TRANSIENT BEHAVIOR OF THE $M/G/1$ WORKLOAD PROCESS

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In this paper we describe the time-dependent moments of the workload process in the $M/G/1$ queue. The kth moment as a function of time can be characterized in terms of a differential equation involving lower moment functions and the time-dependent server-occupation probability. For general initial conditions, we show that the first two moment functions can be represented as the difference of two nondecreasing functions, one of which is the moment function starting at zero. The two nondecreasing components can be regarded as probability cumulative distribution function (cdf's) after appropriate normalization. The normalized moment functions starting empty are called moment cdf's; the other normalized components are called moment-difference cdf's. We establish relations among these cdf's using stationary-excess relations. We apply these relations to calculate moments and derivatives at the origin of these cdf's. We also obtain results for the covariance function of the stationary workload process. It is interesting that these various time-dependent characteristics can be described directly in terms of the steady-state workload distribution.

In this paper, we derive some simple descriptions of the transient behavior of the classical $M/G/1$ queue. In particular, we focus on the workload process $\{W(t): t \geq 0\}$ (also known as the unfinished work process and the virtual waiting time process), which is convenient to analyze because it is a Markov process. Our main results describe the time-dependent probability that the server is busy, $P(W(t) > 0)$, the time-dependent moments of the workload process, $E[W(t)^k]$, and the covariance function of the stationary workload process.

The transient behavior of the $M/G/1$ queue (and more general models) has been studied extensively, so that there is a substantial literature, including the early papers by Kendall (1951, 1953), Takács (1955, 1962b), Beneš (1957), and Keilson and Kooharian (1960); the advanced books by Takács (1962a, 1967), Beneš (1963), Prabhu (1965, 1980), Kingman (1972), Cohen (1982), Asmussen (1987), and Neuts (1989), and the more recent papers by Ott (1977a, b), Harrison (1977), Middleton (1979), Rosenkrantz (1983), Blanc and van Doorn (1986), Cox and Isham (1986), Gaver and Jacobs (1990), Baccelli and Makowski (1989a, b), and Kella and Whitt (1991). A good basic reference is Kleinrock (1975).

Nevertheless, we believe that we have something to contribute. *We focus on relatively simple exact relations and approximations that are convenient for engineering applications.* In particular, we extend previous work for the same purpose in Abate and Whitt (1987a–c, 1988a–d). Our earlier work described the transient behavior of one-dimensional reflected Brownian motion (RBM) and various processes associated with the $M/M/1$ queue. The $M/M/1$ workload process was discussed in Section 6 of Abate and Whitt (1988b). Since RBM and the $M/G/1$ processes can serve as rough approximations for many other queueing processes, these results help describe how a large class of queueing processes approach steady state. These results provide simple analytical approximations in the spirit of the empirical work by Odoni and Roth (1983). The RBM and $M/M/1$ approximations have also been applied to gain additional insight into queueing simulations in Whitt (1989, 1991).

There are two main objectives in relation to our previous work. First, *we want to see how the theory for RBM and $M/M/1$ extends to the $M/G/1$ model.* As indicated in Remark 6.3(b) of Abate and Whitt (1988b), much of the theory does extend and now we provide the details. Second, *we want to see how well*

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the RBM and $M/M/1$ approximations work for the $M/G/1$ model. We focus on the first objective in this paper; we intend to focus on the second objective in a sequel. Our approach to approximations is discussed in Section 1 of Abate and Whitt (1987a), Section 8 of Abate and Whitt (1988b), and Abate and Whitt (1988c).

Moment CDFs and Moment-Difference CDFs

As in our previous work, *the special case of starting out empty plays an important role.* We represent the kth moment function starting at x as

$$\begin{aligned} m_k(t,x) &\equiv E[W(t)^k \mid W(0)=x] \\ &= m_k(t,0) + d_k(t,x) \end{aligned} \tag{1}$$

and we show that the kth moment function starting empty $m_k(t, 0)$ is *nondecreasing* in t for all k, while the kth moment difference function $d_k(t, x)$ is *nonincreasing* in t for $k = 1, 2$. Indeed, except for the monotonicity of $d_2(t, x)$, which is covered by Theorem 11 here, this result was obtained for the $M/G/1$ workload process and other reflected Lévy processes without negative jumps in Theorem 7.3 of Abate and Whitt (1987b). (It is important to add the condition of no negative jumps there!)

Since the functions $m_k(t, 0)$ and $d_k(t, x)$ are monotone (the last only for $k = 1$ and 2), we can express them as probability cumulative distribution functions (cdf's) after appropriate normalization. For $m_k(t, 0)$, we just divide by the steady-state limit $m_k(\infty) \equiv m_k(\infty, x)$. Looking at the *moment cdf's*

$$H_k(t) \equiv m_k(t,0)/m_k(\infty), \ t \geq 0, \tag{2}$$

is convenient for interpretation, because we separate the steady-state value $m_k(\infty)$ from the proportion of the steady-state value attained at time t. Moreover, as before, the moment cdf's have nice probabilistic structure. See Section 1 of Abate and Whitt (1987a) for more discussion.

Much of the probabilistic structure is expressed via the stationary-excess operator. For any cdf F on the positive real line with mean f_1, let F_e be the associated *stationary-excess cdf* (or equilibrium residual lifetime cdf) defined by

$$F_e(t) = f_1^{-1} \int_0^t [1 - F(u)]du, \ t \geq 0; \tag{3}$$

e.g., see p. 193 of Karlin and Taylor (1975), Whitt (1985), and pp. 319 and 337 of Abate and Whitt (1988b). Let f_k and f_{ek} be the kth moments of F and F_e, respectively. Then

$$f_{ek} = f_{k+1}/(k+1)f_1. \tag{4}$$

Let $F^{(k)}(0)$ and $F_e^{(k)}(0)$ be the kth (right) derivatives at the origin of F and F_e, respectively. Then

$$F_e^{(1)}(0) = 1/f_1 \text{ and } F_e^{(k+1)}(0) = -F_e^{(1)}(0)F^{(k)}(0). \tag{5}$$

By Theorem 7.3 of Abate and Whitt (1987b),

$$\begin{aligned} H_1(t) &= \int_0^\infty \frac{P(W(\infty) > y)}{E[W(\infty)]} P(T_{y0} \leq t)dy \\ &= \int_0^\infty F_{y0}(t)\, dV_e(y), \end{aligned} \tag{6}$$

where $V(t) = P(W(\infty) \leq t)$, V_e is the stationary-excess cdf associated with V, T_{y0} is the first passage time from y to 0 and F_{y0} is its cdf, whose Laplace-Stieltjes transform is given by (33). Moreover, here we show that the second-moment cdf is the stationary-excess of the first-moment cdf, just as it is for the $M/M/1$ workload process (see Theorem 5 of Abate and Whitt 1988b and Theorem 2).

Paralleling (2), we also form the two *moment-difference cdf's*

$$\begin{aligned} G_1(t,x) &= 1 - \frac{d_1(t,x)}{x} \quad \text{and} \\ G_2(t,x) &= 1 - \frac{d_2(t,x)}{x^2}. \end{aligned} \tag{7}$$

The moment-difference cdf's also have nice structure. Indeed, by Theorem 7.3 of Abate and Whitt (1987b),

$$G_1(t,x) = \frac{1}{x} \int_0^x P[T_{y0} \leq t]dy, \tag{8}$$

where T_{y0} is the first-passage time from y to 0. Here we show that the second-moment-difference cdf G_2 is the stationary-excess of the first-moment-difference cdf G_1 (see Theorem 11). From (4), (6), and (8), we see that the moments of H_i and G_i for $i = 1, 2$ can be determined in terms of the moments $E[T_{y0}^k]$; we determine the first four in Theorem 7.

Just as we did before for RBM and $M/M/1$, in this paper we also derive the moments and derivatives at the origin of the $M/G/1$ moment cdf's and moment-difference cdf's, so that we can also derive approximations for these cdf's just as we did for the $M/M/1$ cdf's in our previous work; e.g., we can fit a cdf to the special characteristics. See Abate and Whitt (1987a, 1988c) and Johnson and Taaffe (1989, 1990, 1991) for more discussion.

Expressions in Terms of the Steady-State Workload Moments

An interesting feature of the $M/G/1$ model in contrast to many other stochastic models is that the steady-state workload distribution depends on all the ingredients of the model, in particular, the full service-time distribution (see Whitt 1983). Thus, the steady-state workload distribution determines the service-time distribution and, in principle, the transient behavior. Consistent with this property, we show that the moments of the moment cdf's have relatively nice expressions directly in terms of the steady-state workload moments $\nu_k \equiv m_k(\infty)$. (In part, this is explained by (6).) The steady-state moments, in turn, can be expressed in terms of the service-time moments via the Takács (1962b) recurrence formula, (20) below.

