td>
<td>0.437</td>
<td>0.439</td>
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</table>

$\mathbb{E}[W]$ (UB) | $R_s \backslash R_a$ | 1 | 5 | 10 | 20 |
<table>
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<tr>
<th></th>
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<td>1.30</td>
<td>1.34</td>
<td>1.39</td>
<td>1.41</td>
</tr>
</tbody>
</table>

$\mathbb{E}[W]$ (LB) | $R_s \backslash R_a$ | 1 | 5 | 10 | 20 |
<table>
<thead>
<tr>
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<th></th>
<th></th>
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<td>1.34</td>
<td>1.30</td>
<td>1.31</td>
<td>1.33</td>
</tr>
</tbody>
</table>

Table 24  The improved LB and UB based on information of $M/M/2$ ($\mu_s, \mu_s \geq \theta_w$)

c_a^2 = c_s^2 = 1, \theta_w = 0.4286, \rho = 0.7

<table>
<thead>
<tr>
<th>$\theta_W(F_U, G_L)$</th>
<th>$R_s \backslash R_a$</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
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<td>0.417</td>
<td>0.414</td>
</tr>
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<tr>
<td></td>
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<td>0.417</td>
<td>0.413</td>
<td>0.411</td>
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</tbody>
</table>

$\mathbb{E}[W]$ (UB) | $R_s \backslash R_a$ | 1 | 5 | 10 | 20 |
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$\mathbb{E}[W]$ (LB) | $R_s \backslash R_a$ | 1 | 5 | 10 | 20 |
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Tables 23 and 24 show that the method for producing approximate intervals of likely values for the mean $\mathbb{E}[W]$ remain effective for $K = 2$. 
References


