

Information design possibility set

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Abstract

Let \mathcal{V} be the set of all combinations of expected value of finite objective functions from designing information. I showed that \mathcal{V} is a compact and convex set implementable by signal structures with finite support when unknown states are finite. Moreover, $\mathcal{V}(\mu)$ as a correspondence of prior is continuous. This result can be applied to develop a concavification method of Lagrange multipliers that works with general constrained optimization. It also provides tractability to a wide range of information design problems.

Keywords: concavification, method of Lagrange multipliers, Bayes persuasion, information design

1. Introduction

Let X be a non-empty finite set (state space). $\Delta(X) \in \mathbb{R}^{|X|}$ is the set of all probability measure on X . Let μ denote elements in $\Delta(X)$. $\Delta^2(X)$ is the set of all probability measures (standard Borel measurability) on $\Delta(X)$. Let P denote elements in $\Delta^2(X)$. Let $\{V^i\}_{i=1}^n$ be a finite set of continuous function on $\Delta(X)$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. Let $D(\mu) : \Delta(X) \rightarrow \mathbb{R}^n$ be closed valued.

My objective is to solve the following constrained maximization problem:

$$\begin{aligned} \sup_{P \in \Delta^2(X)} f \left(E_P[V^1], \dots, E_P[V^n] \right) & \quad (1) \\ \text{s.t. } \begin{cases} \left(E_P[V^1], \dots, E_P[V^n] \right) \in D(\mu) \\ E_P[v] = \mu \end{cases} \end{aligned}$$

Suppose $n = 1$ and $D \equiv \mathbb{R}$, then [Equation \(1\)](#) can be solved by concavifying $V^1(\mu)$ ([Kamenica and Gentzkow \(2011\)](#), [Aumann et al. \(1995\)](#)). And [Theorem 8](#) implies that it is without loss to consider optimal information structure involving signal number no more than $|X|$. This gives tractability both analytically and computationally. However, even when $n = 2$, with a general f or a nontrivial constraint D , concavification no longer works and we might need to search over an infinite dimensional space to solve [Equation \(1\)](#).

To solve [Equation \(1\)](#), I studied the set of all possible combinations of expected valuation that can be implemented by designing information P . In [Section 3](#), I proved a

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two-step concavification method: First, the information design possibility set itself can be implemented by combining finite number of information structures that implement its extreme points. Second, each extreme point can be implemented by concavifying a linear combination of V^i 's, thus involves only finite number of signals.

The general concavification method developed in this paper can be applied to a wide range of information design problems. In [Section 4](#), I first provide two applications in static information acquisition and dynamic information acquisition to show that the optimal solutions have a nice Lagrange multiplier characterization. Then I provide an application of persuading receivers with outside options to illustrate how the Lagrange characterization can simplify the optimization problem. Finally I provide an application of [Lemma 1](#) in a setup of screening using information structures, to illustrate how the dimensionality of the problem can be reduced to make the problem tractable.

2. Information possibility set

Notations used: given a convex set C , $\text{ext}(C)$ is set of extreme points of C , $\text{ext}_k(C)$ is set of k -extreme points of C ¹. $\text{exp}(C)$ is set of exposed points of C . $F(C)$ is set of faces of C .

Definition 1. *Information possibility set $\mathcal{V}(\mu) \in \mathbb{R}^n$ is defined as:*

$$\mathcal{V}(\mu) = \left\{ \left(E_P[V^1], \dots, E_P[V^n] \right) \mid P \in \Delta^2(X), E_P[v] = \mu \right\}$$

Lemma 1. $\forall \mu$, $\mathcal{V}(\mu)$ is a compact and convex set. $\forall v \in \text{ext}_k(\mathcal{V}(\mu))$, there exists $P \in \Delta^2(X)$ such that:

$$\begin{cases} v = (E_P[V^1], \dots, E_P[V^n]) \\ |\text{supp}(P)| \leq (k+1)|X| \end{cases}$$

Proof. First of all, we prove that $\mathcal{V}(\mu)$ is compact and convex.

- **Boundedness:** $\forall P \in \Delta^2(X)$, $\min_{\mu \in \Delta(X)} V^i(\mu) \leq E_P[V^i] \leq \max_{\mu \in \Delta(X)} V^i(\mu)$. Therefore, $\forall v \in \mathcal{V}(\mu)$, it is bounded from 0 by $\max_{\mu \in \Delta(X), i} |V^i(\mu)|$ by sup norm. So $\mathcal{V}(\mu)$ is a bounded set.
- **Convexity:** $\forall v_1, v_2 \in \mathcal{V}(\mu)$, there exists $P_1, P_2 \in \Delta^2(X)$ s.t. $v_i = (E_{P_i}[V^1], \dots, E_{P_i}[V^n])$. Since $\Delta^2(X)$ is a linear space and expectation operator is linear functional, $\forall \beta \in [0, 1]$, $P_\beta = \beta P_1 + (1 - \beta)P_2 \in \Delta^2(X)$ and:

$$\begin{aligned} v_\beta &= \left(E_{P_\beta}[V^1], \dots, E_{P_\beta}[V^n] \right) \\ &= \beta \left(E_{P_1}[V^1], \dots, E_{P_1}[V^n] \right) + (1 - \beta) \left(E_{P_2}[V^1], \dots, E_{P_2}[V^n] \right) \\ &= \beta v_1 + (1 - \beta)v_2 \end{aligned}$$

Therefore, $\beta v_1 + (1 - \beta)v_2 \in \mathcal{V}(\mu)$ so $\mathcal{V}(\mu)$ is a convex set.

