TWO-MOMENT APPROXIMATIONS FOR MAXIMA

SUPPLEMENT

by

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Abstract from the Main Paper

We introduce and investigate approximations for the probability distribution of the maximum of n iid nonnegative random variables, in terms of the number n and the first few moments of the underlying probability distribution, assuming the distribution is unbounded above but does not have a heavy tail. Since the mean of the underlying distribution can immediately be factored out, we focus on the effect of the squared coefficient of variation (SCV, c^2, variance divided by the square of the mean). Our starting point is the classical extreme-value theory for representative distributions with the given SCV - mixtures of exponentials for $c^2 \geq 1$, convolutions of exponentials for $c^2 \leq 1$ and gamma for all c^2. We develop approximations for the asymptotic parameters and evaluate their performance. We show that there is a minimum threshold n^*, depending on the underlying distribution, with $n \geq n^*$ required in order for the asymptotic extreme-value approximations to be effective. The threshold n^* tends to increase as c^2 increases above 1 or decreases below 1.

1 Introduction

This paper is a supplement to Crow et al. (2005), a paper with the same title. Here we give additional tables with additional numerical results.

The problem is a classical one in probability theory: There are n independent and identically distributed (iid) nonnegative random variables - $Z_1, \ldots Z_n$ - each distributed as a random variable Z with a cumulative distribution function (cdf) F, and we are interested in the probability distribution of the *maximum* $M_n \equiv \max\{Z_1, Z_2, \ldots, Z_n\}$.

Given the cdf F, we can easily *numerically calculate* the cdf of M_n, because

$$P(M_n \le x) = F(x)^n, \quad x \ge 0 \ . \tag{1}$$

We also can numerically calculate the moments via

$$E[M_n^k] = \int_0^\infty kx^{k-1}[1 - F(x)^n]\, dx \ ; \tag{2}$$

e.g., see p. 150 of Feller (1971); and we can calculate quantiles ($x_{(n,q)}$ such that $P(M_n \le x_{(n,q)}) = q$) by performing binary search with the cdf in (1).

However, suppose that we have only a *partial characterization* of the cdf F. In particular, suppose that we know only its first two moments - $m_k \equiv E[Z^k]$ for $k = 1, 2$ - or, equivalently, only its mean $m_1 \equiv E[Z]$ and its *squared coefficient of variation* $c^2 \equiv c_Z^2$ (SCV, variance divided by the square of the mean). What can we say about the distribution of M_n now?

In the main paper we develop approximations for the distribution of M_n based on the mean m_1 and SCV c^2. We focus especially on the quantiles $x_{(n,q)}$. In the following sections we give additional tables of numerical results for several underlying probability distributions. In Section 2 we give numerical results for the hyperexponential (H_2) distribution (mixture of two exponential distributions), all of which have $c^2 > 1$. In Section 3 we give numerical results for convolutions of exponential distributions, all of which have $c^2 < 1$. In Section 4 we give numerical results for the gamma distribution, which has only two parameters, but covers the full range of possible SCV values: $0 < c^2 < \infty$. The gamma distribution covers the exponential distribution as a special case. In all other cases, the gamma distribution fails to have a pure-exponential tail.

2 The H_2 Distribution

In this section we assume that the underlying cdf F is a mixture of two exponential distributions, and thus H_2. These distributions have $c^2 > 1$ and so tend to be more variable than the exponential distribution. In this section we give additional numerical results for the H_2 distribution. In successive subsections we consider three different quantiles of the distribution of the maximum, M_n: $q = 0.25$, $q = 0.50$, and $q = 0.75$.

2.1 Quantile $q = 0.25$

Below are tables featuring the comparison of exact values with approximations for the $q = 0.25$ quantile of the cdf of the maximum of n iid H_2 random variables with mean 1 and $SCV = \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.9	1.9	2.2	-2.5	1.0	2.0	-7.2	0.6	2.0	2.0	3.2
20	2.7	3.0	3.3	0.0	2.6	3.3	-3.1	2.6	3.4	3.4	4.6
100	6.0	6.5	6.3	5.8	6.4	6.3	6.6	7.5	6.6	6.6	7.8
1000	14.0	11.9	10.6	14.0	11.9	10.6	20.4	14.4	11.2	11.2	12.4
10000	22.3	17.3	14.8	22.3	17.3	14.8	34.2	21.3	15.8	15.8	17.0
100000	30.5	22.8	19.1	30.5	22.8	19.1	48.0	28.2	20.4	20.4	21.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	
1000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	
10000	698.1	2113.2	4031.4				416.7	1666.7	3750.0	3750.0	
100000	6981.0	21132.5	40314.4				4166.7	16666.7	37500.0	37500.0	

Table 1: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 2$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.7	1.6	1.9	-11.1	-0.9	1.7	-17.1	-1.6	1.6	0.5	3.7
20	2.4	2.8	4.0	-5.8	2.2	3.9	-10.2	1.8	3.9	3.3	6.4
100	6.7	9.3	9.1	6.6	9.3	9.1	5.9	9.9	9.3	9.7	12.9
1000	24.2	19.5	16.5	24.2	19.5	16.5	28.9	21.4	17.0	18.9	22.1
10000	41.9	29.7	23.9	41.9	29.7	23.9	51.9	32.9	24.6	28.1	31.3
100000	59.6	39.9	31.4	59.6	39.9	31.4	75.0	44.4	32.3	37.3	40.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	
1000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	
10000	325.8	1127.0	2325.8				250.0	1000.0	2250.0	1562.5	
100000	3257.7	11270.2	23257.7				2500.0	10000.0	22500.0	15625.0	

Table 2: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 4$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.6	1.3	1.0	-34.0	-7.2	-0.5	-41.4	-8.2	-0.6	-5.4	1.8
20	2.2	2.0	3.8	-23.1	-1.4	3.6	-28.9	-2.0	3.5	0.1	7.3
100	4.7	12.3	13.1	2.2	12.3	13.1	0.0	12.5	13.2	13.0	20.2
1000	38.4	31.8	26.7	38.3	31.8	26.7	41.5	33.2	27.0	31.4	38.6
10000	74.5	51.3	40.3	74.5	51.3	40.3	82.9	53.9	40.8	49.8	57.0
100000	110.7	70.8	53.9	110.7	70.8	53.9	124.4	74.7	54.6	68.3	75.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	
1000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	
10000	159.1	590.4	1270.2				138.9	555.6	1250.0	703.1	
100000	1591.0	5904.1	12702.1				1388.9	5555.6	12500.0	7031.2	

Table 3: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 8$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.6	1.2	0.7	-91.0	-25.0	-8.2	-99.8	-26.4	-8.4	-22.9	-7.5
20	2.1	1.7	1.3	-69.0	-13.6	-0.4	-76.3	-14.6	-0.5	-11.8	3.6
100	3.8	13.0	17.7	-17.9	12.9	17.7	-21.6	12.8	17.7	14.0	29.3
1000	55.2	50.9	43.6	55.1	50.9	43.6	56.7	51.9	43.8	50.8	66.2
10000	128.2	88.8	69.4	128.2	88.8	69.4	135.0	91.1	69.9	87.7	103.0
100000	201.3	126.8	95.3	201.3	126.8	95.3	213.3	130.2	96.0	124.5	139.8
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	
1000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	
10000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	
100000	787.8	3033.2	6670.2				735.3	2941.2	6617.6	3320.3	