For one example, let h_{kj} be the jth moment of the moment cdf H_k in (2), let $\nu_k \equiv m_k(\infty) \equiv E[W(\infty)^k]$ be the kth moment of the steady-state workload cdf V, and let ν_{ek} be the kth moment of the steady-state workload stationary-excess cdf V_e. Let the service rate be 1 and let the arrival rate and traffic intensity be ρ, which we assume is less than 1. Then, by the corollary to Theorem 6,

$$h_{11} = \frac{\nu_{e1}}{1-\rho} = \left(\frac{1}{1-\rho}\right)\left(\frac{\nu_2}{2\nu_1}\right). \tag{9}$$

Note that h_{11} provides a summary description of the time it takes for the mean $E[W(t)|W(0) = 0]$ to approach its steady-state value $m_1(\infty)$.

For a second example, let $\{W^*(t)\colon t \geq 0\}$ be a stationary version of the workload process, with $W^*(0) \stackrel{d}{=} W(\infty)$, where $\stackrel{d}{=}$ denotes equality in distribution. Then the *covariance function* is

$$C_w(t) = E[W^*(0)W^*(t)] - (E[W^*(0)])^2,\ t \geq 0, \tag{10}$$

and the *asymptotic variance* is

$$\sigma_w^2 \equiv 2\int_0^\infty C_w(t)dt \tag{11}$$

(e.g., see p. 1345 of Whitt 1989). In the same spirit as (9), we show that

$$\sigma_w^2 = \frac{\nu_3 - \nu_2\nu_1}{1-\rho} \tag{12}$$

(see Theorem 10). Formula 12 extends Beneš (1957), Ott (1977a), and Theorem 8 of Abate and Whitt (1988b). Note that

$$\sigma_w^2/\text{Var } W^*(0) \equiv (\nu_3 - \nu_2\nu_1)/[(1-\rho)(\nu_2 - \nu_1^2)]$$

provides a summary description for the time t it takes for the dependence between $W^*(0)$ and $W^*(t)$ in the stationary version to die out. Note that this summary measure differs from h_{11} in (9), but both are of order $(1-\rho)^{-2}$ as $\rho \to 1$.

Organization of this Paper

Here is how the rest of this paper is organized. In Section 1 we define the $M/G/1$ workload process and introduce our notation. In Section 2 we present a simple derivation of differential equations for the $M/G/1$ moment functions. This produces a nice simple derivation of the Takács (1962b) recurrence relation for the steady-state moments. In Section 3 we apply the differential equations to establish the key relations among the moment cdf's. In Section 4 we review the relations among the basic transforms describing the $M/G/1$ transient behavior. In Section 5 we apply these transform relations to derive the moments of the moment cdf's. In Section 6 we describe the covariance function in (10). In Section 7 we establish properties of the moment-difference cdf's in (7). In Section 8 we mention complementary-cdf cdf's. Finally, in Section 9 we present previously omitted proofs.

1. THE $M/G/1$ MODEL

In this section we quickly review the $M/G/1$ model and introduce our notation. As usual, the $M/G/1$ queue has a single server, unlimited waiting space, a Poisson arrival process, and i.i.d. service times that are independent of the arrival process. The standard queue discipline is first-in first-out, but because we are focusing on the workload process, the specific queue discipline will not matter.

Let $A \equiv \{A(t)\colon t \geq 0\}$ be the Poisson arrival counting process and let it have intensity ρ. Let $\{S_n\colon n \geq 1\}$ be the i.i.d. sequence of service times and let S be a generic service-time random variable (having the distribution of S_1). We assume that S has cdf G with mean 1. Thus, the traffic intensity is ρ, the same as the arrival rate. We are interested in the stable case, so we assume that $\rho < 1$.

Let the *total input process* be $X \equiv \{X(t)\colon t \geq 0\}$, where

$$X(t) = S_1 + \cdots + S_{A(t)},\ t \geq 0, \tag{13}$$

with $S_0 = 0$. Note that $X(t)$ represents the total input of work in the interval $(0, t]$. The process X is a compound Poisson process. Let the *net input process* be $Y \equiv \{Y(t)\colon t \geq 0\}$, where

$$Y(t) = X(t) - t,\ t \geq 0. \tag{14}$$

Let the *workload process* be $W \equiv \{W(t)\colon t \geq 0\}$, defined by

$$W(t) = \begin{cases} Y(t) + W(0) & \text{if } \inf_{0 \le s \le t} Y(s) > -W(0) \\ Y(t) - \inf_{0 \le s \le t} Y(s) & \text{if } \inf_{0 \le s \le t} Y(s) \le -W(0), \end{cases} \tag{15}$$

where $W(0)$ is an initial workload that is independent of $\{A(t): t \geq 0\}$ and $\{S_n: n \geq 1\}$. Note that W is obtained from Y and $W(0)$ by simply applying the one-dimensional, one-sided reflection map (e.g., see p. 19 of Harrison 1985).

It is significant that Y is a Lévy process without negative jumps. The results here hold when Y is replaced by another Lévy process without negative jumps, but we do not discuss that case (see Harrison 1977, Middleton 1979, Prabhu 1980, and Kella and Whitt 1991 for related material).

2. THE MOMENT DIFFERENTIAL EQUATION

Let $m_k(t) \equiv m_k(t, x)$ be the kth moment function defined in (1) and let $p_0(t)$ be the *emptiness probability function*, i.e.,

$$p_0(t) \equiv p_0(t, x) = P(W(t) = 0 | W(0) = x). \tag{16}$$

In this section we will obtain simple expressions for the derivatives of the moment functions m_k in terms of the emptiness probability p_0. We focus on the emptiness probability itself in Section 4. Thus, the emptiness probability is fundamental. This idea does not seem to be as well known as it should be, but it certainly is not new. Indeed, this idea is a major theme in Beneš (1963).

To describe the transient behavior of the workload process, it is customary, following Takács (1955, 1962a, b), to start with an integrodifferential equation for the cdf $P(W(t) \le x)$ or its Laplace transform, but we will show that it is relatively easy to treat the moment functions directly. (This observation has been made with the closure approximations for queues with time-dependent arrival rates, e.g., see Rothkopf and Oren (1979). The results in this section also hold for time-dependent arrival rates.)

First, we establish (review) necessary and sufficient conditions for the moment functions to be finite, for fixed t and in the limit as $t \to \infty$. Let $\Rightarrow$ denote convergence in distribution. All omitted proofs appear in Section 9.

Proposition 1. a. *Here $m_k(t) < \infty$ if and only if $m_k(0) < \infty$ and $E[S^k] < \infty$.*

b. *$W(t) \Rightarrow W(\infty)$ as $t \to \infty$ where $P(W(\infty) < \infty) = 1$.*

c. *If $m_k(0) < \infty$, then $m_k(t) \to m_k(\infty) \equiv E[W(\infty)^k]$ as $t \to \infty$, where $m_k(\infty) < \infty$ if and only if $E[S^{k+1}] < \infty$.*

Note from Proposition 1 that one higher moment of S must be finite to have $m_k(\infty) < \infty$ than is required to have $m_k(t) < \infty$ for $t < \infty$.

We now consider the derivative with respect to time of the kth moment function, denoted by $m'_k(t)$. An expression for the derivative of the first moment function follows from a basic *conservation law*, i.e., rate-in equals rate-out (e.g., see p. 55 of Takács 1962a). In particular, since the rate-in of work is $t^{-1}EX(t) = \rho$ and the rate-out at time t is $1 - p_0(t)$,

$$m'_1(t) = \rho - 1 + p_0(t), \; t > 0, \tag{17}$$

or, equivalently,

$$m_1(t) = m_1(0) + (\rho - 1)t + \int_0^t p_0(u)du, \; t > 0. \tag{18}$$

Since $W(t) \Rightarrow W(\infty)$ as $t \to \infty$, $p_0(t) \to p_0(\infty)$ as $t \to \infty$. By Little's law ($L = \lambda W$) applied to the server, we know that $p_0(\infty) = 1 - \rho$. Hence, $m'_1(t) \to 0$ as $t \to \infty$.

Our main result in this section is a higher-moment extension of (17). When we let $t \to \infty$, we immediately obtain the Takács (1962b) recurrence formula for the steady-state moments (see (5.112) on p. 201 of Kleinrock 1975, Lemoine 1976, and p. 185 of Asmussen 1987). The first steady-state moment is the Pollaczek-Khintchine mean value formula for the workload. The proof is very simple except for a few technical details; we sketch it here. We provide the extra technical details in Section 9. Let $m_0(\infty) = 1$ and let $\stackrel{d}{=}$ denote equality in distribution.