¹ $\text{ext}(C) = \bigcup_{k < n} \text{ext}_k(C)$ and $C = \bigcup_{k \leq n} \text{ext}_k(C)$.

- **Closeness:** $\Delta(X)$ is a finite dimensional simplex. If we consider Prokhorov metric on $\Delta^2(X)$, then $\Delta^2(X)$ is a complete and separable space (Theorem 6.8 of Billingsley (2013)). Now since $\Delta(X)$ is compact, by Theorem 9, $\Delta^2(X)$ is a compact, complete and separable space with Prokhorov metric. Prokhorov metric induces a topology equivalent to weak* topology (by Theorem 6.8 of Billingsley (2013)). So $\forall v_k \in \mathcal{V}(\mu)$, if $v_k \rightarrow v$, then consider the sequence P_k such that $v_k = E_{P_k}[(V^i)]$. By compactness of $\Delta^2(X)$, pick a subsequence $P_k \xrightarrow{w^*} P$. Then $\forall V^i$, since V^i is continuous, $E_{P_k}[V^i] \rightarrow E_P[V^i]$. So $v \in \mathcal{V}(\mu)$ and $\mathcal{V}(\mu)$ is a closed set.
- **Compactness:** $\mathcal{V}(\mu)$ is a finite dimensional bounded and closed set, so it is compact.

$\forall v \in \text{ext}_k(\mathcal{V}(\mu))$, v is an interior point of a k -dimensional face F of $\mathcal{V}(\mu)$. Then by Theorem 7, $v \in \text{conv}(\text{ext}(F))$. By Theorem 8, there exists $\{v_j\}_{j=1}^{k+1} \subset \text{ext}(F)$ and $\sum \pi_j = 1$ s.t. $\sum \pi_j v_j = v$. By Lemma 4, $\{v_j\} \subset \text{ext}(\mathcal{V}(\mu))$. The next step is to prove that $\forall j$, there exists $P_j \in \Delta^2(X)$ s.t. $v_j = (E_{P_j}[V^1], \dots, E_{P_j}[V^n])$ and $|\text{supp}(P_j)| \leq |X|$.

Lemma 2. $\forall \mu, \forall v \in \text{exp}(\mathcal{V}(\mu)), \exists P \in \Delta^2(X)$ and $|\text{supp}(P)| \leq |X|$ s.t. $v = E_P[(V^i)]$.

Proof. By definition of exposed points, there exists a linear function $l \in L(\mathbb{R}^n)$ s.t.

$$l(v) > l(v') \quad \forall v' \in \mathcal{V}(x), v' \neq v$$

In finite dimensional space, a linear function $l(v)$ can be equivalently written as $\sum \lambda_i v_i + c$. Consider the following maximization problem:

$$\begin{aligned} \sup_{P \in \Delta^2(X)} E_P \left[\sum \lambda_i V^i + c \right] \\ \text{s.t. } E_P[v] = \mu \end{aligned} \quad (2)$$

By Theorem 8, Equation (2) can be solved by convexifying the graph of $\sum \lambda_i V^i(\mu) + c$. The maximum is achieved by a P s.t. $|\text{supp}(P)| \leq |X|$. Of course $E_P[(V^i)] \in \mathcal{V}(\mu)$. Then by definition of l , $l(v) \geq E_P[\sum \lambda_i V^i + c]$. On the other hand, there exists $P' \in \Delta^2(X)$ s.t. $v = E_{P'}[(V^i)]$, by optimality of P , $l(v) \leq E_P[\sum \lambda_i V^i + c]$. Therefore, since v is the unique element in $\mathcal{V}(\mu)$ achieving $l(v)$, we have $E_P[(V^i)] = v$ and $|\text{supp}(P)| \leq |X|$. Q.E.D.

$\forall v^j \in \text{ext}(\mathcal{V}(\mu))$, by Theorem 6, there exists $\{v^{jl}\}_{l=1}^{\infty} \subset \text{exp}(\mathcal{V}(\mu))$ and $\lim_{l \rightarrow \infty} v^{jl} = v^j$. By Lemma 2, there exists $P^{jl} \in \Delta^2(X)$ s.t. $|\text{supp}(P^{jl})| \leq |X|$ and $v^{jl} = E_{P^{jl}}[(V^i)]$. Now each P^{jl} can be represented as $(p_t^{jl}, \mu_t^{jl})_{t=1}^{|X|} \in \mathbb{R}^{2|X|}$, where:

$$\begin{cases} \sum_t p_t^{jl} = 1 \\ \sum_t p_t^{jl} \mu_t^{jl} = \mu \\ \sum_t p_t^{jl} V^i(\mu_t^{jl}) = v_i^{jl} \quad \forall i \end{cases}$$

Since (p_t^l, μ_t^l) is in finite dimensional vector space, there exists a subsequence converging to (p_t^j, μ_t^j) when $l \rightarrow \infty$. Therefore, since V^i is each continuous, it is easy to verify that:

$$\begin{cases} \sum_t p_t^j = 1 \\ \sum_t p_t^j \mu_t^j = \mu \\ \sum_t p_t^j V^i(\mu_t^j) = v_i^j \forall i \end{cases}$$

Therefore, v^j is implemented by $P^j \in \Delta^2(X)$ and $|\text{supp}(P^j)| \leq |X|$. So $P = \sum \pi_j P^j \in \Delta^2(X)$ and $|\text{supp}(P)| \leq (k+1) \cdot |X|$. By linearity of expectation operator, $E_P[(V^i)] = \sum \pi_j E_{P^j}[(V^i)] = \sum \pi_j v_i^j = v$. Q.E.D.