Table 4: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 16$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.6	1.1	0.6	-227.2	-71.4	-30.7	-237.6	-73.1	-30.9	-68.9	-37.2
20	2.1	1.5	0.9	-183.1	-48.9	-15.5	-191.9	-50.2	-15.6	-46.7	-15.0
100	3.5	4.2	20.0	-80.5	3.4	19.8	-85.6	2.9	19.8	4.8	36.5
1000	66.3	78.2	70.3	66.3	78.2	70.3	66.3	78.9	70.5	78.5	110.1
10000	213.1	153.0	120.7	213.1	153.0	120.7	218.3	154.9	121.1	152.2	183.8
100000	359.8	227.8	171.2	359.8	227.8	171.2	370.3	230.9	171.8	225.9	257.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	
1000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	
10000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	
100000	392.2	1538.8	3422.5				378.8	1515.2	3409.1	1611.3	

Table 5: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 32$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.5	1.1	0.6	-544.2	-186.0	-89.9	-556.1	-188.0	-90.2	-183.1	-118.8
20	2.1	1.4	0.8	-455.7	-141.3	-59.9	-466.0	-142.9	-60.1	-138.7	-74.4
100	3.3	2.5	10.0	-250.1	-37.5	9.7	-256.8	-38.3	9.6	-35.7	28.6
1000	44.1	111.1	109.3	44.0	111.0	109.3	42.5	111.4	109.4	111.7	175.9
10000	338.2	259.5	208.9	338.2	259.5	208.8	341.9	261.1	209.2	259.0	323.3
100000	632.3	408.0	308.4	632.3	408.0	308.4	641.2	410.7	309.0	406.4	470.7
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	
1000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	
10000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	
100000	195.7	775.2	1734.2				192.3	769.2	1730.8	793.5	

Table 6: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 64$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.5	1.0	0.5	-1267.1	-459.1	-237.6	-1280.5	-461.4	-237.9	-455.9	-326.3
20	2.0	1.4	0.7	-1089.8	-370.1	-178.0	-1101.7	-372.0	-178.3	-367.1	-237.6
100	3.3	2.3	1.3	-678.2	-163.3	-39.7	-686.5	-164.4	-39.9	-161.1	-31.6
1000	5.8	132.7	158.2	-89.3	132.6	158.1	-92.4	132.6	158.2	133.6	263.1
10000	499.5	428.5	355.9	499.5	428.5	355.9	501.7	429.7	356.2	428.3	557.9
100000	1088.4	724.4	553.7	1088.4	724.4	553.7	1095.7	726.7	554.2	723.1	852.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	
1000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	
10000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	
100000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	

Table 7: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 128$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.5	1.0	0.5	-2890.4	-1093.8	-591.6	-2905.4	-1096.4	-592.0	-1090.2	-830.1
20	2.0	1.4	0.7	-2535.7	-916.0	-472.9	-2549.1	-918.3	-473.3	-912.7	-652.7
100	3.2	2.2	1.2	-1712.1	-503.2	-197.3	-1721.9	-504.7	-197.5	-500.7	-240.6
1000	5.3	87.6	197.1	-533.8	87.4	197.0	-538.4	87.1	197.0	88.8	348.8
10000	644.6	678.1	591.3	644.6	678.0	591.3	645.2	678.9	591.5	678.2	938.3
100000	1822.9	1268.7	985.6	1822.9	1268.6	985.6	1828.7	1270.6	986.0	1267.7	1527.7
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	
1000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	
10000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	
100000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	

Table 8: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 256$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	1.5	1.0	0.5	-6492.2	-2540.2	-1417.7	-6508.7	-2543.2	-1418.1	-2536.2	-2015.
20	2.0	1.4	0.7	-5782.6	-2185.0	-1180.7	-5797.5	-2187.6	-1181.1	-2181.3	-1660.
100	3.2	2.2	1.1	-4134.9	-1360.1	-630.4	-4146.3	-1362.0	-630.6	-1357.3	-836.
1000	5.1	3.9	157.1	-1777.6	-180.1	156.9	-1783.8	-180.7	156.9	-178.4	342.
10000	579.7	1000.1	944.2	579.6	1000.0	944.2	578.7	1000.5	944.3	1000.5	1521.
100000	2936.9	2180.1	1731.5	2936.9	2180.1	1731.5	2941.1	2181.7	1731.8	2179.5	2700.
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
1000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	
10000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	
100000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	

Table 9: Approximations for the $q = 0.25$ quantile for H_2 with $SCV = 512$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 4.0$.

2.2 Quantile $q = 0.50$

Below are comparisons of exact values with approximations for the $q = 0.50$ quantile of the cdf of the maximum of n iid H_2 random variables with mean 1 and $SCV = \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.7	3.0	3.3	0.0	2.6	3.3	-3.1	2.6	3.4	3.4	3.2
20	3.7	4.4	4.6	2.5	4.3	4.6	1.1	4.7	4.8	4.8	4.6
100	8.3	8.1	7.6	8.3	8.1	7.6	10.8	9.5	8.0	8.0	7.8
1000	16.5	13.5	11.8	16.5	13.5	11.8	24.6	16.4	12.6	12.6	12.4
10000	24.8	19.0	16.1	24.8	19.0	16.1	38.4	23.4	17.2	17.2	17.0
100000	33.0	24.4	20.4	33.0	24.4	20.4	52.2	30.3	21.8	21.8	21.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	
1000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	
10000	698.1	2113.2	4031.4				416.7	1666.7	3750.0	3750.0	
100000	6981.0	21132.5	40314.4				4166.7	16666.7	37500.0	37500.0	

Table 10: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 2$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.4	2.8	4.0	-5.8	2.2	3.9	-10.2	1.8	3.9	3.3	3.7
20	3.3	5.3	6.2	-0.5	5.2	6.1	-3.3	5.3	6.2	6.0	6.4
100	11.9	12.4	11.3	11.9	12.4	11.3	12.8	13.3	11.6	12.5	12.9
1000	29.5	22.6	18.8	29.5	22.6	18.8	35.9	24.9	19.3	21.7	22.1
10000	47.2	32.8	26.2	47.2	32.8	26.2	58.9	36.4	27.0	30.9	31.3
100000	64.9	43.0	33.6	64.9	43.0	33.6	81.9	47.9	34.6	40.1	40.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	
1000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	
10000	325.8	1127.0	2325.8				250.0	1000.0	2250.0	1562.5	
100000	3257.7	11270.2	23257.7				2500.0	10000.0	22500.0	15625.0	

Table 11: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 4$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.2	2.0	3.8	-23.1	-1.4	3.6	-28.9	-2.0	3.5	0.1	1.8
20	2.9	4.7	7.8	-12.2	4.5	7.7	-16.5	4.2	7.7	5.7	7.3
100	13.1	18.2	17.2	13.1	18.1	17.2	12.5	18.7	17.4	18.5	20.2
1000	49.2	37.6	30.8	49.2	37.6	30.8	54.0	39.5	31.2	37.0	38.6
10000	85.4	57.1	44.4	85.4	57.1	44.4	95.4	60.2	45.0	55.4	57.0
100000	121.6	76.6	58.0	121.6	76.6	58.0	136.8	80.9	58.8	73.8	75.5
	exact np						approx. np				
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	
1000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	
10000	159.1	590.4	1270.2				138.9	555.6	1250.0	703.1	
100000	1591.0	5904.1	12702.1				1388.9	5555.6	12500.0	7031.2	