Theorem 1. a. *If $m_k(t) < \infty$ for some k, $k \geq 2$, then the derivative $m'_k(t)$ exists and*

$$m'_k(t) = \rho E[S^k] - (1 - \rho)k m_{k-1}(t) + \rho \sum_{j=2}^{k-1} \binom{k}{j} E[S^j] m_{k-j}(t). \tag{19}$$

b. *If $m_{k+1}(0) < \infty$ and $E[S^{k+1}] < \infty$ for some k, $k \geq 1$, then $m'_{k+1}(t) \to 0$ as $t \to \infty$ and*

$$m_k(\infty) = \frac{\rho}{1-\rho} \sum_{j=1}^{k} \binom{k}{j} \frac{E(S^{j+1})}{j+1} m_{k-j}(\infty), \; k \geq 1. \tag{20}$$

Sketch of Proof. (See Section 9 for more.)

Part a. We calculate $m_k(t + \epsilon) - m_k(t)$ to order ϵ by conditioning and unconditioning on $W(t)$. We say that $f(\epsilon) = o(\epsilon)$ if $f(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$. Note that

$X(t+\epsilon) - X(t)$ is independent of $W(t)$. Moreover, $X(t+\epsilon) - X(t) \stackrel{d}{=} X(\epsilon)$. To order ϵ, there is either one arrival in $A(\epsilon)$ or none. Hence, ignoring complications when $0 < x < \epsilon$, we have

$$E[W(t+\epsilon)^k | W(t) = x > 0] = \rho\epsilon E[(x + S - \epsilon)^k] + (1 - \rho\epsilon)E[(x-\epsilon)^k] + o(\epsilon)$$
$$= \rho\epsilon \sum_{j=0}^{k} \binom{k}{j} x^j E[S^{k-j}] + (1-\rho\epsilon)x^k - \epsilon k x^{k-1} + o(\epsilon)$$

and

$$E[W(t+\epsilon)^k | W(t) = 0] = \rho\epsilon E[S^k] + o(\epsilon).$$

Next, upon unconditioning, ignoring the problems involving interchanging the expectation with the limit as $\epsilon \to 0$, we obtain

$$m_k(t+\epsilon) - m_k(t) = \rho\epsilon \sum_{j=0}^{k} \binom{k}{j} E[S^j] m_{k-j}(t) + (1-\rho\epsilon)m_k(t) - \epsilon k m_{k-1}(t) - m_k(t) + o(\epsilon). \quad (21)$$

We obtain (19) from (21) by noting that the three terms involving $m_k(t)$ cancel, combining the two terms involving $m_{k-1}(t)$, pulling out the term involving $m_0(t)$, dividing by ϵ and letting $\epsilon \to 0$.

Part b. We apply mathematical induction. When we are considering $m'_{k+1}(t)$ given the condition $E[W(0)^{k+1}] < \infty$ and $E[S^{k+1}] < \infty$, we have $m_j(t) \to m_j(\infty) < \infty$ for each $j < k$ by the induction assumption, because $E[W(0)^{j+1}] < \infty$ and $E[S^{j+1}] < \infty$ for all $j \leqslant k$. Assuming now that $m'_{k+1}(t) \to 0$ as $t \to \infty$ (which we will prove later), we see from (19), with k replaced by $k+1$, that $m_k(t) \to m_k(\infty)$ as $t \to \infty$, where

$$m_k(\infty) = \frac{\rho}{(k+1)(1-\rho)} E[S^{k+1}] + \frac{\rho}{(k+1)(1-\rho)} \sum_{j=2}^{k} \binom{k+1}{j} E[S^j] m_{k+1-j}(\infty)$$
$$= \frac{\rho}{1-\rho} \sum_{j=2}^{k+1} \frac{1}{k+1} \binom{k+1}{j} E[S^j] m_{k+1-j}(\infty),$$

which becomes (20) upon making the change of variables $l = j - 1$; e.g., then

$$\frac{1}{k+1}\binom{k+1}{j} = \frac{1}{l+1}\binom{k}{l}.$$

From (17) and (19), we see that the moment functions m_k depend on the arrival rate ρ and the service-time distribution only via ρ, the service-time moments and the emptiness probability $p_0(t)$. Moreover, (17) and (19) provide a recursive formula for $m_k(t)$ in terms of ρ, $E[S^j]$, $1 \leqslant j \leqslant k$, and p_0.

We conclude this section by mentioning that the proof of Theorem 1 also applies to the $M/M/1$ queue length moments, so that we can obtain different proofs of our previous results.

3. THE MOMENT CDFs

We now focus on the special case in which we start empty, i.e., $P(W(0) = 0) = 1$. Then, as we show below, $W(t)$ is stochastically increasing in t and we can regard appropriately normalized moment functions as probability cumulative distribution functions on the positive half-line. (A real-valued function F on the positive half-line is a cdf if it is a nonnegative and nondecreasing with $F(\infty) \equiv \lim_{t\to\infty} F(t) = 1$. By convention, we take it to be right-continuous.

Recall that one random variable Z_1 is *stochastically less than or equal to* another Z_2, denoted by $Z_1 \leqslant_{st} Z_2$, if $E[g(Z_1)] \leqslant E[g(Z_2)]$ for all nondecreasing real-valued functions g for which both expectations exist. A family of random variables $\{Z(t): t \geqslant 0\}$ is *stochastically increasing* if $Z(t_1) \leqslant_{st} Z(t_2)$ for $0 \leqslant t_1 < t_2$. The following result is well known, but worth emphasis.

Proposition 2. *If* $P(W(0) = 0) = 1$, *then the workload process* $\{W(t): t \geqslant 0\}$ *is stochastically increasing.*

Proof. Since the net input process $\{Y(t): t \geqslant 0\}$ has stationary independent increments

$$W(t) \stackrel{d}{=} M(t) \equiv \sup_{0 \leqslant u \leqslant t} \{Y(u)\}, \quad t \geqslant 0, \quad (22)$$

when $W(0) = 0$ by (15). Obviously $M(t)$ is nondecreasing in t w.p.1. Hence, $E[g(M(t))]$ is nondecreasing in t for each nondecreasing real-valued function g. Finally, by (22), $E[g(W(t))] = E[g(M(t))]$.

Henceforth, in this section we assume that $P(W(0) = 0) = 1$. For emphasis, we thus write p_{00} for p_0. As a consequence of Proposition 2, we can form cdf's associated with the moment functions $m_k(t)$ as defined in (2) whenever $m_k(\infty) < \infty$. Moreover, as a consequence of Proposition 2, $p_{00}(t)$ is nonincreasing in t. Since $p_{00}(0) = 1$ and $p_{00}(\infty) = 1 - \rho$, we can form a 0th-moment or *server-occupation cdf* H_0 by setting

$$H_0(t) = [1 - p_{00}(t)]/\rho, \quad t \geqslant 0. \quad (23)$$

It is significant that the emptiness probability function p_{00} is a well-studied object. In particular, it is a standard p function associated with a regenerative phenomenon in the sense of Kingman (see p. 38 there). It follows from Theorem 2.3 on p. 32 of Kingman that p_{00} is Lipschitz continuous with modulus ρ, that is,

$$|p_{00}(t_2) - p_{00}(t_1)| \leq \rho |t_2 - t_1| \tag{24}$$

for all positive t_1 and t_2, because $p'_{00}(0) = -\rho$. Consequently, p_{00} is absolutely continuous with respect to Lebesgue measure, which implies the same for H_0 in (23), so that H_0 has a density h_0 with

$$H_0(t) = \int_0^t h_0(u)du, \ t \geq 0, \tag{25}$$

and $0 \leq h_0(t) \leq 1$ for all t. However, as illustrated by considering the case of deterministic service times (see p. 39 of Kingman), H_0 is not necessarily differentiable at all t. (These important properties of the emptiness probability function p_{00} were also obtained directly by Ott 1977a.)

As in our previous papers, we relate the different moment cdf's to each other by using the stationary-excess operator in (3). Our main result in this section follows directly from Theorem 1. It is a generalization of the $M/M/1$ result in Theorem 5 of Abate and Whitt (1988b). Recall that ν_k is the kth moment of $V(t) = P(W(\infty) \leq t)$.

Theorem 2

a. *If* $E[S^{k+1}] < \infty$, *then* H_k *is a proper cdf.*
b. $H_1 = H_{0e}$.
c. $H_2 = H_{1e}$.
d. $H_3 = (1 + \alpha_3)H_{2e} - \alpha_3 H_2$, *where* $\alpha_3 = 3\nu_1\nu_2/\nu_3$.
e. $H_4 = (1 + \alpha_4 + \beta_4)H_{3e} - \alpha_4 H_3 - \beta_4 H_2$, *where*

$$\alpha_4 = \frac{4\nu_1\nu_3}{\nu_4} \quad \text{and} \quad \beta_4 = \frac{6\nu_2^2}{\nu_4}.$$

Proof. By Proposition 2, H_k is a proper cdf provided that $\nu_k \equiv m_k(\infty) < \infty$, which holds if and only if $E[S^{k+1}] < \infty$ by Proposition 1. To obtain the explicit expressions, apply (13) and Theorem 1, noting that $h_k(t) \equiv m'_k(t)/m_k(\infty)$ is the probability density function of the kth moment cdf H_k, while

$$h_{ke}(t) = h_{ke}(0)[1 - H_k(t)], \ t \geq 0, \tag{26}$$

is the probability density function of h_{ke} by (3)–(5). For example, from (17),

$$h_1(t)m_1(\infty) = m'_1(t) = \rho - 1 + p_{00}(t)$$
$$= \rho[1 - H_0(t)],$$

so that by (26),

$$h_1(t)m_1(\infty)/\rho = h_{0e}(t)/h_{0e}(0),$$

$$\mu_{0e} \equiv \int_0^\infty x \, dH_0(s)$$
$$= 1/h_1(0) = m_1(\infty)/\rho$$

and, indeed, $h_1 = h_{0e}$. The various expressions, including the constants in parts d and e, are obtained by algebraic manipulation. Given the stated results, it is easy to see how to group terms to verify the formulas.