Lemma 3. Correspondence $\mathcal{V} : \Delta(X) \rightarrow \mathbb{R}^n$ is continuous. $\text{Gr}(\mathcal{V})$ is convex and compact.

Proof.

- Boundedness: $\Delta(X)$ is a bounded set. $\forall \mu \in \Delta(X)$, \mathcal{V} is uniformly bounded by radius $\max_{\mu \in \Delta(X), i} |V^i(\mu)|$ by sup norm. So $\text{Gr}(\mathcal{V})$ is bounded.
- Convexity: $\forall (\mu_1, v_1), (\mu_2, v_2) \in \text{Gr}(\mathcal{V})$. $\forall \alpha \in [0, 1]$. Since $\Delta(X)$ is convex, $\mu_\alpha = \alpha \mu_1 + (1 - \alpha) \mu_2 \in \Delta(X)$. Now we prove that $v_\alpha = \alpha v_1 + (1 - \alpha) v_2 \in \mathcal{V}(\mu_\alpha)$. By definition, there exists $P_1, P_2 \in \Delta^2(X)$ s.t. $E_{P_1}[(V^i)] = v_1$, $E_{P_1}[v] = \mu_1$ and $E_{P_2}[(V^i)] = v_2$, $E_{P_2}[v] = \mu_2$. Define $P_\alpha = \alpha P_1 + (1 - \alpha) P_2$, then by linearity of expectation operator, $E_{P_\alpha}[v] = \alpha E_{P_1}[v] + (1 - \alpha) E_{P_2}[v] = \mu_\alpha$. $E_{P_\alpha}[(V^i)] = \alpha E_{P_1}[(V^i)] + (1 - \alpha) E_{P_2}[(V^i)] = v_\alpha$. Therefore, $(\mu_\alpha, v_\alpha) \in \text{Gr}(\mathcal{V})$.
- Closedness: $\forall \{(\mu_j, v_j)\} \subset \text{Gr}(\mathcal{V})$, suppose $\mu_j \rightarrow \mu$, $v_j \rightarrow v$. Want to show that $\mu \in \Delta(X)$ and $v \in \mathcal{V}(\mu)$. First of all, since $\Delta(X)$ is complete, $\mu \in \Delta(X)$. Now by **Lemma 1**, there exists (p_j, v_j) such that:

$$\begin{cases} \sum_{k=1}^{(n+1)|X|} p_j^k = 1 \\ \sum_{k=1}^{(n+1)|X|} p_j^k v_j^k = \mu_j \\ \sum_{k=1}^{(n+1)|X|} p_j^k V^i(v_j^k) = v_j^i \end{cases}$$

Now since $p_j \in \Delta((n+1)|X|)$ and $v_j \in \Delta(X)$ are both compact spaces. Consider standard Euclidean metric on product space $\Delta((n+1)|X|) \times \Delta(X)^{(n+1)|X|}$, it is also compact. Therefore there exists converging subsequence $p_j \rightarrow p$ and $v_j^k \rightarrow v^k$. Then

$$\begin{cases} \sum_{k=1}^{(n+1)|X|} p^k = \lim_{j \rightarrow \infty} \sum_{k=1}^{(n+1)|X|} p_j^k = 1 \\ \sum_{k=1}^{(n+1)|X|} p^k v^k = \lim_{j \rightarrow \infty} \sum_{k=1}^{(n+1)|X|} p_j^k v_j^k = \lim_{j \rightarrow \infty} \mu_j = \mu \\ \sum_{k=1}^{(n+1)|X|} p^k V^i(v^k) = \lim_{j \rightarrow \infty} \sum_{k=1}^{(n+1)|X|} p_j^k V^i(v_j^k) = \lim_{j \rightarrow \infty} v_j^i = v^i \end{cases}$$

Therefore, (p, v) implements v at μ . So $v \in \mathcal{V}(\mu)$.

- Compactness: Since $\text{Gr}(\mathcal{V})$ is closed and bounded, it is compact.

- **Continuity:** Since $\text{Gr}(\mathcal{V})$ is compact, $\mathcal{V}(\mu)$ is upper hemicontinuous. Now we only need to show lower hemicontinuity. $\forall (\mu_m) \subset \Delta(X), \mu_m \rightarrow \mu \in \Delta(X), \forall v \in \mathcal{V}(\mu)$. By **Lemma 1**, v is implemmented by (p, ν) with support size $(n+1)|X|$. There exists a stochastic matrix q_{jk} such that:

$$\Rightarrow \begin{cases} v_j = \frac{1}{\sum_k \mu_k q_{jk}} (\mu_1 q_{j1}, \dots, \mu_{-1} q_{j,-1}) \\ p_j = \sum_k \mu_k q_{jk} \\ \frac{\partial p_j}{\partial \mu_k} = q_{jk} \\ \frac{\partial v_{jl}}{\partial \mu_k} = \begin{cases} \frac{p_j q_{jl} - \mu_l q_{jl}^2}{p_j^2} & \text{when } k = l \\ \frac{-\mu_l q_{jl} q_{jk}}{p_j^2} & \text{when } k \neq l \end{cases} \end{cases}$$

Therefore, since each $p_j > 0$, when μ_m is sufficiently close to μ , corresponding (p_m, ν_m) will be bounded from (p, ν) by $|\mu - \mu_m|$. By continuity of $V^i, v_m = (\sum p_m V^i(\nu_m)) \rightarrow (\sum p V^i(\nu)) = v$. Therefore, $\mathcal{V}(\mu)$ is both upper hemicontinuous and lower hemicontinuous.