Table 12: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 8$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.1	1.7	1.3	-69.0	-13.6	-0.4	-76.3	-14.6	-0.5	-11.8	-7.5
20	2.7	2.5	7.6	-47.0	-2.2	7.4	-52.7	-2.8	7.3	-0.7	3.6
100	6.0	24.4	25.5	4.1	24.3	25.5	2.0	24.6	25.6	25.1	29.3
1000	77.1	62.3	51.4	77.1	62.3	51.3	80.3	63.7	51.7	61.9	66.2
10000	150.2	100.2	77.2	150.2	100.2	77.2	158.6	102.9	77.8	98.7	103.0
100000	223.3	138.2	103.1	223.3	138.2	103.1	236.9	142.0	103.9	135.6	139.8
	exact np						approx. np				
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	
1000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	
10000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	
100000	787.8	3033.2	6670.2				735.3	2941.2	6617.6	3320.3	

Table 13: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 16$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.1	1.5	0.9	-183.1	-48.9	-15.5	-191.9	-50.2	-15.6	-46.7	-37.2
20	2.6	2.0	1.6	-138.9	-26.4	-0.3	-146.1	-27.3	-0.4	-24.5	-15.0
100	4.3	26.0	35.1	-36.3	25.9	35.0	-39.9	25.8	35.0	27.0	36.5
1000	110.5	100.7	85.5	110.5	100.7	85.5	112.1	101.8	85.7	100.7	110.1
10000	257.2	175.5	135.9	257.2	175.5	135.9	264.1	177.8	136.4	174.4	183.8
100000	404.0	250.4	186.4	404.0	250.4	186.4	416.0	253.8	187.0	248.0	257.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	
1000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	
10000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	
100000	392.2	1538.8	3422.5				378.8	1515.2	3409.1	1611.3	

Table 14: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 32$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.1	1.4	0.8	-455.7	-141.3	-59.9	-466.0	-142.9	-60.1	-138.7	-118.8
20	2.6	1.8	1.0	-367.2	-96.6	-29.9	-375.9	-97.8	-30.1	-94.4	-74.4
100	4.0	7.4	39.8	-161.6	7.2	39.7	-166.7	6.8	39.7	8.7	28.6
1000	132.6	155.7	139.3	132.6	155.7	139.2	132.7	156.4	139.4	156.0	175.9
10000	426.7	304.2	238.8	426.7	304.2	238.8	432.0	306.1	239.2	303.4	323.3
100000	720.9	452.7	338.4	720.9	452.7	338.4	731.3	455.8	339.0	450.7	470.7
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	
1000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	
10000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	
100000	195.7	775.2	1734.2				192.3	769.2	1730.8	793.5	

Table 15: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 64$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.0	1.4	0.7	-1089.8	-370.1	-178.0	-1101.7	-372.0	-178.3	-367.1	-326.3
20	2.6	1.8	0.9	-912.6	-281.0	-118.5	-922.9	-282.6	-118.7	-278.4	-237.6
100	3.8	2.9	20.1	-501.0	-74.2	19.8	-507.6	-75.0	19.8	-72.4	-31.6
1000	88.0	221.7	217.7	87.9	221.7	217.6	86.4	222.0	217.8	222.3	263.1
10000	676.8	517.6	415.5	676.8	517.6	415.5	680.5	519.1	415.8	517.1	557.9
100000	1265.7	813.4	613.3	1265.7	813.4	613.3	1274.6	816.1	613.8	811.8	852.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	
1000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	
10000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	
100000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	

Table 16: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 128$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.0	1.4	0.7	-2535.7	-916.0	-472.9	-2549.1	-918.3	-473.3	-912.7	-830.1
20	2.5	1.7	0.9	-2181.0	-738.2	-354.2	-2192.9	-740.2	-354.5	-735.3	-652.7
100	3.8	2.7	1.5	-1357.4	-325.4	-78.6	-1365.6	-326.5	-78.7	-323.3	-240.6
1000	6.4	265.3	315.8	-179.0	265.2	315.7	-182.1	265.2	315.8	266.2	348.8
10000	999.3	855.8	710.0	999.3	855.8	710.0	1001.4	857.0	710.3	855.7	938.3
100000	2177.7	1446.4	1104.3	2177.7	1446.4	1104.3	2185.0	1448.8	1104.8	1445.1	1527.7
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	
1000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	
10000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	
100000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	

Table 17: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 256$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.0	1.4	0.7	-5782.6	-2185.0	-1180.7	-5797.5	-2187.6	-1181.1	-2181.3	-2015.
20	2.5	1.7	0.9	-5073.0	-1829.7	-943.7	-5086.4	-1832.0	-944.0	-1826.4	-1660.
100	3.8	2.6	1.3	-3425.3	-1004.9	-393.4	-3435.1	-1006.4	-393.6	-1002.4	-836.
1000	5.8	175.4	394.0	-1068.0	175.2	393.9	-1072.6	174.9	393.9	176.5	342.
10000	1289.3	1355.3	1181.2	1289.2	1355.3	1181.2	1289.8	1356.1	1181.4	1355.4	1521.
100000	3646.5	2535.3	1968.5	3646.5	2535.3	1968.5	3652.3	2537.3	1968.9	2534.4	2700.
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
1000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	
10000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	
100000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	

Table 18: Approximations for the $q = 0.50$ quantile for H_2 with $SCV = 512$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 2.0$.

2.3 Quantile $q = 0.75$

Below are tables featuring the comparison of exact values with approximations for the $q = 0.75$ quantile of the cdf of the maximum of n iid H_2 random variables with mean 1 and $SCV = \{2, 4, 8, 16, 32, 64, 128, 256, 512\}$ for four values of n and three values of r. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	4.1	4.8	4.9	3.2	4.7	4.9	2.2	5.3	5.1	5.1	3.2
20	5.9	6.4	6.2	5.7	6.4	6.2	6.4	7.3	6.5	6.5	4.6
100	11.4	10.2	9.2	11.4	10.2	9.2	16.0	12.2	9.7	9.7	7.8
1000	19.7	15.6	13.5	19.7	15.6	13.5	29.9	19.1	14.3	14.3	12.4
10000	27.9	21.1	17.8	27.9	21.1	17.8	43.7	26.0	19.0	19.0	17.0
100000	36.2	26.5	22.0	36.2	26.5	22.0	57.5	32.9	23.6	23.6	21.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.7	2.1	4.0				0.4	1.7	3.8	3.8	
20	1.4	4.2	8.1				0.8	3.3	7.5	7.5	
100	7.0	21.1	40.3				4.2	16.7	37.5	37.5	
1000	69.8	211.3	403.1				41.7	166.7	375.0	375.0	
10000	698.1	2113.2	4031.4				416.7	1666.7	3750.0	3750.0	
100000	6981.0	21132.5	40314.4				4166.7	16666.7	37500.0	37500.0	