From Theorem 2, we see that the moment cdf's H_k for $k \leq 4$ can be expressed directly in terms of the 0th-moment cdf H_0. Moreover, by (4) and (5), the moments of H_k and the derivatives of H_k at $t = 0$ can be expressed directly in terms of the corresponding quantities of H_0.

4. BASIC LAPLACE TRANSFORM RELATIONS

In Section 3 we saw that the moment cdf's H_k can be expressed in terms of the emptiness function p_{00} or the associated server-occupation cdf H_0. In this section we review the basic Laplace transform relations that enable us to determine p_{00} and H_0. Unfortunately, however, the situation is not quite as simple as in the $M/M/1$ case, because we characterize p_{00} only via a functional equation for its Laplace transform. In very few cases ($M/M/1$ is one) can we obtain a direct expression for this transform. Nevertheless, in the next section we apply these transform relations to determine the moments of H_0 and thus the moments of the moment cdf's H_k for $k \leq 4$. The functional equations can also be solved iteratively to numerically invert the transforms (see sections 1.2 and 2.2 of Neuts 1989, and Abate and Whitt 1992a, b).

For any cdf F, let $\hat{f}$ be its Laplace-Stieltjes transform (LST), defined by $\hat{f}(s) = \int_0^\infty e^{-st}\, dF(t)$, which coincides with the Laplace transform of its density f when $F(t) = \int_0^t f(u)du$ for all t; i.e., then $\hat{f}(s) = \int_0^\infty e^{-st} f(t)dt$. Thus, $\hat{g}$ and $\hat{g}_e$ are the LSTs of the service-time cdf G and the associated service-time stationary-excess cdf G_e, respectively.

As before, let V be the cdf of $W(\infty)$ whose LST is given by the *Pollaczek-Khintchine transform formula*

$$\hat{v}(s) = \frac{1 - \rho}{1 - \rho \hat{g}_e(s)}; \tag{27}$$

(see (5.108) on p. 200 of Kleinrock). Let B be the cdf of a busy-period distribution and recall that its LST $\hat{b}$ is characterized by the *Kendall functional equation*

$$\hat{b}(s) = \hat{g}(s + \rho - \rho \hat{b}(s)); \tag{28}$$

(see (59) in Kendall 1951, the discussion by I. J. Good on p. 182 there, and (5.137) on p. 212 of Kleinrock 1975).

Let η and ζ be two functions defined by

$$\eta(s) = s - \rho + \rho\hat{g}(s) \quad \text{and}$$
$$\zeta(s) = s + \rho - \rho\hat{b}(s). \tag{29}$$

The functions η and ζ are inverse functions in the sense that, for any s,

$$\eta(\zeta(s)) = s, \tag{30}$$

as it is easy to see from (28) and (29). Note that we can rewrite (28) and (29) as a functional equation for ζ, namely

$$s + \rho - \zeta(s) = \rho\hat{g}(\zeta(s)). \tag{31}$$

The function ζ in (29) is known to be the exponent of the first passage time LST. In particular, as before, let T_{x0} be the first passage time from x to 0 and let F_{x0} be its cdf. The cdf F_{x0} is related to the probability of emptiness $p_{x0}(t) \equiv P(W(t) = 0 | W(0) = x)$ by

$$p_{x0}(t) = (F_{x0} * p_{00})(t) \equiv \int_0^t p_{00}(t-u)dF_{x0}(u), \tag{32}$$

where $*$ denotes convolution, as is easy to verify by first principles; i.e., to be at 0 at time t you have to reach 0 for a first time somewhere in the interval $(0, t]$.

The LST of the first-passage time cdf F_{x0} is

$$\hat{f}_{x0}(s) \equiv \int_0^\infty e^{-st}\, dF_{x0}(t) = e^{-x\zeta(s)} \tag{33}$$

for ζ in (29), and the Laplace transform of $p_{x0}(t)$ is

$$\hat{p}_{x0}(s) \equiv \int_0^\infty e^{-st} p_{x0}(t)dt = \frac{e^{-x\zeta(s)}}{\zeta(s)}; \tag{34}$$

(see (9) on p. 52 of Takács 1962a, p. 229 of Kleinrock 1975, and p. 70 of Prabhu 1980). Hence, if $W(0)$ has cdf F_0, then

$$\hat{p}_0(s) \equiv \int_0^\infty e^{-st} p_0(t)dt = \frac{\hat{f}_0(\zeta(s))}{\zeta(s)}; \tag{35}$$

(see (9) on p. 52 of Takács 1962a).

As a consequence of (34), we have the Laplace transform of the emptiness function p_{00}, i.e.,

$$\hat{p}_{00}(s) \equiv \int_0^\infty e^{-st} p_{00}(t)dt = \frac{1}{\zeta(s)}$$
$$= \frac{1}{s + \rho - \rho\hat{b}(s)}. \tag{36}$$

The final expression confirms that p_{00} is a standard p function associated with a regenerative phenomenon (see (4) on p. 38 of Kingman).

By combining (28), (29), and (36), i.e., by replacing $\hat{b}(s)$ by $\hat{g}(\zeta(s))$ in (36), we see that $\hat{p}_{00}$ satisfies the functional equation

$$\hat{p}_{00}(s) = \frac{1}{s + \rho - \rho\hat{g}(1/\hat{p}_{00}(s))}. \tag{37}$$

The functional equations (28), (31), and (37) are obviously equivalent; i.e., a solution to one yields a solution to all.

To do further analysis, it is convenient to introduce an additional random quantity. As in our previous papers, let $T_{\epsilon 0}$ be the *first passage time to 0 starting in equilibrium*; let $F_{\epsilon 0}(t)$ and $\hat{f}_{\epsilon 0}(s)$ be its cdf and LST, respectively. Consistent with previous notation, let $(f_{\epsilon 0})_k$ be the kth moment of $F_{\epsilon 0}$.

Theorem 3. a. *The LST of the equilibrium time to emptiness is*

$$\hat{f}_{\epsilon 0}(s) = 1 - \rho + \rho\hat{b}_e(s) = \frac{(1-\rho)\zeta(s)}{s} = \hat{v}(\zeta(s))$$
$$= \hat{v}(s\hat{f}_{\epsilon 0}(s)/(1-\rho));$$

b. $(F_{\epsilon 0})_e = B_{ee}$;
c. $(f_{\epsilon 0})_k = \rho b_{ek}$ *for all* $k \geq 1$.

Proof. (Part a) By first principles, in equilibrium the probability that the server is idle is $1 - \rho$ and, given that the server is busy, the remaining busy period has the busy-period stationary-excess distribution, so that we obtain the first formula. Alternatively, from first principles and (33),

$$\hat{f}_{\epsilon 0}(s) = \int_0^\infty \hat{f}_{x0}(s)dV(s) = \int_0^\infty e^{-x\zeta(s)}dV(x)$$
$$= \hat{v}(\zeta(s)), \tag{38}$$

but from (27) and (29),

$$\hat{v}(s) = \frac{(1-\rho)s}{\eta(s)}, \tag{39}$$

so that, by (30),

$$\hat{f}_{\epsilon 0}(s) = \frac{(1-\rho)\zeta(s)}{\eta(\zeta(s))} = \frac{(1-\rho)\zeta(s)}{s}$$
$$= \frac{(1-\rho)(s + \rho - \rho\hat{b}(s))}{s} = 1 - \rho + \rho\hat{b}_e(s),$$

with the last step holding because

$$\hat{b}_e(s) = (1-\rho)\frac{(1-\hat{b}(s))}{s}. \quad (41)$$

We have thus established all formulas except the last one. However, from the second formula, we see that $\hat{\zeta}(s) = s\hat{f}_{\epsilon 0}(s)/(1-\rho)$, which establishes the final formula.

For part b, apply (3), noting that $1 - F_{\epsilon 0}(t) = \rho[1 - B_e(t)]$. For part c, apply the first formula in part a.

We now obtain expressions for the LSTs $\hat{h}_0(s)$ and $\hat{h}_1(s)$. See Corollary 5.2.1 and Theorem 9.1 of Abate and Whitt (1988a) for related $M/M/1$ results. Note that $\hat{h}_0(s) = \hat{b}(s)$ in that case.