Q.E.D.

3. Main theorem

3.1. Existence and finite support

Theorem 1. Let X be a non-empty state space, $\{V^i\}_{i=1}^n \subset C\Delta(X), f \in \mathbb{C}\mathbb{R}^n, \forall \mu \in \Delta(X)$, suppose $\mathcal{V}(\mu) \cap D(\mu) \neq \emptyset$, then there exists $P^* \in \Delta^2(X)$ solving **Equation (1)** and $|\text{supp}(P^*)| \leq (n+1) \cdot |X|$.

Proof. By definition of $\mathcal{V}(\mu)$, **Equation (1)** is equivalent to the following problem:

$$\sup_{v \in D \cap \mathcal{V}(\mu)} f(v) \quad (3)$$

By **Lemma 1**, $\mathcal{V}(\mu)$ is a compact set. Then $\mathcal{V}(\mu) \cap D(\mu)$ is compact and non-empty. By Wierestrass's theorem, there exists $v^* \in \mathcal{V}(\mu) \cap D(\mu)$ solving **Equation (3)**. Then by **Lemma 1**, there exists $P^* \in \Delta^2(X)$ s.t. $v^* = (E_{P^*}[V^1], \dots, E_{P^*}[V^n])$ and $|\text{supp}(P^*)| \leq (n+1) \cdot |X|$. Therefore, P^* solves **Equation (1)**. Q.E.D.

3.2. Necessary condition of optimizer

Theorem 2. Let X be a non-empty state space, $\{V^i\}_{i=1}^n \subset C\Delta(X), f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable. Let $D \equiv \mathbb{R}^n$. Then a necessary condition for P^* solving **Equation (1)** is:

$$P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} \nabla f(E_{P^*}[V^1], \dots, E_{P^*}[V^n]) \cdot (E_P[V^1], \dots, E_P[V^n]) \quad (4)$$

Proof. Solving Equation (1) is equivalent to solving Equation (3). Suppose by contradiction that Equation (4) is violated at optimal P^* . Then it is equivalently saying that there exists $v \in \mathcal{V}(\mu)$ such that:

$$\nabla f(v^*) \cdot v^* < \nabla f(v^*) \cdot v$$

By Lemma 1, $\mathcal{V}(\mu)$ is a convex set. Therefore $v_\alpha = (1 - \alpha)v^* + \alpha v \in \mathcal{V}(\mu)$. Consider $h(\alpha) = f(v_\alpha)$. Then $h'(0) = \nabla f(v^*) \cdot (v - v^*) > 0$. So there exists $\alpha' > 0$ s.t. $h(\alpha') > h(0)$. Then $f(v^*) < f(v_{\alpha'})$. Contradicting optimality of v^* . Q.E.D.

Theorem 3. Let X be a non-empty state space, $\{V^i\}_{i=1}^{n+m} \subset C\Delta(X)$, $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ is constant in last m arguments. Let $D \equiv \{v | v^i \geq 0 \forall i > n\}$. Then there exists P^* solving Equation (1) and $\lambda \in B_{m+n}$ such that:

$$P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} E_P \left[\sum \lambda^i V^i \right]$$

Proof. $\forall P^*$ solving Equation (1), let v^* be corresponding value. Define:

$$v_\alpha = v^* + \alpha \left(\underbrace{0, \dots, 0}_n, \underbrace{1, \dots, 1}_m \right)$$

Then by definition $f(v_\alpha) = f(v^*)$. $v_0 = v^* \in \mathcal{V}(\mu)$. Since $\mathcal{V}(\mu)$ is bounded, for large enough α , $v_\alpha \notin \mathcal{V}(\mu)$. Then since $\mathcal{V}(\mu)$ is compact, there exists α s.t. $v_\alpha \in \partial\mathcal{V}(\mu)$. Since $\mathcal{V}(\mu)$ is convex, there exists $l \in L(\mathbb{R}^{m+n})$ s.t. $v_\alpha \in \arg \max_{v \in \mathcal{V}(\mu)} l(v)$. Let $l = \sum \lambda^i v^i$, then:

$$v_\alpha \in \arg \max_{v \in \mathcal{V}(\mu)} \sum \lambda^i v^i$$

Let P_α be the corresponding information structure implementing v_α (existence of P_α guaranteed by Lemma 1). Then

$$P_\alpha \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} E_P \left[\sum \lambda^i V^i \right]$$

Since $f(v_\alpha) = f(v^*)$, P_α solves Equation (1) as well. Q.E.D.