Table 19: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 2$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	3.7	6.1	6.8	1.0	6.1	6.7	-1.4	6.2	6.9	6.8	3.7
20	6.5	9.2	9.0	6.3	9.1	9.0	5.5	9.7	9.2	9.5	6.4
100	18.6	16.3	14.2	18.6	16.3	14.2	21.6	17.7	14.5	16.0	12.9
1000	36.3	26.5	21.6	36.3	26.5	21.6	44.6	29.3	22.2	25.2	22.1
10000	54.0	36.7	29.0	54.0	36.7	29.0	67.7	40.8	29.9	34.4	31.3
100000	71.6	46.9	36.4	71.6	46.9	36.4	90.7	52.3	37.6	43.6	40.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.3	1.1	2.3				0.2	1.0	2.2	1.6	
20	0.7	2.3	4.7				0.5	2.0	4.5	3.1	
100	3.3	11.3	23.3				2.5	10.0	22.5	15.6	
1000	32.6	112.7	232.6				25.0	100.0	225.0	156.2	
10000	325.8	1127.0	2325.8				250.0	1000.0	2250.0	1562.5	
100000	3257.7	11270.2	23257.7				2500.0	10000.0	22500.0	15625.0	

Table 20: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 4$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	3.2	6.2	8.9	-9.3	6.1	8.8	-13.1	5.9	8.8	7.1	1.8
20	4.6	12.0	12.9	1.6	12.0	12.9	-0.6	12.2	13.0	12.7	7.3
100	26.9	25.6	22.4	26.9	25.6	22.4	28.3	26.6	22.6	25.6	20.2
1000	63.1	45.1	36.0	63.1	45.1	36.0	69.8	47.4	36.4	44.0	38.6
10000	99.2	64.6	49.6	99.2	64.6	49.6	111.2	68.1	50.3	62.4	57.0
100000	135.4	84.1	63.2	135.4	84.1	63.2	152.7	88.8	64.1	80.8	75.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.2	0.6	1.3				0.1	0.6	1.2	0.7	
20	0.3	1.2	2.5				0.3	1.1	2.5	1.4	
100	1.6	5.9	12.7				1.4	5.6	12.5	7.0	
1000	15.9	59.0	127.0				13.9	55.6	125.0	70.3	
10000	159.1	590.4	1270.2				138.9	555.6	1250.0	703.1	
100000	1591.0	5904.1	12702.1				1388.9	5555.6	12500.0	7031.2	

Table 21: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 8$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.9	3.0	9.6	-41.1	0.9	9.5	-46.4	0.4	9.4	2.3	-7.5
20	3.7	12.4	17.3	-19.1	12.3	17.2	-22.8	12.2	17.3	13.4	3.6
100	32.0	38.9	35.4	32.0	38.8	35.3	31.9	39.5	35.5	39.1	29.3
1000	105.0	76.8	61.2	105.0	76.8	61.2	110.2	78.7	61.6	76.0	66.2
10000	178.1	114.7	87.1	178.1	114.7	87.1	188.5	117.8	87.7	112.8	103.0
100000	251.2	152.7	113.0	251.2	152.7	113.0	266.8	157.0	113.8	149.7	139.8
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
20	0.2	0.6	1.3				0.1	0.6	1.3	0.7	
100	0.8	3.0	6.7				0.7	2.9	6.6	3.3	
1000	7.9	30.3	66.7				7.4	29.4	66.2	33.2	
10000	78.8	303.3	667.0				73.5	294.1	661.8	332.0	
100000	787.8	3033.2	6670.2				735.3	2941.2	6617.6	3320.3	

Table 22: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 16$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.8	2.2	4.1	-127.0	-20.3	3.8	-133.8	-21.2	3.7	-18.5	-37.2
20	3.4	3.8	19.2	-82.8	2.2	19.0	-88.1	1.7	19.0	3.6	-15.0
100	19.8	54.5	54.3	19.8	54.5	54.3	18.2	54.8	54.4	55.1	36.5
1000	166.5	129.3	104.7	166.5	129.3	104.7	170.1	130.8	105.0	128.8	110.1
10000	313.3	204.1	155.2	313.3	204.1	155.2	322.1	206.8	155.7	202.5	183.8
100000	460.1	278.9	205.6	460.1	278.9	205.6	474.1	282.8	206.4	276.2	257.5
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
20	0.1	0.3	0.7				0.1	0.3	0.7	0.3	
100	0.4	1.5	3.4				0.4	1.5	3.4	1.6	
1000	3.9	15.4	34.2				3.8	15.2	34.1	16.1	
10000	39.2	153.9	342.3				37.9	151.5	340.9	161.1	
100000	392.2	1538.8	3422.5				378.8	1515.2	3409.1	1611.3	

Table 23: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 32$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.7	1.9	1.1	-343.4	-84.6	-21.9	-351.7	-85.7	-22.0	-82.4	-118.8
20	3.3	2.5	8.4	-254.8	-39.9	8.1	-261.6	-40.7	8.0	-38.1	-74.4
100	5.2	64.0	77.8	-49.2	63.9	77.7	-52.4	63.9	77.8	64.9	28.6
1000	244.9	212.5	177.3	244.9	212.4	177.3	247.0	213.6	177.5	212.3	175.9
10000	539.1	361.0	276.9	539.1	361.0	276.9	546.3	363.3	277.3	359.7	323.3
100000	833.2	509.5	376.4	833.2	509.5	376.4	845.6	512.9	377.1	507.0	470.7
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
20	0.0	0.2	0.3				0.0	0.2	0.3	0.2	
100	0.2	0.8	1.7				0.2	0.8	1.7	0.8	
1000	2.0	7.8	17.3				1.9	7.7	17.3	7.9	
10000	19.6	77.5	173.4				19.2	76.9	173.1	79.3	
100000	195.7	775.2	1734.2				192.3	769.2	1730.8	793.5	

Table 24: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 64$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.7	1.9	1.0	-864.9	-257.1	-102.5	-874.8	-258.6	-102.6	-254.6	-326.3
20	3.2	2.3	1.3	-687.7	-168.0	-42.9	-696.0	-169.2	-43.0	-165.9	-237.6
100	4.7	39.0	95.5	-276.1	38.8	95.4	-280.8	38.5	95.4	40.2	-31.6
1000	312.9	334.7	293.2	312.8	334.7	293.2	313.3	335.5	293.4	334.9	263.1
10000	901.7	630.6	491.0	901.7	630.6	491.0	907.4	632.5	491.4	629.6	557.9
100000	1490.6	926.4	688.9	1490.6	926.4	688.9	1501.4	929.6	689.4	924.3	852.6
	exact np						approx. np				
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
20	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
100	0.1	0.4	0.9				0.1	0.4	0.9	0.4	
1000	1.0	3.9	8.7				1.0	3.9	8.7	3.9	
10000	9.8	38.9	87.3				9.7	38.8	87.2	39.4	
100000	97.8	389.1	872.9				96.9	387.6	872.1	393.7	