Theorem 4

a. $\hat{h}_0(s) = \dfrac{\hat{b}_e(s)}{\hat{f}_{\epsilon 0}(s)} = \dfrac{\hat{b}_e(s)}{1-\rho+\rho\hat{b}_e(s)}$.

b. $h_1(s) = \hat{v}_e(\zeta(s)) = \hat{h}_{0e}(s) = \dfrac{\hat{b}_{ee}(s)}{\hat{f}_{\epsilon 0}(s)} = \dfrac{(\hat{f}_{\epsilon 0})_e(s)}{\hat{f}_{\epsilon 0}(s)}$.

Proof. (Part a) By (23),

$$\hat{h}_0(s) = \frac{1 - s\hat{p}_{00}(s)}{\rho}, \quad (42)$$

so that, by (36) and (41),

$$\hat{h}_0(s) = \frac{1}{\rho}\left[1 - \frac{s}{s+\rho-\rho\hat{b}(s)}\right] = \frac{\hat{b}_e(s)}{1-\rho+\rho\hat{b}_e(s)},$$

which yields the first formula, by Theorem 3a.

(Part b) By (6),

$$\hat{h}_1(s) = \int_0^\infty \hat{f}_{y0}(s)dV_e(y) = \int_0^\infty e^{-y\zeta(s)}dV_e(y)$$
$$= \hat{v}_e(\zeta(s)).$$

By Theorem 2b, $\hat{h}_1 = \hat{h}_{0e}$. By (3) and the result from part a, for some constant c,

$$\hat{h}_{0e}(s) = \frac{c}{s}[1 - \hat{h}_0(s)] = \frac{c}{s}\left[1 - \frac{\hat{b}_e(s)}{1-\rho+\rho\hat{b}_e(s)}\right]$$
$$= \frac{c\hat{b}_{ee}(s)}{\hat{f}_{\epsilon 0}(s)}.$$

However, since $\hat{h}_{0e}(s)\hat{f}_{\epsilon 0}(s)$ and $\hat{b}_{ee}(s)$ are proper cdf's, we must have $c = 1$ and the desired result.

Remark 1. The results in Theorem 4 suggest that $1/\hat{f}_{\epsilon 0}(s)$ might be the LST of a bonafide cdf, but this is not true. Indeed, $1/\hat{f}_{\epsilon 0}(s) = s\hat{p}_{00}(s)/(1-\rho)$ is the Laplace transform of $[\delta_0 + p'_{00}(t)]/(1-\rho)$, where δ_0 denotes a unit point mass at 0 and $-\rho \le p'_{00}(t) \le 0$.

5. MOMENTS OF THE MOMENT CDFs

Even though the $M/G/1$ transient quantities of interest are only characterized implicitly via transform functional equations, we can obtain the moments by differentiating. For the busy-period functional equation (28), this involves a reversion of series, as nicely described on p. 148 of Cox and Smith (1961).

As before, we will denote the kth moment of a cdf F or its LST $\hat{f}$ by f_k. Since the steady-state workload $W(\infty)$ has cdf V, this means that its kth moment is denoted by v_k as well as $m_k(\infty)$. We show that it is convenient to express the moments b_{ek}, h_{0k}, and h_{1k} in terms of the moments v_k. To interpret the results that follow, recall that $g_1 = 1$ and, from (4) and (20),

$$v_k = \frac{\rho}{1-\rho}\sum_{j=1}^{k}\binom{k}{j} g_{ej}v_{k-j}. \quad (43)$$

We first apply the relation

$$\hat{f}_{\epsilon 0}(s) = \hat{v}(s\hat{f}_{\epsilon 0}(s)/(1-\rho)) \quad (44)$$

in Theorem 3a to express the moments b_{ek} in terms of the moments v_k. Recall from Theorem 3c that $(f_{\epsilon 0})_k = \rho b_{ek}$. Let $(f_{\epsilon 0}^{*j})_k$ be the kth moment of the j-fold convolution of $F_{\epsilon 0}$, i.e., of the transform $\hat{f}_{\epsilon 0}(s)^j$. We give a recursive expression for $(f_{\epsilon 0})_k = \rho b_{ek}$ for all k and then a convenient explicit expression for the first four moments of the busy-period stationary-excess cdf B_e.

Theorem 5

a. $(f_{\epsilon 0})_k = \rho b_{ek} = \displaystyle\sum_{j=1}^{k}\binom{k}{j}\frac{v_j}{(1-\rho)^j}(f_{\epsilon 0}^{*j})_{k-j}$, $k \ge 1$,

b. $b_{e1} = \dfrac{v_1}{\rho(1-\rho)}$,

c. $b_{e2} = \dfrac{v_2 + 2v_1^2}{\rho(1-\rho)^2}$,

d. $b_{e3} = \dfrac{v_3 + 9v_2v_1 + 6v_1^3}{\rho(1-\rho)^3}$,

e. $b_{e4} = \dfrac{v_4 + 16v_3v_1 + 12v_2^2 + 72v_2v_1^2 + 24v_1^4}{\rho(1-\rho)^4}$.

Proof. From Theorem 3a,

$$\hat{f}_{\epsilon 0}(s) = \sum_{k=0}^{\infty}(-1)^k\frac{\rho b_{ek}s^k}{k!}, \quad (45)$$

while

$$\hat{v}(s) = \sum_{k=0}^{\infty} (-1)^k \frac{v_k s^k}{k!}. \tag{46}$$

Combining (44)–(46) and rearranging terms yields the results.

From Theorem 5a, we see that $(f_{e0})_k$ is monotone in $(v_1, \ldots, v_k)$, which, in turn, is monotone in $(g_1, \cdots, g_{k+1})$ by (20). (We then think of the arrival rate as fixed instead of the mean service time.)

Since $b_1 = (1-\rho)^{-1}$ and $b_{ek} = b_{k+1}/(k+1)b_1$, we have the following corollary to Theorem 5.

Corollary

a. $b_2 = \dfrac{2v_1}{\rho(1-\rho)^2}$;

b. $b_3 = \dfrac{3(v_2 + 2v_1^2)}{\rho(1-\rho)^3}$;

c. $b_4 = \dfrac{4(v_3 + 9v_2v_1 + 6v_1^3)}{\rho(1-\rho)^4}$;

d. $b_5 = \dfrac{5(v_4 + 16v_3v_1 + 12v_2^2 + 72v_2v_1^2 + 24v_1^4)}{\rho(1-\rho)^5}$.

Similarly, we apply the relation $\hat{f}_{e0}(s)\hat{h}_0(s) = \hat{b}_e(s)$ in Theorem 4a to obtain expressions for the moments h_{0k}. As in Theorem 5, we give a recursive expression for h_{0k} for all k and then convenient explicit expressions for the first four.

Theorem 6

a. $h_{0k} = b_{ek} - \rho \sum_{j=1}^{k} \binom{k}{j} b_{ej} h_{0(k-j)}, \; k \geq 1$;

b. $h_{01} = (1-\rho)b_{e1} = \dfrac{v_1}{\rho}$;

c. $h_{02} = (1-\rho)b_{e2} - 2\rho(1-\rho)b_{e1}^2 = \dfrac{v_2}{\rho(1-\rho)}$;

d. $h_{03} = \dfrac{v_3 + 3v_2v_1}{\rho(1-\rho)^2}$;

e. $h_{04} = \dfrac{v_4 + 8v_3v_1}{\rho(1-\rho)^3} + 12v_2v_1^2 + 6v_2^2$.

Note that ρh_{01} represents the expected total server utilization lost by starting at 0 instead of in steady state; i.e., by (23),

$$\rho h_{01} = \rho \int_0^{\infty} [1 - H_0(t)dt$$

$$= \int_0^{\infty} [\rho - (1 - \rho_{00}(t))]dt$$

$$= \lim_{t\to\infty} E\left[\int_0^t (1_{\{W^*(u)>0\}} - 1_{\{W(u)>0 | W(0)=0\}})du\right],$$

where 1_B is the indicator function of the set B and W^* is the stationary version, as in (10).

By combining (4), Theorems 2b and 6, we obtain expressions for the first three first-moment cdf moments h_{1k}.

Corollary

a. $h_{11} = \dfrac{1}{1-\rho}\left(\dfrac{v_2}{2v_1}\right) = \dfrac{v_{e1}}{1-\rho}$;

b. $h_{12} = \dfrac{1}{(1-\rho)^2}\left(\dfrac{v_3}{3v_1} + v_2\right) = \dfrac{v_{e2} + v_2}{(1-\rho)^2}$;

c. $h_{13} = \dfrac{1}{(1-\rho)^3}\left(\dfrac{v_4}{4v_1} + 2v_3 + 3v_2v_1 + \dfrac{3v_2^2}{2v_1}\right)$.