3.3. Convex optimization

Theorem 4. Let X be a non-empty state space, $\{V^i\}_{i=1}^n \subset C\Delta(X)$, $D \equiv \{v | g(v) \geq 0\}$. If both f and g are quasi-concave and continuous, then there exists P^* solving Equation (1), $v^* = (E_P[V^i])$ and $\lambda \in B_n$ such that:

$$\begin{cases} P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} E_P \left[\sum \lambda^i V^i \right] \\ v^* \in \arg \min_{f(v) \geq f(v^*), v \in D} \lambda \cdot v \end{cases}$$

Proof. First, by **Theorem 1**, P^* solving **Equation (1)** exists. Then by optimality of P^* :

$$\mathcal{V}(\mu) \cap \{v | v \in D, f(v) > f(v^*)\} = \emptyset$$

Since f and g are quasi-convex, $\{v | v \in D, f(v) > f(v^*)\}$ is a convex set. Then by separating hyperplane theorem, there exists c and λ s.t. $\forall v \in \mathcal{V}(\mu), v' \in D$ and $f(v') > f(v^*)$:

$$\lambda \cdot v \leq c \text{ and } \lambda \cdot v' > c$$

By continuity of f and g , $v^* \in \text{cl}(\{v | v \in D, f(v) > f(v^*)\})$. So $\lambda \cdot v^* = c$. Then it is easy to verify that λ satisfies the conditions in **Theorem 4**. Q.E.D.

Corollary 4.1. Let X be a non-empty state space, $\{V^i\}_{i=1}^n \subset C\Delta(X)$, $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi-concave. Let $D \equiv \{v | g(v) \geq 0\}$, g is quasi-concave. If f and g are both differentiable, then there exists P^* solving **Equation (1)**, $v^* = (E_P[V^i])$ and $\gamma, \eta \geq 0$ such that:

$$P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} (\eta \nabla f(v^*) + \gamma \cdot Jg(v^*)) \cdot (E_P[V^1], \dots, E_P[V^n])$$

Proof. By **Theorem 4**:

$$v^* \in \arg \min_{f(v) \geq f(v^*), v \in D} \lambda \cdot v \tag{5}$$

It is easy to verify that **Equation (5)** as a dual problem is a convex optimization problem. Since both f and g are differentiable, by Kuhn-Tucker condition, there exists $\gamma, \eta \geq 0$ such that:

$$\lambda - \eta \cdot \nabla f(v^*) - \gamma \cdot Jg(v^*) = 0$$

Then by definition of λ :

$$P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} (\eta \nabla f(v^*) + \gamma \cdot Jg(v^*)) \cdot (E_P[V^1], \dots, E_P[V^n])$$

Q.E.D.

3.4. Maximum theorem

Theorem 5. Let X be a non-empty state space, $\{V^i\}_{i=1}^n \subset C\Delta(X)$, $f \in C\mathbb{R}^n$. Suppose $D(\mu)$ is a continuous correspondence and $\forall \mu \in \Delta(X)$, $\mathcal{V}(\mu) \cap D(\mu) \neq \emptyset$. Let $\kappa(\mu)$ be maximum in **Equation (1)** and $\mathcal{P}(\mu)$ be solution to **Equation (1)**, then $\kappa(\mu)$ is continuous and $\mathcal{P}(\mu)$ is compact-valued and upper hemicontinuous².

² with respect to Prokhorov metric.

Proof. [Theorem 5](#) is an application of maximum theorem. Since by [Lemma 3](#) $\mathcal{V}(\mu)$ and $D(\mu)$ are both continuous, $\mathcal{V}(\mu) \cap D(\mu)$ is non-empty, compact valued and continuous. [Equation \(1\)](#) is equivalent to maximizing $f(v)$ on $\mathcal{V}(\mu) \cap D(\mu)$. Therefore, by maximum theorem, $\kappa(\mu)$ is continuous and the argmax correspondence $V^*(\mu)$ is compact-valued and upper hemicontinuous.

Now we show that $\mathcal{P}(\mu)$ is compact valued and upper hemicontinuous.

- compactness: (sequential compactness will be sufficient) $\forall \{P_m\} \subset \mathcal{P}(\mu)$, consider $v_m = E_{P_m}[(V^i)]$. Then $v_m \in V^*(\mu)$, so there exists subsequence (without loss assume to be v) m itself) $v_m \rightarrow v \in V^*(\mu)$. Then since $\Delta^2(X)$ is compact by [Theorem 9](#), there exists subsequence $P_m \xrightarrow{w^*} P \in \Delta^2(X)$. Then $E_P[(V^i)] = \lim E_{P_m}[(V^i)] = \lim v_m = v \in V^*(\mu)$. So $P \in \mathcal{P}(\mu)$.
- upper hemicontinuity: $\forall \mu_m \rightarrow \mu, P_m \xrightarrow{w^*} P$ and $P_m \in \mathcal{P}(\mu_m)$. Then $v_m = E_{P_m}[(V^i)] \in V^*(\mu_m)$. By definition of w^* convergence, $v_m \rightarrow v = E_P[(V^i)]$. By upper hemicontinuity of $V^*(\mu)$, $v \in V^*(\mu)$. Therefore, $P \in \mathcal{P}(\mu)$.

Q.E.D.