Table 25: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 128$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.7	1.8	0.9	-2085.7	-690.4	-322.3	-2097.1	-692.3	-322.6	-687.6	-830.1
20	3.2	2.2	1.2	-1731.0	-512.6	-203.6	-1740.9	-514.2	-203.8	-510.2	-652.7
100	4.5	3.5	72.2	-907.4	-99.8	72.0	-913.6	-100.5	71.9	-98.1	-240.6
1000	271.1	490.8	466.3	271.0	490.8	466.3	269.9	491.2	466.4	491.3	348.8
10000	1449.3	1081.4	860.6	1449.3	1081.4	860.6	1453.4	1083.0	860.9	1080.8	938.3
100000	2627.7	1672.0	1254.9	2627.7	1672.0	1254.9	2637.0	1674.8	1255.5	1670.2	1527.7
	exact np						approx. np				
$n \backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.1				0.0	0.0	0.1	0.0	
100	0.0	0.2	0.4				0.0	0.2	0.4	0.2	
1000	0.5	1.9	4.4				0.5	1.9	4.4	2.0	
10000	4.9	19.5	43.8				4.9	19.5	43.8	19.6	
100000	48.9	194.9	438.0				48.6	194.6	437.7	196.1	

Table 26: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 256$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

	exact (1.1)			asymp. (3.6)			product (4.13)			simple (1.9)	crude (1.8)
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75		
10	2.7	1.8	0.9	-4882.3	-1734.3	-880.0	-4895.3	-1736.5	-880.3	-1731.1	-2015.1
20	3.2	2.2	1.1	-4172.7	-1379.0	-643.0	-4184.1	-1380.9	-643.3	-1376.2	-1660.2
100	4.5	3.1	1.8	-2525.0	-554.2	-92.7	-2532.8	-555.2	-92.8	-552.2	-836.2
1000	7.5	625.9	694.6	-167.8	625.9	694.6	-170.4	626.0	694.7	626.8	342.7
10000	2189.5	1805.9	1481.9	2189.5	1805.9	1481.9	2192.1	1807.2	1482.1	1805.7	1521.7
100000	4546.8	2986.0	2269.2	4546.8	2986.0	2269.2	4554.5	2988.4	2269.6	2984.6	2700.6
	exact np						approx. np				
$n\backslash r$	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75	$n\psi(c^2)$	in (1.12)
10	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
20	0.0	0.0	0.0				0.0	0.0	0.0	0.0	
100	0.0	0.1	0.2				0.0	0.1	0.2	0.1	
1000	0.2	1.0	2.2				0.2	1.0	2.2	1.0	
10000	2.4	9.8	21.9				2.4	9.7	21.9	9.8	
100000	24.4	97.6	219.4				24.4	97.5	219.3	97.8	

Table 27: Approximations for the $q = 0.75$ quantile for H_2 with $SCV = 512$. Also displayed are exact values and approximations for np, indicating when the asymptotics should be used. The problematic values of n are those for which $np < 1/q = 1.33333$.

3 Convolutions of Exponential Distributions

In this section we assume that the underlying cdf F is the convolution of two or more exponential distributions, all with different means. These distributions have $c^2 < 1$ and so tend to be less variable than the exponential distribution. These distributions also all have a pure-exponential tail.

We can achieve any SCV value between $1/n$ and 1 with a convolution of n exponentials; e.g., see Aldous and Shepp (1987). If we restrict attention to $n = 2$, there are only two parameters, which we can match to the mean and the SCV. We get the equations

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 \quad \text{and} \quad \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} = c^2 \ , \tag{3}$$

with the constraints: $\lambda_1 < \lambda_2$ and $0.5 < c^2 < 1$. That yields a quadratic equation for λ_1^{-1} in terms of c^2.

In the first subsection we consider the convolution of *two* exponential distributions, which have SCV values $0.5 < c^2 < 1$. In the second subsection we consider convolutions of four exponentials, which yield SCV values $0.25 < c^2 < 1$.

3.1 Convolutions of Two Exponentials: $0.50 < SCV < 1$

We present numerical results for a wide range of c^2 in the tables below, in particular, for c^2 in $\{0.99, 0.9, 0.8, 0.7, 0.6, 0.55, 0.51, 0.501, 0.5001\}$. We include the cases with $c^2 \leq 0.51$ to deliberately include difficult cases, in which λ_2^{-1} is close to λ_1^{-1}. In this case the extreme-value approximation is not initially so good, but eventually becomes good.

Except in the relatively pathological cases with c^2 close to 0.5, the extreme-value asymptotics, i.e., (3.6), are spectacular for all values of n. However, the asymptotic approximation begins to perform poorly for smaller n as c^2 decreases toward 0.5. The simple rough approximation in (1.9) is also spectacular for all n when c^2 is high, e.g., above 0.7. For lower c^2, (1.9) produces an error for very large n. For $10 \leq n \leq 1000$, (1.9) is consistently good for all c^2, being much better than the asymptotic approximation in (3.6) for the last "pathological" cases with c^2 very close to 0.5.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.69	2.66	2.66	3.55	3.54	3.54	2.29
20	3.37	3.35	3.35	4.23	4.23	4.23	2.98
100	4.96	4.95	4.95	5.83	5.83	5.83	4.57
1000	7.24	7.24	7.24	8.12	8.12	8.12	6.85
100000	11.82	11.82	11.82	12.70	12.70	12.70	11.41
1000000	14.12	14.12	14.12	14.99	14.99	14.99	13.69

Table 28: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.99$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.9950$, $\lambda_2^{-1} = 0.0050$, $C_{1,2} = 1.0051$ and $C_{2,2} = -0.0051$.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.62	2.58	2.58	3.43	3.42	3.42	2.17
20	3.26	3.24	3.24	4.08	4.07	4.08	2.79
100	4.77	4.76	4.77	5.60	5.60	5.60	4.24
1000	6.94	6.94	6.95	7.78	7.78	7.79	6.31
100000	11.31	11.31	11.32	12.14	12.14	12.16	10.46
1000000	13.49	13.49	13.51	14.32	14.32	14.34	12.53

Table 29: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.90$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.9472$, $\lambda_2^{-1} = 0.0528$, $C_{1,2} = 1.0590$ and $C_{2,2} = -0.0590$.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.52	2.49	2.49	3.28	3.27	3.28	2.02
20	3.12	3.10	3.11	3.89	3.88	3.90	2.58
100	4.54	4.53	4.55	5.31	5.31	5.34	3.86
1000	6.58	6.57	6.61	7.36	7.36	7.40	5.70
100000	10.66	10.66	10.73	11.44	11.44	11.52	9.39
1000000	12.70	12.70	12.79	13.48	13.48	13.58	11.23

Table 30: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.8$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.8873$, $\lambda_2^{-1} = 0.1127$, $C_{1,2} = 1.1455$ and $C_{2,2} = -0.1455$.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.42	2.39	2.40	3.12	3.10	3.13	1.86
20	2.97	2.95	2.98	3.68	3.67	3.71	2.35
100	4.27	4.27	4.32	4.99	4.98	5.06	3.47
1000	6.15	6.15	6.25	6.86	6.86	6.99	5.09
100000	9.90	9.90	10.10	10.62	10.62	10.84	8.31
1000000	11.78	11.78	12.03	12.50	12.50	12.76	9.92

Table 31: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.7$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.81623$, $\lambda_2^{-1} = 0.18377$, $C_{1,2} = 1.2906$ and $C_{2,2} = -0.29057$.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.30	2.28	2.29	2.93	2.92	2.97	1.69
20	2.79	2.78	2.83	3.42	3.42	3.51	2.10
100	3.95	3.95	4.08	4.58	4.58	4.76	3.07
1000	5.61	5.61	5.86	6.25	6.25	6.54	4.45
100000	8.94	8.94	9.43	9.58	9.58	10.11	7.21
1000000	10.61	10.61	11.21	11.25	11.25	11.89	8.60

Table 32: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.6$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.7236$, $\lambda_2^{-1} = 0.2764$, $C_{1,2} = 1.6180$ and $C_{2,2} = -0.6180$.