Theorems 5 and 6 and their corollaries obviously can be extended to higher moments, but we have yet to discover general expressions for the kth moment. Such general expressions (of a sort) follow from (6), however. For this purpose, we describe the moments of the first passage time from x to 0, T_{x0} (see p. 79 of Prabhu 1980).

Theorem 7. *The first four moments of the first passage time T_{x0} are:*

a. $(f_{x0})_1 = \dfrac{x}{1-\rho}$;

b. $(f_{x0})_2 = \dfrac{x}{(1-\rho)^2}(2v_1 + x)$;

c. $(f_{x0})_3 = \dfrac{x}{(1-\rho)^3}(3v_2 + 6v_1(v_1 + x) + x^2)$;

d. $(f_{x0})_4 = \dfrac{x}{(1-\rho)^4}(4v_3 + 36v_2v_1 + 24v_1^3 + 12v_2x + 36v_1^2x + 12v_1x^2 + x^3)$.

Proof. Differentiate the transform in (33) and re-express in terms of the moments v_i.

Combining (6) and Theorem 7, we obtain an alternate proof of the corollary to Theorem 6. We also obtain the following general result directly from (6).

Theorem 8. *For all positive integers k,*

$$h_{1k} = \int_0^\infty E[T_{y0}^k]dV_e(y).$$

General expressions in terms of g_i or v_i for arbitrary k in Theorems 5–8 remain a mathematically interesting open problem.

We can also apply (6) and Theorem 2 to describe the derivatives of H_k at the origin.

Theorem 9. a. *For all* $y > 0$,

$$F_{y0}(t) = 0,\ 0 \le t < y, \tag{47}$$

so that $F_{y0}^{(j)} = 0$ *and* $H_1^{(j)}(0) = 0$ *for all* $j \ge 1$.

b. $H_2^{(1)}(0) = (1 - \rho)/v_{e1}$, *while* $H_2^{(j)}(0) = 0$ *for all* $j \ge 2$.

Proof. a. Note that $W(t)$ decreases at most at rate 1, so that $P[T_{y0} < y] = 0$; (47) with (6) implies the rest.

b. Apply (5) and Theorem 2c.

We can also use Theorem 2, (4), and (5) to obtain $H_k^{(j)}(0)$ for $k \ne 1$ and $j \ge 1$.

6. THE COVARIANCE FUNCTION

Let $C_w(t)$ be the covariance function of the stationary workload process, as defined in (10), and let $c_w(t)$ be the associated *correlation function* defined by

$$C_w(t) = \text{Var}\ (W(\infty))c_w(t),\ t \ge 0. \tag{48}$$

The functions $C_w(t)$ and $c_w(t)$ were studied by Beneš (1957) and Ott (1977a, b). Indeed, Ott derived many structural properties for $C_w(t)$, including the fact that $C_w(t)$ and $C_w'(t)$ are monotone, which implies that $1 - c_w(t)$ is a bonafide cdf provided that $E[S^3] < \infty$, so that Var $(W(\infty)) < \infty$. In this section, we complement these results by providing some additional structure.

For any cdf F with mean f_1, let F^* be the cdf defined by

$$F^*(t) = f_1^{-1} \int_0^t u dF(u),\ t \ge 0. \tag{49}$$

and let $\hat{f}^*$ be its LST. Note that F^* is the stationary total-life distribution associated with F (see p. 195 of Karlin and Taylor). The distribution B^*, where B is the busy-period cdf, plays a key role, as noted by Ott (1977a) (see (2.23) there).

Theorem 10. $1 - c_w(t) = U_e(t)$ *and*

$$1 - \rho \frac{(c_s^2 + 1)}{2} c_w'(t) = U(t), \tag{50}$$

where U is the cdf with LST

$$\hat{u}(s) = \frac{\hat{b}_e^*(s)}{(1 - \rho + \rho\hat{b}^*(s))\hat{f}_{\epsilon 0}(s)} \tag{51}$$

and the first two moments

$$u_1 = \frac{v_2 - v_1^2}{(1 - \rho)v_1} \quad \text{and} \quad u_2 = \frac{v_3 - v_2 v_1}{(1 - \rho)^2 v_1}, \tag{52}$$

so that (12) *holds, i.e.,*

$$\sigma_w^2 \equiv 2 \int_0^\infty C_w(t)dt = 2\ \text{Var}\ W(\infty)u_{e1}$$
$$= \frac{v_3 - v_2 v_1}{1 - \rho} = \frac{\rho}{(1 - \rho)^4}$$
$$\cdot \left[\frac{(1 - \rho)^2}{2} g_{e3} + \frac{5}{2}(1 - \rho)\rho g_{e1} g_{e2} + 2\rho^2 g_{e1}^3\right]. \tag{53}$$

Remark 2. Note that (53) agrees with (2.16) of Ott (1977a). In the $M/M/1$ case, $g_{ek} = g_k$ for all k, $g_3 = 6$, $g_2 = 2$, and $g_1 = 1$, so that $\sigma_w^2 = \rho(3 - \rho)/(1 - \rho)^4$. Also $\hat{u}(s) = b(s)h_1(s)$.

Proof. Let $\hat{m}_1(s, x)$ be the Laplace transform of the moment function $m_1(t, x)$ starting in x. Thus, the conservation law in (18) can be expressed as

$$\hat{m}_1(s, x) = \frac{x}{s} - \frac{(1 - \rho)}{s^2} + \frac{\hat{p}_{x0}(s)}{s}. \tag{54}$$

As in Theorem 8 of Abate and Whitt (1988b), we can express the Laplace transform $\hat{C}_w(s)$ by

$$s\hat{C}_w(s) = \int_0^\infty s\hat{m}_1(s, x)x dV(x) - v_1^2. \tag{55}$$

Combining (54) and (55), we obtain

$$s\hat{C}_w(s) = \text{Var}\ W(\infty) - \frac{(1 - \rho)v_1}{s}$$
$$+ \int_0^\infty \hat{p}_{x0}(s)x\ dV(x). \tag{56}$$

Letting

$$\hat{u}(s) = \frac{s}{(1 - \rho)v_1} \int_0^\infty \hat{p}_{x0}(s)x dV(x), \tag{57}$$

we see that

$$s\hat{C}_w(s) = \text{Var}\ W(\infty)\left[1 - \frac{(1 - \rho)v_1}{\text{Var}\ W(\infty)} \frac{(1 - \hat{u}(s))}{s}\right], \tag{58}$$

so that it suffices to show that $\hat{u}$ defined in (57) coincides with $\hat{u}$ defined in (51) with the first two moments in (52).

Starting from $\hat{u}$ defined in (57), note that $\hat{p}_{x0}(s) = \zeta(s)^{-1}e^{-x\zeta(s)}$ by (34), so that

$$\hat{u}(s) = \frac{s}{(1-\rho)v_1}\int_0^\infty \frac{e^{-x\zeta(s)}}{\zeta(s)} x dV(s)$$
$$= \frac{s}{(1-\rho)v_1}\left(\frac{-1}{\zeta(s)}\right)\frac{d}{d\zeta}\int_0^\infty e^{-x\zeta(s)}dV(s)$$
$$= \frac{s}{(1-\rho)v_1}\left(\frac{-1}{\zeta(s)}\right)\hat{v}'(\zeta(s)). \tag{59}$$

However, by Theorem 3a, $\hat{f}_{\epsilon 0}(s) = \hat{v}(\zeta(s))$, so that

$$\hat{v}'(\zeta) = \frac{d}{d\zeta}\hat{f}_{\epsilon 0}(s) = \frac{d}{d\zeta}(1-\rho+\rho\hat{b}_e(s))\frac{1}{(d\zeta/ds)}$$
$$= \frac{\rho\hat{b}'_e(s)}{1-\rho\hat{b}'(s)} = \frac{-\rho b_{e1}\hat{b}^*_e(s)}{1+\rho b_1\hat{b}^*(s)}. \tag{60}$$

Hence,

$$\hat{u}(s) = \frac{s}{(1-\rho)v_1}\left(\frac{1}{\zeta(s)}\right)\left(\frac{\rho b_{e1}\hat{b}^*_e(s)}{1+\rho b_1\hat{b}^*(s)}\right)$$
$$= \frac{b^*_e(s)}{(1-\rho+\rho\hat{b}^*(s))\hat{f}_{\epsilon 0}(s)},$$

with the second line holding because $(1-\rho)\zeta(s)/s = \hat{f}_{\epsilon 0}(s)$ by Theorem 3a and $\rho b_{e1}/v_1 = 1$. Finally, the moments u_k are obtained by expanding the terms $\hat{b}^*_e(s)$, $\hat{b}^*(s)$, and $\hat{f}_{\epsilon 0}(s)$ in (51). For this purpose, note that $b^*_k = b_{k+1}/b_1 = (k+1)b_{ek}$, $b^*_{ek} = b_{e(k+1)}/b_{e1} = 2b_{k+2}/(k+2)b_2$ and $(f_{\epsilon 0})_k = \rho b_{ek} = \rho b_{k+1}/(k+1)b_1$. Consequently,

$$u_1 = \frac{b_{e2}}{b_{e1}} - 3\rho b_{e1} = \frac{v_2 - v_1^2}{(1-\rho)v_1}$$

and

$$u_2 = \frac{b_{e3}}{b_{e1}} - 10\rho b_{e2} + 14\rho^2 b_{e1}^2 = \frac{v_3 - v_2 v_1}{(1-\rho)^2 v_1}.$$

7. MOMENT-DIFFERENCE CDFs

As noted at the outset, the first two moment-difference functions $d_k(t, x)$ in (1) are monotone, so that we can define associated moment-difference cdf's as in (7). The results beyond Theorem 7.3 of Abate and Whitt (1987b) are contained in the following theorem.