4. Applications

4.1. Costly Information acquisition

A direct application of [Theorem 1](#) is to costly information acquisition problem. Consider a variant of rational inattention model. Decision utility at each belief is $F(\mu) = \max_a E_\mu[u(a, x)]$. Information measure of any experiment P is $I(P|\mu) = E_P[H(\mu) - H(v)]$ where H is the entropy function. Assume cost of experiments are convex in their measure, the decision problem can be written as:

$$\sup_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} E_P[F(v)] - f(E_P[H(\mu) - H(v)]) \quad (6)$$

In a standard rational inattention problem, f is linear. Then standard concavification method suggests that optimal experiment involves signals no more than $|X|$. The reason why we want to deviate from a linear f is that standard RI has two kind of debatable predictions: 1) prior invariant choice of optimal posteriors (see [Caplin and Dean \(2013\)](#)). 2) no dynamics if we allow repeated experiments (see [Steiner et al. \(2017\)](#)). However, when f is more general, say convex, we knew little about how to solve [Equation \(6\)](#). [Theorem 2](#) becomes useful.

Proposition 1. *There exists P^* solving [Equation \(6\)](#), $|\text{supp}(P^*)| = 2|X|$. Moreover, if f is differentiable, P^* solves:*

$$P^* \in \arg \max_{\substack{P \in \Delta^2(X) \\ E_P[v] = \mu}} E_P[F(v) - f'(E_{P^*}[H(\mu) - H(v)]) \cdot H(v)]$$

4.2. Dynamic information design

Consider the following Bellman equation:

$$V(\mu) = \max \left\{ F(\mu), \sup_{P \in \Delta^2(X)} e^{-\rho dt} E_P[V(v)] - f(E_P[H(\mu) - H(v)]) \right\} \quad (7)$$

$$\text{s.t.} \begin{cases} E_P[v] = \mu \\ E_P[H(\mu) - H(v)] \leq C \end{cases}$$

Proposition 2. *If $F, H \in C\Delta(X)$, $f \in C\mathbb{R}$. $F(x), f(x), C \geq 0$. Then there exists unique $V \in C\Delta(X)$ solving Equation (7).*

Proof. Let $\mathcal{Z} = \{V \in C\Delta(X) | F \leq V \leq \text{co}(F)\}$. We define operator:

$$T(V)(\mu) = \max \left\{ F(\mu), \sup_{P \in \Delta^2(X)} e^{-\rho dt} E_P[V(v)] - f(E_P[H(\mu) - H(v)]) \right\} \quad (8)$$

$$\text{s.t.} \begin{cases} E_P[v] = \mu \\ E_P[H(\mu) - H(v)] \leq C \end{cases}$$

By **Theorem 1**, the max operator is well defined. When $P = \delta_\mu$, $E_P[v] = \mu$ and $E_P[H(\mu) - H(v)] = 0$ so the sup operator is also well defined. Now we prove that T is a contraction mapping on (\mathcal{Z}, L_∞) .

- $T(\mathcal{Z}) \subset \mathcal{Z}$: First of all, given the outer max operator in **Equation (8)**, $T(V)(\mu) \geq F(\mu)$. Then $\forall P \in \Delta^2(X)$ such that $E_P[v] = \mu$ and $E_P[H(\mu) - H(v)] \leq C$:

$$\begin{aligned} & e^{-\rho dt} E_P[V(v)] - f(E_P[H(\mu) - H(v)]) \\ & \leq e^{-\rho dt} E_P[V(v)] \\ & \leq E_P[\text{co}(F)(v)] \\ & = \text{co}(F)(\mu) \end{aligned}$$

First inequality is from f being non-negative, second inequality is from V being non-negative, $e^{-\rho dt} < 1$ and $V \leq \text{co}(F)$. Last equality is from $\text{co}(F)$ being linear. Last step is to show $T(\mathcal{Z})(\mu) \in C\Delta(X)$. This is directly implied by **Theorem 5**.

- $T(V)$ is monotonic: Suppose $U(\mu) \geq 0$ and $U + V \in \mathcal{Z}$. If $T(V)(\mu) = F(\mu)$, then by construction $T(V + U) \geq F(\mu) = T(V)(\mu)$. If $T(V)(\mu) > F(\mu)$, let P be solution to **Equation (8)** at μ for V :

$$\begin{aligned} T(V + U)(\mu) & \geq e^{-\rho dt} E_P[V(v) + U(v)] - f(E_P[H(\mu) - H(v)]) \\ & = T(V)(\mu) + e^{-\rho dt} E_P[U(v)] \\ & \geq T(V)(\mu) \end{aligned}$$

And constraints $E_P[H(\mu) - H(v)] \leq C$ and $E_P[v] = \mu$ are independent of choice of V so still satisfied.

- $T(V)$ is contraction. We claim that $T(V + \alpha)(\mu) \leq T(V)(\mu) + e^{-\rho dt} \alpha$. Suppose not true at μ . Obviously $T(V + \alpha)(\mu) > F(\mu)$. Then let P be the solution of [Equation \(8\)](#) at μ for $V + \alpha$.

$$\begin{aligned}
T(V)(\mu) &\geq e^{-\rho dt} E_P[V(v)] - f(E_P[H(\mu) - H(v)]) \\
&= e^{-\rho dt} E_P[V(v) + \alpha] - f(E_P[H(\mu) - H(v)]) - e^{-\rho dt} \alpha \\
&= T(V + \alpha)(\mu) - e^{-\rho dt} \alpha \\
&> T(V)(\mu)
\end{aligned}$$

Similar to last part, constraints $E_P[H(\mu) - H(v)] \leq C$ and $E_P[v] = \mu$ are still satisfied. Contradiction.

Therefore, by Blackwell condition, $T(V)$ is a contraction mapping on \mathcal{Z} . There exists a unique solution $V \in \mathcal{Z}$ solving the fixed point problem $T(V) = V$. Q.E.D.