	$q=0.50$			$q=0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.25	2.24	2.24	2.82	2.82	2.89	1.60
20	2.70	2.70	2.75	3.27	3.27	3.40	1.98
100	3.75	3.75	3.95	4.33	4.33	4.60	2.86
1000	5.27	5.27	5.65	5.85	5.85	6.31	4.13
100000	8.30	8.30	9.07	8.88	8.88	9.72	6.66
1000000	9.82	9.82	10.78	10.39	10.39	11.43	7.93

Table 33: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.55$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.65811$, $\lambda_2^{-1} = 0.34189$, $C_{1,2} = 2.0811$ and $C_{2,2} = -1.0811$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.20	2.32	2.19	2.73	2.82	2.82	1.52
20	2.62	2.72	2.69	3.15	3.22	3.31	1.87
100	3.58	3.63	3.84	4.09	4.14	4.46	2.69
1000	4.92	4.95	5.48	5.43	5.45	6.11	3.87
100000	7.57	7.58	8.77	8.07	8.08	9.40	6.21
1000000	8.89	8.89	10.41	9.39	9.39	11.04	7.39

Table 34: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.51$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.5707$, $\lambda_2^{-1} = 0.4293$, $C_{1,2} = 4.0355$ and $C_{2,2} = -3.0355$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.1949	2.6781	2.1814	2.7135	3.1375	2.8039	1.4999
20	2.6045	3.0402	2.6720	3.1165	3.4996	3.2945	1.8471
100	3.5361	3.8809	3.8112	4.0347	4.3403	4.4337	2.6535
1000	4.8303	5.0837	5.4410	5.3161	5.5431	6.0635	3.8071
100000	7.3424	7.4893	8.7006	7.8151	7.9486	9.3231	6.1142
1000000	8.5779	8.6920	10.3304	9.0473	9.1514	10.9529	7.2678

Table 35: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.501$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.52236$, $\lambda_2^{-1} = 0.47764$, $C_{1,2} = 11.680$ and $C_{2,2} = -10.680$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.1940	3.1685	2.1803	2.7114	3.6144	2.8022	1.4981
20	2.6027	3.5200	2.6705	3.1134	3.9659	3.2924	1.8447
100	3.5315	4.3361	3.8087	4.0286	4.7820	4.4306	2.6496
1000	4.8205	5.5036	5.4370	5.3040	5.9495	6.0589	3.8011
100000	7.3172	7.8388	8.6937	7.7866	8.2847	9.3156	6.1042
1000000	8.5424	9.0064	10.3220	9.0076	9.4523	10.9439	7.2557

Table 36: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of two exponential distributions, having overall mean 1 and $SCV = 0.5001$ for six values of n. The distribution parameters are $\lambda_1^{-1} = 0.50707$, $\lambda_2^{-1} = 0.49293$, $C_{1,2} = 35.855$ and $C_{2,2} = -34.855$.

3.2 Convolutions of Four Exponentials: $0.25 < SCV < 1.0$

To obtain values of c^2 less than 0.5, we need to consider convolutions of more exponential distributions (sums of more independent exponential random variables). To illustrate, below we give results for the sum of 4 independent exponential random variables with means 0.4, 0.3, 0.2 and 0.1. Here the SCV is $c^2 = 0.3$ and the mean is again 1. (The mean is the sum of the means, while the variance is the sum of the variances, where the variance of an exponential is the square of its mean.)

As before, the asymptotic extreme-value results are excellent, although there is about 5% error for small n. The simple rough approximation is good for smaller n, but begins to deviate for larger n. There is a 13% error for $n = 1000$. Overall, the simple rough approximation in (1.9) is reasonable.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.91	2.01	1.91	2.29	2.37	2.40	1.05
20	2.21	2.29	2.29	2.58	2.64	2.78	1.26
100	2.89	2.94	3.18	3.25	3.29	3.66	1.74
1000	3.84	3.86	4.44	4.19	4.21	4.92	2.43
100000	5.69	5.70	6.96	6.05	6.05	7.44	3.82
1000000	6.62	6.62	8.22	6.97	6.97	8.70	4.51

Table 37: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.4, 0.3, 0.2 and 0.1, having overall mean 1 and $SCV = 0.30$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 10.667$.

We now consider other distributions that are convolutions of four exponential distributions, again all with different means. The remaining tables are constructed from iid random variables, each distributed as the convolution of four exponential distributions with individual means $0.25 + 2w$, $0.25 + w$, $0.25 - w$ and $0.25 - 2w$, for a specified w, such that $2w < 0.25$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	2.07	2.17	2.06	2.53	2.61	2.62	1.30
20	2.43	2.51	2.50	2.89	2.95	3.06	1.58
100	3.27	3.32	3.52	3.72	3.75	4.08	2.23
1000	4.44	4.46	4.99	4.88	4.90	5.55	3.16
100000	6.75	6.76	7.91	7.19	7.19	8.47	5.01
1000000	7.90	7.90	9.38	8.34	8.34	9.93	5.94

Table 38: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.4980, 0.3740, 0.1260 and 0.0020, (from $w = 0.1240$) having overall mean 1.00 and $SCV = 0.4038$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 5.3981e + 00$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.99	2.11	1.99	2.41	2.51	2.51	1.17
20	2.32	2.43	2.40	2.74	2.82	2.92	1.42
100	3.09	3.15	3.35	3.50	3.55	3.87	1.98
1000	4.15	4.19	4.71	4.56	4.58	5.23	2.79
100000	6.25	6.26	7.44	6.65	6.65	7.96	4.40
1000000	7.29	7.29	8.80	7.69	7.69	9.32	5.20

Table 39: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.4500, 0.3500, 0.1500 and 0.0500, (from $w = 0.1000$) having overall mean 1.00 and $SCV = 0.3500$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 7.5938e + 00$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.93	2.09	1.94	2.33	2.45	2.43	1.09
20	2.24	2.37	2.32	2.63	2.73	2.82	1.30
100	2.95	3.03	3.23	3.33	3.39	3.72	1.81
1000	3.93	3.97	4.52	4.30	4.33	5.01	2.53
100000	5.85	5.86	7.10	6.21	6.22	7.59	3.98
1000000	6.80	6.81	8.39	7.16	7.17	8.88	4.70

Table 40: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.4100, 0.3300, 0.1700 and 0.0900, (from $w = 0.0800$) having overall mean 1.00 and $SCV = 0.3140$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 1.1218e + 01$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.89	2.09	1.89	2.25	2.41	2.36	1.02
20	2.18	2.34	2.26	2.53	2.67	2.73	1.21
100	2.83	2.94	3.12	3.17	3.26	3.59	1.68
1000	3.72	3.79	4.36	4.06	4.12	4.83	2.33
100000	5.47	5.50	6.82	5.80	5.82	7.29	3.65
1000000	6.33	6.35	8.05	6.66	6.67	8.52	4.31