Theorem 11

a. $d_2(t, x)$ *is decreasing and convex.*
b. $G_2 = G_{1e}$.

Proof. Let $d'_k(t, x) = d/dt\, d_k(t, x)$ and $m'(t, x) = d/dt\, m_k(t, x)$. From Theorem 1a,

$$d'_2(t, x) \equiv m'_2(t, x) - m'_2(t, 0)$$
$$= 2(1-\rho)[m_1(t_1 0) - m_1(t, x)]$$
$$= -2(1-\rho)d_1(t, x).$$

Since $d_1(t, x)$ is positive and decreasing in t, $d_2(t, x)$ is decreasing and convex in t. Moreover, from (7),

$$G'_2(t, x) \equiv \frac{d}{dt}G_2(t, x) = \frac{-d'_2(t, x)}{x^2}$$
$$= \frac{2(1-\rho)d_1(t, x)}{x^2}$$
$$= \frac{2(1-\rho)}{x}[1 - G_1(t, x)],$$

Since $G'_2(t, x) = c[1 - G_1(t, x)]$ for some constant c, $G_2 = G_{1e}$ (and the first moment of G_1 must be $g^x_{11} = x/2(1-\rho)$).

Remark 3. It is not difficult to see that $d_3(t, x)$ is not monotone and $d_1(t, x)$ is not convex, using Theorem 1a and (17).

From (8) and Theorems 7 and 11, it is easy to compute the moments of $G_i(t, x)$ for $i = 1, 2$. Let the kth moment of $G_i(t, x)$ be denoted by g^x_{ik}. We summarize the results in the following theorem.

Theorem 12

a. *For all* $x > 0$ *and* $k \geq 1$,
$$g^x_{1k} = 1/x\int_0^x E[T^k_{y0}]dy;$$
b. $$g^x_{11} = \frac{x}{2(1-\rho)};$$
c. $$g^x_{12} = \frac{x}{(1-\rho)^2}\left(v_1 + \frac{x}{3}\right);$$
d. $$g^x_{13} = \frac{x}{(1-\rho)^3}\left(\frac{3v_2}{2} + 3v_1^2 + 2v_1 x + \frac{x^2}{4}\right);$$
e. $$g^x_{14} = \frac{x}{(1-\rho)^4}\left(2v_3 + 18v_2v_1 + 12v_1^3 + 4v_2 x + 12v_1^2 x + 3v_1 x^2 + \frac{x^3}{5}\right).$$

Similarly, we can compute the derivatives at the origin. Let $G^{(j)}_k(t, x)$ be the jth derivative with respect to t of $G_k(t, x)$ in (7) evaluated at t.

Theorem 13. a. *For all* $x > 0$,

$$G^{(1)}_1(t, x) = \frac{1-\rho-p_{x0}(t)}{x}, \tag{61}$$

so that $G_1^{(1)}(0, x) = (1 - \rho)/x$.

b. *For all* $x > 0$,

$$G_2^{(1)}(t, x) = \frac{2(1-\rho)}{x}[1 - G_1(t, x)], \qquad (62)$$

and

$$G_2^{(2)}(t, x) = -\frac{2(1-\rho)}{x^2}[1 - \rho - p_{x0}(t)], \qquad (63)$$

so that $G_2^{(1)}(0, x) = 2(1 - \rho)/x$ *and* $G_2^{(2)}(0, x) = -2(1 - \rho)^2/x^2$.

Proof. a. Note that

$$\begin{aligned} G_1^{(1)}(t, x) &= \frac{d}{dt}\left[1 - \frac{d_1(t, x)}{x}\right] \\ &= \frac{-m'_1(t, x) + m'_1(t, 0)}{x} \\ &= \frac{\rho - 1 - p_{x0}(t)}{x} \text{ by (17);} \end{aligned}$$

b. Apply Theorems 11b and 12b.

Remark 4. Additional properties of the moment-difference cdf's can be obtained as in Section 10 of Abate and Whitt (1987b); e.g., the cdf's $G_1(t, x)$ are stochastically increasing in x.

8. COMPLEMENTARY-CDF CDF's

As in subsection 1.7 of Abate and Whitt (1987a), we can focus on the full-time dependent distribution starting empty instead of the time-dependent moments starting empty, by considering complementary-cdf cdf's. For this purpose, let

$$H_y(t) = \frac{P(W(t) > y | W(0) = 0)}{P(W(\infty) > y)}, \quad t \geq 0. \qquad (64)$$

To characterize the complementary-cdf cdf's, let T_{0y} be the first passage time from 0 to the open interval (y, ∞) by the net input process Y in (14). Since $\rho < 1$, $Y(t) \to -\infty$ as $t \to \infty$, so that T_{0y} has a defective distribution, i.e., $P(T_{0y} < \infty) < 1$. However, the complementary-cdf cdf's can be expressed in terms of the conditional cdf's of T_{0y} given that $T_{0y} < \infty$.

Theorem 14. *For each* $y > 0$, H_y *is a bonafide cdf and*

$$H_y(t) = \frac{P(T_{0y} \leq t)}{P(T_{0y} < \infty)} = P(T_{0y} \leq t | T_{0y} < \infty), \; t > 0.$$

Proof. By Proposition 2, $P(W(t) > y | W(0) = 0)$ is nondecreasing in t, and by (22),

$$P(W(t) > y | W(0) = 0) = P(M(t) > y) \qquad (65)$$

and

$$P(W(\infty) > y) = P(M(\infty) > y). \qquad (66)$$

Moreover, $M(t) > y$ if and only if $T_{0y} \leq t$, which implies that $M(\infty) > y$ if and only if $T_{0y} < \infty$. (We use the fact that T_{0y} is the first passage time to the open interval (y, ∞).) Consequently,

$$P(M(t) > y) = P(T_{0y} \leq t) \qquad (67)$$

and

$$P(M(\infty) > y) = P(T_{0y} < \infty). \qquad (68)$$

Combining (65)–(68) yields the result.

Unfortunately, however, the complementary-cdf cdf's are more complicated than the moment cdf's; e.g., we have yet to determine the moments of H_y. The situation is much nicer for RBM (see subsection 1.7 of Abate and Whitt 1987a).

9. REMAINING PROOFS

In this section we provide the remaining proofs. We start with some lemmas needed in the proof of Proposition 1.

Lemma 1. *For all positive integers n and k*,

$$E\left[\left(\sum_{j=1}^{n} S_j\right)^k\right] \leq n^k E[S^k].$$

Proof. By convexity and Jensen's inequality (p. 47 of Chung 1974),

$$E\left[\left(n^{-1}\sum_{j=1}^{n} S_j\right)^k\right] \leq n^{-1}\sum_{j=1}^{n} E[S_j^k] = E[S^k].$$

Lemma 2. *For all* $t > 0$ *and all positive integers* k, $E[X(t)^k] \leq E[A(t)^k]E[S^k]$.