4.3. Persuade voters with outside options

Consider a politician who can strategically design a public signal to voters to influence their voting behavior (the setup in [Alonso and Câmara \(2016\)](#)).

Voting game: There are $n \geq 1$ voters who chooses from a binary policy set $A = \{a_0, a_1\}$. There are two states $X = \{x_0, x_1\}$. Each voter gets Bernoulli utility $u_i(a, x)$ from voting for the policy a . Assume that a_1 is unanimously preferred to a_0 when x_1 is the true state and vice versa. The politician has state independent utility over policies and prefers a_1 strictly to a_0 . I assume that a_0 is a default policy. For a_1 to be proved, the politician needs more than m ($m \leq n$) voters to voter for a_1 . The politician can design a signal structure to influence voters' decisions. Equivalently, I assume that the politician chooses a distribution over posterior beliefs $P \in \Delta^2(X)$.

Outside option: Different from [Alonso and Câmara \(2016\)](#), where number of potential voters is fixed, I assume that each voter has opportunity cost c_i of participating in the voting game. Therefore, to approve the new policy, the politician should first attract at least m voters to the game and then persuade them to vote for a_1 .

To simplify notation, I write all voter's utility as functions of belief $F_i(\mu)$. Let $\bar{\mu}_i$ be the threshold belief for each voter to vote for a_1 . The politician's optimization problem can be written as:

$$\begin{aligned}
&\sup_{i_1, \dots, i_k, P} E_P \left[\mathbf{1}_{\#\{\mu \geq \bar{\mu}_{i_j}\} \geq m} \right] & (9) \\
&\text{s.t.} \quad \begin{cases} E_P[F_{i_j}] \geq c_{i_j} \\ E_P[v] = \mu \end{cases}
\end{aligned}$$

Notice that in [Equation \(9\)](#), the politician doesn't necessarily need to exclude voters outside of $\{i_1, \dots, i_k\}$, so the maximum from [Equation \(9\)](#) must be weakly larger than the politician's optimal utility. On the other hand, for any strategy in [Equation \(9\)](#), potentially including more voters to the voting game can only make the politician better off. So [Equation \(9\)](#) exactly characterizes the politician's optimization problem.

For any voter, except for $\bar{\mu}_i$, there is another critical belief $\tilde{\mu}_i$:

$$\frac{\tilde{\mu}_i - \mu}{\tilde{\mu}_i} F_i(0) + \frac{\mu}{\tilde{\mu}_i} F_i(\tilde{\mu}_i) = c_i$$

Suppose voter observes information structure inducing posterior belief 0 and $\tilde{\mu}_i$, then the voter is exactly indifferent between paying the opportunity cost and entering the voting game and not.

Proposition 3. Let μ^* be the smallest belief s.t. $\#\{i|\bar{\mu}_i \geq \mu^*\} \geq m$ and $\#\{i|\tilde{\mu}_i \geq \mu^*\} \geq m$, then the optimal strategy for Equation (9) is:

$$\begin{cases} P(0) = \frac{\mu^* - \mu}{\mu^*} \\ P(\mu^*) = \frac{\mu}{\mu^*} \end{cases}$$

and $\{i_1, \dots, i_k\} = \{i | \min\{\tilde{\mu}_i, \bar{\mu}_i\} \geq \mu^*\}$.

Proposition 3 states that when voters must pay opportunity cost to enter the voting game, then there are potentially two pivotal voters. One is the one who's most difficult to persuade to adopt a_1 , and the other is the one who's most difficult to attract to the voting game. Both *difficulty* levels are measured by the location of the critical beliefs.

Proof. The key step of proving **Proposition 3** is to apply **Corollary 4.1** to Equation (9). Notice that the objective function is Equation (9) is in fact an indicator function with some threshold belief level (say μ' , which is the lowest belief to persuade at least m voters to vote for a_1). So **Corollary 4.1** is directly applicable to Equation (9), and the objective function is in the form of:

$$\sum \lambda_{ij} \max\{0, \mu - \bar{\mu}_{ij}\} + \mathbf{1}_{\mu \geq \mu'} \quad (10)$$

It is easy to see that Equation (10) is a convex function on $\mu \in [0, \mu']$ and a linear function on $\mu \in [\mu', 1]$ (there is no point to include voters who will never vote for a_1). So optimal persuasion strategy must induce either belief 0 or interior belief $\nu > \mu'$. Of course since at least m voters are included and persuaded, $\nu \geq \mu^*$. On the other hand, it is easy to verify that the strategy define by μ^* induces at least m voters to participate, so μ^* is optimal. Q.E.D.