Table 41: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.3700, 0.3100, 0.1900 and 0.1300, (from $w = 0.0600$) having overall mean 1.00 and $SCV = 0.2860$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 1.9542e + 01$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.85	2.15	1.86	2.19	2.44	2.31	0.96
20	2.12	2.38	2.22	2.46	2.67	2.67	1.15
100	2.73	2.91	3.05	3.04	3.20	3.50	1.58
1000	3.55	3.67	4.24	3.86	3.96	4.69	2.19
100000	5.13	5.19	6.61	5.43	5.48	7.06	3.41
1000000	5.91	5.95	7.80	6.20	6.24	8.25	4.03

Table 42: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.3300, 0.2900, 0.2100 and 0.1700, (from $w = 0.0400$) having overall mean 1.00 and $SCV = 0.2660$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 4.6793e + 01$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.83	2.38	1.84	2.16	2.63	2.28	0.93
20	2.09	2.58	2.19	2.41	2.84	2.63	1.11
100	2.66	3.05	3.00	2.96	3.30	3.44	1.52
1000	3.43	3.72	4.16	3.71	3.97	4.61	2.10
100000	4.88	5.05	6.48	5.15	5.31	6.93	3.27
1000000	5.58	5.72	7.64	5.85	5.97	8.09	3.86

Table 43: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.2900, 0.2700, 0.2300 and 0.2100, (from $w = 0.0200$) having overall mean 1.00 and $SCV = 0.2540$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 2.5405e + 02$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.83	2.72	1.84	2.15	2.96	2.28	0.92
20	2.08	2.91	2.18	2.39	3.14	2.62	1.10
100	2.64	3.34	2.99	2.93	3.58	3.43	1.50
1000	3.39	3.96	4.14	3.67	4.20	4.58	2.08
100000	4.80	5.21	6.45	5.06	5.44	6.89	3.24
1000000	5.48	5.83	7.60	5.74	6.07	8.04	3.81

Table 44: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.2700, 0.2600, 0.2400 and 0.2300, (from $w = 0.0100$) having overall mean 1.00 and $SCV = 0.2510$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 1.6402e + 03$.

	$q = 0.50$			$q = 0.75$			
n	exact	asymp. (3.6)	(1.9)	exact	asymp. (3.6)	(1.9)	(1.8)
10	1.83	4.23	1.83	2.15	4.45	2.27	0.92
20	2.08	4.40	2.18	2.39	4.62	2.62	1.10
100	2.64	4.81	2.99	2.93	5.03	3.43	1.50
1000	3.38	5.39	4.14	3.66	5.61	4.58	2.07
100000	4.77	6.55	6.44	5.03	6.77	6.88	3.22
1000000	5.44	7.13	7.59	5.70	7.35	8.03	3.80

Table 45: A comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid random variables, each distributed as the convolution of four exponential distributions with individual means 0.2520, 0.2510, 0.2490 and 0.2480, (from $w = 0.0010$) having overall mean 1.00 and $SCV = 0.2500$, for six values of n. The remaining asymptotic parameter is $p = C_{1,4} = 1.3336e + 06$.

4 The Gamma Distribution

In this final section, we consider the gamma distribution, which covers the full range of possible SCV values: $0 < c^2 < \infty$. We consider the cases $c^2 > 1$ and $c^2 < 1$ in turn in the following subsections.

4.1 $SCV > 1$

The tables here provide a comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid gamma random variables with mean 1 and $SCV = \{2, 4, 6, 8, 16, 32, 64\}$ for six values of n. The approximations are the asymptotic approximation in (7.5), the associated simple rough approximation in (1.9), the exact values for H_2 with $r = 0.5$, and the crude approximation in (1.8).

¿From the results, we see that the various approximations all perform quite well for smaller SCV values such as $c^2 = 2.0$, but problems arise as c^2 increases. For large c^2, all the approximations based on asymptotics produce negative values for the smaller values of n. When is larghe enough, the asymptotic approximation becomes reasonable, but it is never spectacular, presumably because the gamma distribution fails to have a pure-exponential tail. Fortunately, the H_2-exact approximation is roughly reasonable for all values of n.

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	3.4	3.4	3.4	3.0	4.8	5.1	5.1	4.8	3.2
20	4.5	4.5	4.8	4.4	6.0	6.2	6.5	6.4	4.6
100	7.3	7.3	8.0	8.1	8.9	9.0	9.7	10.2	7.8
1000	11.5	11.5	12.6	13.5	13.1	13.2	14.3	15.6	12.4
100000	20.2	20.2	21.8	24.4	21.9	21.9	23.6	26.5	21.6
100000000	33.6	33.5	35.6	40.8	35.3	35.3	37.4	42.9	35.5

Table 46: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 2$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	4.0	3.0	3.3	2.8	6.5	6.5	6.8	6.1	3.7
20	5.9	5.0	6.0	5.3	8.6	8.5	9.5	9.2	6.4
100	10.9	10.2	12.5	12.4	13.9	13.7	16.0	16.3	12.9
1000	18.8	18.1	21.7	22.6	21.9	21.7	25.2	26.5	22.1
100000	35.5	35.0	40.1	43.0	38.8	38.6	43.6	46.9	40.5
100000000	61.6	61.3	67.7	73.7	65.0	64.8	71.2	77.6	68.1

Table 47: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 4$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	4.4	-0.6	0.1	2.0	8.4	6.4	7.1	6.2	1.8
20	7.5	3.1	5.7	4.7	12.1	10.1	12.7	12.0	7.3
100	16.4	12.9	18.5	18.2	21.8	20.0	25.6	25.6	20.2
1000	31.1	28.5	37.0	37.6	37.1	35.5	44.0	45.1	38.6
100000	63.6	61.8	73.8	76.6	70.0	68.8	80.8	84.1	75.5
100000000	115.0	113.7	129.1	135.1	121.7	120.8	136.1	142.6	130.7

Table 48: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 8$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	3.9	-13.6	-11.8	1.7	9.9	0.4	2.3	3.0	-7.5
20	8.4	-6.5	-0.7	2.5	16.2	7.6	13.4	12.4	3.6
100	23.8	12.8	25.1	24.4	33.9	26.9	39.1	38.9	29.3
1000	51.6	43.6	61.9	62.3	63.1	57.6	76.0	76.8	66.2
100000	114.9	109.6	135.6	138.2	127.5	123.7	149.7	152.7	139.8
100000000	216.7	213.1	246.1	252.1	229.9	227.1	260.2	266.6	250.4

Table 49: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 16$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	2.1	-50.8	-46.7	1.5	9.6	-22.7	-18.5	2.2	-37.2
20	7.5	-36.8	-24.5	2.0	19.4	-8.6	3.6	3.8	-15.0
100	32.5	1.4	27.0	26.0	50.7	29.5	55.1	54.5	36.5
1000	84.2	62.5	100.7	100.7	106.4	90.7	128.8	129.3	110.1
100000	207.8	194.0	248.0	250.4	232.8	222.2	276.2	278.9	257.5
100000000	409.6	400.5	469.1	474.8	435.9	428.7	497.2	503.4	478.6