Proof. Conditioning on $A(t)$ and applying Lemma 1, we obtain

$$E[X(t)^k] = EE\left[\left(\sum_{j=1}^{A(t)} S_j\right)^k \Bigg| A(t)\right] \leq E[A(t)^k]E[S^k].$$

Proof of Proposition 1. a. Let 1_B be the indicator function of the set B. Note that

$$\begin{aligned} &[W(0) - t\}^+ + 1_{\{A(t)-A(t-1) \geq 1\}}(S - 1)^+ \\ &\leq W(t) \leq W(0) + X(t), \end{aligned}$$

so that

$$\sum_{j=0}^{k}\binom{k}{j}[(W(0)-t)^{+j}](1-e^{-\rho})E[(s-1)^{+(k-j)}]$$

$$\leq m_k(t) \leq \sum_{j=0}^{k}\binom{k}{j}m_j(0)E[X(t)^{k-j}]. \quad (69)$$

Since

$$E[|Z|^r]^{1/r} \leq E[|Z|^s]^{1/s} \quad \text{for } 1 \leq r < s \quad (70)$$

for any random variable Z, the right side of (69) is finite, and, thus, $m_k(t) < \infty$ if $m_k(0) < \infty$ and $E[X(t)^k] < \infty$, but $E[X(t)^k] < \infty$ if $E[S^k]$ by Lemma 2. On the other hand, by (70), for the left side of (69) to be finite it is necessary and sufficient that $E[(W(0) - t)^{+k}] < \infty$ and $E[(S - 1)^{+k}] < \infty$. However, it is easy to see that if $E[S^k] = \infty$, then $E[(S - 1)^{+k}] = \infty$. Similarly, if $E[(W(0) - t)^{+k}] = \infty$, then $E[W(0)^k] = \infty$.

b. Let $W(t, x)$ be the workload process with $W(0) = x$. By Proposition 2, $W(t, 0)$ is stochastically increasing, so that $W(t, 0) \Rightarrow W(\infty)$ and $m_k(t, 0) \to m_k(\infty)$ as $t \to \infty$ for all k. Since $\rho < 1$, we can apply the strong law of large numbers to deduce that $Y(t) \to -\infty$ w.p.1 as $t \to \infty$. Hence, $W(\infty) \stackrel{d}{=} \sup_{t \geq 0} \{Y(t)\}$ is a proper random variable. By the coupling argument in Theorem 7.3 of Abate and Whitt (1987b), $D(t, x) \equiv W(t, x) - W(t, 0)$ has decreasing sample paths with $D(t, x) \to 0$ w.p.1 as $t \to \infty$ for all x, because $Y(t) \to -\infty$ w.p.1. Consequently, $W(t) \Rightarrow W(\infty)$ as $t \to \infty$ for all $W(0)$.

c. Since $E[D(t, x)^k] \leq m_k(0) < \infty$ and $W(t, x)^k = (W(t, 0) + D(t, x))^k$,

$$m_k(t, x) = \sum_{j=0}^{k}\binom{k}{j}E[W(t, 0)^j]E[D(t, x)^{k-j}]$$

and $m_k(t) \to m_k(\infty)$ as $t \to \infty$ for all $W(0)$. It thus remains to show that $m_k(\infty) < \infty$ if and only if $E[S^{k+1}] < \infty$. For this final result, we apply the classical random walk arguments (see Kiefer and Wolfowitz 1956, Lemoine 1976, and Chapter VIII of Asmussen 1987). In particular, we can apply PASTA (Poisson Arrivals See Time Averages) to see that $W(\infty)$ is distributed the same as the stationary distribution of the discrete-time waiting-time process. Then we apply Theorem 2.1 on p. 184 of Asmussen, noting that the condition $E[(X^+)^{k+1}] < \infty$ there is equivalent to $E[S^{k+1}] < \infty$.

We now prove a lemma to be used in the proof of Theorem 1.

Lemma 3. *For all positive integers k, $E[X(\epsilon)^k] = \lambda \epsilon E[S^k] + o(\epsilon)$ as $\epsilon \to 0$.*

Proof. Conditioning on $A(\epsilon)$, we obtain

$$E[X(\epsilon)^k] = E\left[E\left[\left(\sum_{j=1}^{A(\epsilon)} S_j\right)^k \middle| A(\epsilon)\right]\right]$$

$$= E[S_1^k]P(A(\epsilon) = 1)$$

$$+ \sum_{m=2}^{\infty} E\left[\left(\sum_{j=1}^{m} S_j\right)^k\right]P(A(\epsilon) = m),$$

where

$$E[S_1^k]P(A(\epsilon) = 1) = E[S^k](\lambda\epsilon + o(\epsilon)) \quad (71)$$

and, by Lemma 1,

$$\sum_{m=1}^{\infty} E\left[\left(\sum_{j=1}^{m} S_j\right)^k\right]P(A(\epsilon) = m)$$

$$\leq \sum_{m=2}^{\infty} m^k E[S^k]\frac{(\lambda\epsilon)^m e^{-\lambda\epsilon}}{m!}$$

$$\leq \epsilon^2 E[S^k]e^{\lambda}\sum_{m=2}^{\infty} m^k \frac{\lambda^m e^{-\lambda}}{m!}$$

$$\leq \epsilon^2 E[S^k]e^{\lambda}E[A(1)^k] = O(\epsilon^2). \quad (72)$$

Combining (71) and (72) gives the desired result.

Proof of Theorem 1. a. The main idea of the proof was sketched in Section 2. To be rigorous, we now bound $m_k(t + \epsilon) - m_k(t)$ above and below by quantities that we can analyze more easily. The upper bound has the input of work $X(t + \epsilon) - X(t)$ in $(t, t + \epsilon]$ come at the end of the interval; the lower bound has it occur at the beginning of the interval. We write $X(\epsilon)$ for $X(t + \epsilon) - X(t)$ below, with the understanding that it is independent of $W(t)$. In particular, note that

$$m_k(t + \epsilon) - m_k(t) \leq E[((W(t) - \epsilon)^+ + X(\epsilon))^k] - E[W(t)],$$

where

$$E[(W(t) - \epsilon)^+ + X(s))^k]$$

$$= \sum_{j=0}^{k}\binom{k}{j}E[(W(t) - \epsilon)^{+j}]E[X(\epsilon)^{k-j}]$$

$$= E[(W(t) - \epsilon)^{+k}] + \sum_{j=0}^{k-1}\binom{k}{j}E[(W(t) - \epsilon)^{+j}]$$

$$\cdot (\rho\epsilon E[S^{k-j}] + o(\epsilon) \text{ (by Lemma 3)}$$

$$= E[W(t)^k] - k\epsilon E[W(t)^{k-1}]$$

$$+ \rho\epsilon\sum_{j=0}^{k-1}\binom{k}{j}E[W(t)^j]E[S^{k-j}] + o(\epsilon),$$

so that

$$\frac{E[((W(t)-\epsilon)^+ + X(\epsilon))^k] - E[W(t)^k]}{\epsilon}$$

$$\to \rho \sum_{j=0}^{k-1} \binom{k}{j} m_j(t) E[S^{k-j}] - k m_{k-1}(t)$$

as $\epsilon \to 0$, as in (17). Next, note that

$$m_k(t+\epsilon) - m_k(t)$$

$$\geq E[(W(t) + X(\epsilon) - \epsilon]^{+k}] - E[W(t)^k],$$

where

$$E[(W(t) + X(\epsilon) - \epsilon)^{+k}]$$

$$= E[(W(t) + X(\epsilon) - \epsilon)^{+k} | W(t) > 0] P(W(t) > 0)$$

$$+ E[(X(\epsilon) - \epsilon)^{+k}] P(W(t) = 0).$$

For $\epsilon < x$,

$$E[(W(t) + X(\epsilon) - \epsilon)^{+k} | W(t) = x]$$

$$= E[(x + X(\epsilon) - \epsilon)^k]$$

$$= \sum_{j=0}^{k} \binom{k}{j} x^j E[(X(\epsilon) - \epsilon)^{k-j}] = x^k + kx^{k-1}(\rho\epsilon - \epsilon)$$

$$+ \sum_{j=0}^{k-2} \binom{k}{j} x^j [\rho \epsilon E[S^{k-j}] + o(\epsilon)]$$

by Lemma 3, while $E[(X(\epsilon) - \epsilon)^{+k}] = \rho\epsilon E[S^k] + o(\epsilon)$ by Lemma 3. Hence,

$$m_k(t+\epsilon) - m_k(t) \geq \rho\epsilon \sum_{j=1}^{k-2} \binom{k}{j} m_j(t) E[S^{k-j}]$$

$$- (1-\rho)\epsilon k m_{k-1}(t) + \rho\epsilon E[S^k] + o(\epsilon).$$

Since the upper and lower bounds have identical limits, the derivative exists and equals the common limit.

b. By part a, $m'_{k+1}(t)$ exists and has the form (19) with k replaced by $(k+1)$. By Proposition 1, $m_j(t) \to m_j(\infty) < \infty$ for all $j \leq k$. Hence, by (19), $m'_{k+1}(t)$ converges to a finite limit, say $m'_{k+1}(\infty)$. If $E[S^{k+2}] < \infty$, then $m_{k+1}(t) \to m_{k+1}(\infty)$ by Proposition 1 and $m'_{k+1}(\infty)$ must be 0. However, the situation is more complicated if $E[S^{k+1}] < \infty = E[S^{k+2}]$, because then $m_{k+1}(\infty) = \infty$. We treat this case by truncating the service-time distribution and taking limits. Let $S_x = \min\{S, x\}$. For each $x > 0$, $E[S_x^k] < \infty$ for all k. Let $W^x(t)$ and $m_k^x(t)$ be $W(t)$ and $m_k(t)$ when the service-time distribution is S_x. It is easy to see that $W^x(t)$ approaches $W(t)$ from below w.p.1 as $x \to \infty$. Moreover, $m_k^x(\infty) \to m_k(\infty)$ as $x \to \infty$ for each k. Hence, by (15) with $t = \infty$, $0 = m_{k+1}^{x'}(\infty) \to m'_{k+1}(\infty)$ as $x \to \infty$.

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