4.4. Screening with information

Consider a problem of Bayesian persuasion with unknown receiver types. Let Θ be the set of receiver types, X be the finite set of states and A be the set of actions. $\forall \theta \in \Theta$, decision utility at each belief is $F_\theta(\mu) = \max_a E_\mu[u(a, x, \theta)]$. Sender's utility at each belief given receiver type θ is $V_\theta(\mu)$. Assume that the type distribution is $\pi(\theta) \in \Delta(\Theta)$. The sender can screen the receivers by providing a menu of information structures. Then by revelation principle, sender's optimization problem is:

$$\sup_{P_\theta \in \Theta \times \Delta^2(X)} \int E_{P_\theta}[V_\theta] \pi(\theta) d\theta \quad (11)$$

$$\text{s.t. } \begin{cases} E_{P_\theta}[F_\theta] \geq E_{P_{\theta'}}[F_\theta] \quad \forall \theta, \theta' \in \Theta \\ E_{P_\theta}[v] = \mu \quad \forall \theta \in \Theta \end{cases}$$

When Θ and A are both infinite, solving [Equation \(11\)](#) is difficult due to the dimensionality of strategy space. When A is finite, it is WLOG to restrict the sender to use direct message which suggests the actions being played conditional on the state. Then [Equation \(11\)](#) reduces to a screening problem with finite dimensional strategy function (plus a few more obedience constraints). In the remaining case where Θ is finite but A is infinite, it is still unclear whether it is WLOG to consider only finite dimensional screening mechanisms.

Now consider the finite Θ case. Suppose $\Theta = \{1, \dots, N\}$. Define:

$$\begin{cases} \mathcal{V}_i(\mu) = \left\{ E_P[V_i], E_P[F_1], \dots, E_P[F_N] \mid P \in \Delta^2(X), E_P[v] = \mu \right\} \\ D(\mu) = \left\{ v \in \mathbb{R}^{(N \times (N+1))} \mid v_i^{i+1} \geq v_j^{i+1} \forall i, j \right\} \end{cases}$$

Then [Equation \(11\)](#) is equivalent to the following problem:

$$\sup_{v \in D(\mu) \cap \times_{i=1}^N \mathcal{V}_i(\mu)} \pi_i v_i^1 \quad (12)$$

By [Lemma 1](#), each $\mathcal{V}_i(\mu)$ is compact set. Therefore, $D(\mu) \cap \times \mathcal{V}_i(\mu)$ is compact. It is easy to see that $D(\mu) \cap \times \mathcal{V}_i(\mu)$ is non-empty. By Weierstrass's theorem, there exists v^* solving [Equation \(12\)](#). Then by [Lemma 1](#), there exists $P_i^* \in \Delta^2(X)$ s.t. $v_i^* = (E_{P_i^*}[V_i], E_{P_i^*}[F_1], \dots, E_{P_i^*}[F_N])$ and $|\text{supp}(P_i^*)| \leq (N+2) \cdot |X|$. Therefore, (P_1^*, \dots, P_N^*) solves [Equation \(11\)](#) and we get the following proposition:

Proposition 4. *If Θ is finite, then $\forall \mu \in \Delta(X)$, there exists $(P_1^*, \dots, P_N^*) \in \Delta^2(X)^N$ solving [Equation \(11\)](#) and each $|\text{supp}(P_i^*)| \leq (N+2) \cdot |X|$.*

[Proposition 4](#) states that it is WLOG to consider only mechanisms with finite support when solving [Equation \(11\)](#). Therefore, it is sufficient to maximize over $N(N+2) \cdot |X|$ posterior beliefs and $N(N+2) \cdot |X|$ corresponding probabilities to solve constrained optimization problem [Equation \(11\)](#), which is a computationally tractable problem.

5. Conclusion

In this paper, I study the set of all possible combinations of expected valuations that can be implemented by designing information. I show that the set can be implemented only using information structures with finite realizations, and all extreme points of the set can be characterized using a concavification characterization. I developed a Lagrange method in the information design setup, and applied the results to various applications including static and dynamic information acquisition, persuasion of receivers with outside options and screening using information.

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A. Theorems used in proof

Here are key theorems used for my proof. **Theorem 6** is Straszewicz’s theorem (Straszewicz (1935), see Theorem 18.6 of Rockafellar (1969)). **Theorem 7** is Krein-Milman theorem (see Theorem 3.23 of Rudin (1991)). **Theorem 8** is Carathéodory’s theorem (Carathéodory (1907)). **Theorem 9** is Prokhorov’s theorem (see Theorem 5.1 of Billingsley (2013))

Theorem 6. Let $C \in \mathbb{R}^n$ be a closed convex set, $\text{cl}(\text{exp}(C)) = \text{ext}(C)$.

Theorem 7. Let $C \in \mathbb{R}^n$ be a compact and convex set, $C = \text{conv}(\text{ext}(C))$.

Theorem 8. Let $C \in \mathbb{R}^n$, if $x \in \text{conv}(C)$ then $x \in \text{conv}(R)$ for $R \subset C$, $|R| \leq n + 1$.

Theorem 9. A tight set Π of probability measures on Borel sets of metric topological space \mathcal{X} is relative compacc in weak-* topology.

Lemma 4. Let C be a convex set in \mathbb{R}^n . Then $\forall F \in F(C)$, $\text{ext}(F) \subset \text{ext}(C)$.

Proof. $\forall x \in \text{ext}(F)$ there exists affine f defining face F . $\forall y, z \in C$. Suppose $y \in F$, then $f(x) = f(y)$. If there exists $\alpha \in (0, 1)$ s.t. $\alpha y + (1 - \alpha)z = x$, then $\alpha f(y) + (1 - \alpha)f(z) = f(x) \implies f(z) = f(x) = f(y)$ so $z \in F$. Since $x \in \text{ext}(F)$, $x \in \{y, z\}$. Suppose $y \notin F$, then $f(x) = \alpha f(y) + (1 - \alpha)f(z) < f(x)$ by definition of f , contradiction. To sum up, $x \in \text{ext}(C)$. Q.E.D.