Table 50: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 32$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(1.9)	H_2 ex.	exact	asymp.	(1.9)	H_2 ex.	(1.8)
10	0.4	-147.3	-138.7	1.4	6.4	-91.0	-82.4	1.9	-118.8
20	4.2	-119.5	-94.4	1.8	19.0	-63.3	-38.1	2.5	-74.4
100	39.6	-43.6	8.7	7.4	71.5	12.7	64.9	64.0	28.6
1000	133.7	78.2	156.0	155.7	176.2	134.5	212.3	212.5	175.9
100000	374.5	340.7	450.7	452.7	423.9	397.0	507.0	509.5	470.7
100000000	775.0	753.2	892.8	898.3	827.4	809.5	949.1	955.0	912.8

Table 51: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 64$

4.2 $SCV < 1$

We now turn to the less variable gamma distributions, having $0 < c^2 < 1$. The tables here provide a comparison of exact values with approximations for two quantiles ($q = 0.50$ and 0.75) of the cdf of the maximum of n iid gamma random variables with mean 1 and $SCV = \{1/2, 1/4, 1/8, 1/16, 1/32, 1/64\}$ for five values of n. The approximations are the asymptotic approximation in (7.5), the associated simple approximation (7.10), the simple rough approximation based on the shifted-exponential in (1.9) and the crude approximation in (1.8).

Except for the first case $c^2 = 0.5$, the asymptotic approximation does not do well for the smaller values of n. Just as for $c^2 > 1$, when c^2 gets very small, the asymptotic approximation produces negative estimates for smaller values of n. Again, the asymptotic approximation is not even spectacularly accurate for extremely large n, presumably because the gamma distribution fails to have a pure-exponential tail. On the other hand, the simple rough approximation (1.9) is consistently good for smaller values of n, e.g., $10 \leq n \leq 1000$.

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	2.19	1.75	1.48	2.18	2.71	2.19	1.92	2.80	1.50
20	2.60	2.23	2.09	2.67	3.11	2.67	2.52	3.29	1.84
100	3.53	3.25	3.32	3.81	4.03	3.69	3.76	4.43	2.65
1000	4.82	4.60	4.88	5.44	5.30	5.04	5.32	6.06	3.80
100000	7.31	7.16	7.69	8.69	7.78	7.60	8.13	9.31	6.10
100000000	10.96	10.85	11.61	13.58	11.42	11.29	12.05	14.20	9.56
1.000000e+10	13.36	13.26	14.14	16.83	13.81	13.70	14.58	17.46	11.86
1.000000e+12	15.74	15.66	16.62	20.09	16.19	16.10	17.06	20.71	14.16

Table 52: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/2$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	1.83	0.84	0.12	1.83	2.15	1.06	0.33	2.27	0.92
20	2.08	1.22	0.55	2.18	2.39	1.44	0.77	2.62	1.10
100	2.64	1.94	1.38	2.99	2.93	2.16	1.60	3.43	1.50
1000	3.38	2.82	2.36	4.14	3.66	3.04	2.58	4.58	2.07
100000	4.77	4.35	4.03	6.44	5.03	4.57	4.25	6.88	3.22
100000000	6.75	6.43	6.22	9.89	6.99	6.65	6.44	10.33	4.95
1.000000e+10	8.03	7.75	7.60	12.20	8.27	7.97	7.82	12.64	6.10
1.000000e+12	9.28	9.04	8.93	14.50	9.52	9.26	9.15	14.94	7.25

Table 53: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/4$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	1.57	-0.00	-0.91	1.59	1.77	0.11	-0.80	1.90	0.55
20	1.73	0.31	-0.56	1.84	1.92	0.42	-0.45	2.15	0.63
100	2.08	0.89	0.07	2.40	2.25	1.00	0.18	2.72	0.84
1000	2.52	1.53	0.76	3.22	2.68	1.64	0.87	3.53	1.12
100000	3.33	2.56	1.85	4.85	3.47	2.67	1.96	5.16	1.70
100000000	4.43	3.83	3.18	7.29	4.57	3.94	3.29	7.60	2.56
1.000000e+10	5.13	4.60	3.98	8.92	5.26	4.71	4.09	9.23	3.14
1.000000e+12	5.81	5.34	4.74	10.55	5.94	5.45	4.85	10.86	3.71

Table 54: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/8$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	1.40	-0.79	-1.77	1.42	1.53	-0.74	-1.72	1.64	0.32
20	1.50	-0.50	-1.47	1.59	1.62	-0.45	-1.41	1.81	0.36
100	1.72	-0.00	-0.93	1.99	1.83	0.05	-0.88	2.21	0.46
1000	1.99	0.52	-0.39	2.57	2.09	0.58	-0.33	2.79	0.61
100000	2.48	1.29	0.41	3.72	2.56	1.34	0.47	3.94	0.89
100000000	3.12	2.16	1.32	5.45	3.19	2.22	1.37	5.67	1.32
1.000000e+10	3.51	2.66	1.83	6.60	3.59	2.71	1.88	6.82	1.61
1.000000e+12	3.90	3.12	2.30	7.75	3.97	3.17	2.35	7.97	1.90

Table 55: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/16 = 0.0625$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	1.28	-1.55	-2.55	1.30	1.36	-1.52	-2.52	1.45	0.18
20	1.35	-1.27	-2.26	1.42	1.43	-1.24	-2.24	1.57	0.20
100	1.49	-0.81	-1.78	1.70	1.56	-0.78	-1.76	1.86	0.25
1000	1.66	-0.34	-1.31	2.11	1.72	-0.31	-1.28	2.26	0.32
100000	1.96	0.30	-0.65	2.92	2.01	0.33	-0.62	3.08	0.47
100000000	2.35	0.97	0.03	4.14	2.39	1.00	0.06	4.30	0.68
1.000000e+10	2.58	1.33	0.40	4.96	2.62	1.36	0.43	5.11	0.83
1.000000e+12	2.80	1.65	0.73	5.77	2.85	1.68	0.76	5.93	0.97

Table 56: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/32 = 0.03125$

	$q = 0.50$				$q = 0.75$				
n	exact	asymp.	(7.10)	(1.9)	exact	asymp.	(7.10)	(1.9)	(1.8)
10	1.19	-2.28	-3.28	1.21	1.25	-2.26	-3.27	1.32	0.10
20	1.24	-2.01	-3.01	1.30	1.29	-1.99	-3.00	1.41	0.11
100	1.33	-1.56	-2.55	1.50	1.38	-1.55	-2.54	1.61	0.14
1000	1.45	-1.12	-2.11	1.78	1.49	-1.11	-2.10	1.89	0.17
100000	1.64	-0.55	-1.53	2.36	1.67	-0.54	-1.52	2.47	0.24
100000000	1.88	0.02	-0.95	3.22	1.91	0.03	-0.94	3.33	0.35
1.000000e+10	2.02	0.31	-0.66	3.80	2.05	0.33	-0.64	3.91	0.42
1.000000e+12	2.16	0.56	-0.40	4.37	2.18	0.58	-0.39	4.48	0.50

Table 57: Approximations for the $q = 0.50$ and $q = 0.75$ quantile for Gamma with $SCV = 1/64 = 0.015625$

References

[1] Crow, C. S., IV, D. Goldberg, W. Whitt. 2005. Two-moment approximations for maxima. Available at http://www.columbia.edu/∼ww2040/recent.html